Mathematical induction and universal quantification

In the following, x and x' range over the same domain; similarly, y and y' range over the same domain. Moreover, $<$ is a well-founded relation on the domain of x, i.e. for any P

(0) $(\underline{A}x :: P.x) \equiv (\underline{A}x :: P.x \Leftarrow (\underline{A}x' : x'<x : P.x'))$.

Consider now the obligation to prove for some R

(1) $\qquad (\underline{A}x,y :: R.x.y)$.

Two approaches are now possible.

(i) $\quad$ (1)
$\quad = \quad$ {nesting}
$\quad (\underline{A}y :: (\underline{A}x :: R.x.y))$
$\quad = \quad$ {(0) with $P.x := R.x.y$; unnesting}
$\quad (\underline{A}x,y :: R.x.y \Leftarrow (\underline{A}x' : x'<x : R.x'.y))$

(ii) $\quad$ (1)
$\quad = \quad$ {nesting}
$\quad (\underline{A}x :: (\underline{A}y :: R.x.y))$
$\quad = \quad$ {(0) with $P.x := (\underline{A}y :: R.x.y)$; renaming dummy}
$\quad (\underline{A}x :: (\underline{A}y :: R.x.y) \Leftarrow (\underline{A}x' : x'<x : (\underline{A}y' :: R.x'.y')))$
$\quad = \quad$ {$\Leftarrow Q$ distributes over $\underline{A}$; unnesting}
$\quad (\underline{A}x,y :: R.x.y \Leftarrow (\underline{A}x' : x'<x : (\underline{A}y' :: R.x'.y')))$

Method (i) gives a shorter formula, but ends with a stronger proof obligation than method (ii), which, therefore, is more powerful. The message was driven home to me at last week's session of the ETAC, where we considered for $R.x.y$

$$\neg\, x<y \;\vee\; \neg\, y<x \quad .$$

Approach (i) leads one into a dead end; approach (ii) leads one to observe for any x,y

$$(\underline{A}x' : x'<x : (\underline{A}y' :: \neg\, x'<y' \vee \neg\, y'<x'))$$
$$= \quad \{\text{trading, unnesting}\}$$
$$(\underline{A}x',y' :: \neg\, x'<x \vee \neg\, x'<y' \vee \neg\, y'<x')$$
$$\Rightarrow \quad \{\text{instantiate with } x',y' := y,x\}$$
$$\neg\, y<x \vee \neg\, y<x \vee \neg\, x<y$$
$$= \quad \{\text{pred. calc.}\}$$
$$\neg\, x<y \vee \neg\, y<x \quad .$$

Nuenen, 10 July 1990

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