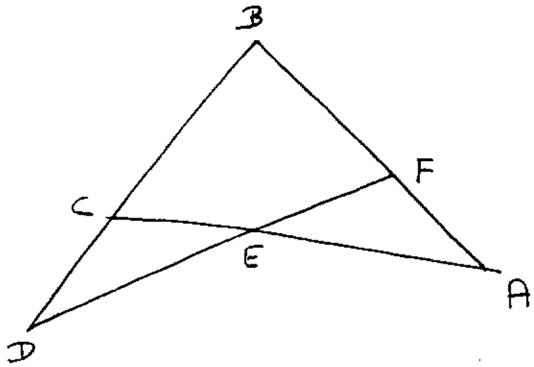
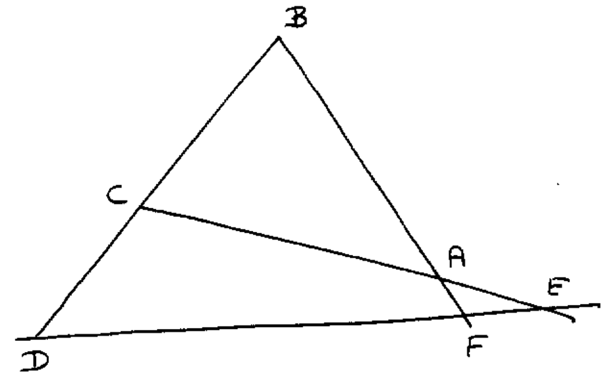
Proving the theorem of Menelaos





For non-degenerate triangle ABC and D, E, F collinear as in above figures the theorem of Menelaos states

$$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1 \quad .$$

In terms of λ, μ, r, defined by

$D = \lambda B + (1-\lambda)C$, $E = \mu C + (1-\mu)A$, $F = rA + (1-r)B$

we have to conclude

$$\frac{1-\lambda}{\lambda} \cdot \frac{1-\mu}{\mu} \cdot \frac{1-r}{r} = -1 \qquad \text{or}$$

(0) $\quad (1-\lambda)\cdot(1-\mu)\cdot(1-r) + \lambda\cdot\mu\cdot r = 0$

from the fact that D, E, and F are collinear. This is expressed by

(1) $\quad$ Det. $P = 0$ where $\quad P = \begin{vmatrix} D_x & D_y & 1 \\ E_x & E_y & 1 \\ F_x & F_y & 1 \end{vmatrix}$

Writing $D_x = \lambda B_x + (1-\lambda)C_x$, etc. we can factorize

$P = Q \cdot R$ where

$$Q = \begin{vmatrix} \lambda & 1-\lambda & 0 \\ 0 & \mu & 1-\mu \\ 1-\nu & 0 & \nu \end{vmatrix} \qquad R = \begin{vmatrix} B_x & B_y & 1 \\ C_x & C_y & 1 \\ A_x & A_y & 1 \end{vmatrix}$$

From this factorization we conclude

(2) $\quad Det.P = (Det.Q) \cdot (Det.R)$.

Triangle ABC being non-degenerate is expressed by

(3) $\quad Det.R \neq 0$,

and from (1), (2), (3) we conclude

$$Det.Q = 0 ,$$

which, in view of Q's definition, equivales (0) .

I designed the above proof in reaction to a very classical argument in which the above two figures had to be dealt with separately. I like the proof for the way R enters the picture. (And just in case you feel tempted to send me shorter proofs of Menelaos's theorem: I know all about barycentric coordinates.)

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