Problem 10406 from The American Mathematical Monthly, Volume 101, Number 8 / October 1994

10406. *Proposed by David C. Fisher, University of Colorado, Denver, CO, Karen L. Collins, Wesleyan University, Middleton, CT, and Lucia B. Krompart, Rochester, MI.*

Show that a path on an m by n square grid which starts at the northwest corner, goes through each point exactly once, and ends at the southeast corner divides the grid into two equal halves: (a) those regions opening north or east; and (b) those regions opening south or west.

(**10406**) A path meeting the conditions of the problem on a 5 by 8 grid is shown in figure 10406 below.

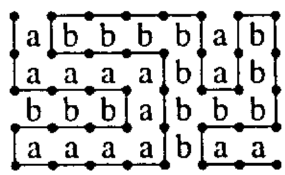
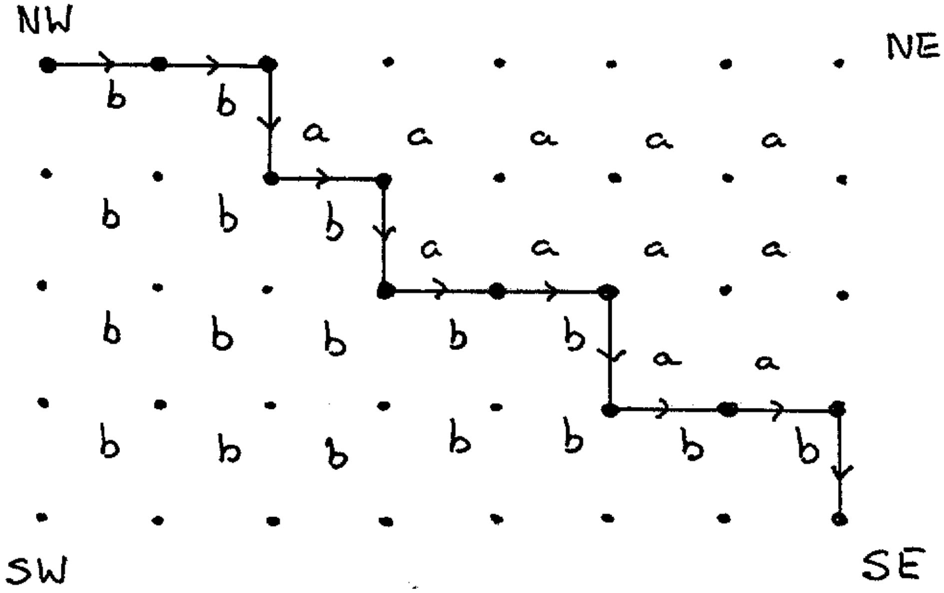


Figure 10406

We confine our attention to paths from NW to SE that do not necessarily go through each point; the off-path points are called free. Directing the paths from NW to SE, I identify (here without proof) the region(s) "a" with those to the left of the path, and the regions "b" with those to its right.

A path, along which turns to the left and turns to the right alternate, is a shortest path; we give a simple example. (in the same 5 by 8 grid).



For the m by n grid with a shortest path we observe

- \# points = $m * n$
- \# unit squares = $(m-1) * (n-1)$
- \# edges on shortest path = $(m-1) + (n-1)$
- \# points on shortest path = $m+n-1$
- \# free points = $(m*n) - (m+n-1)$
  $= (m-1) * (n-1)$

in short: the number of unit squares equals the number of free points. But we can go further: this equality holds for the separate regions! We can establish in the a-region a 1-1 correspondence between each unit square and its NE corner, and in the b-region between each unit square and its SW corner.

Introducing

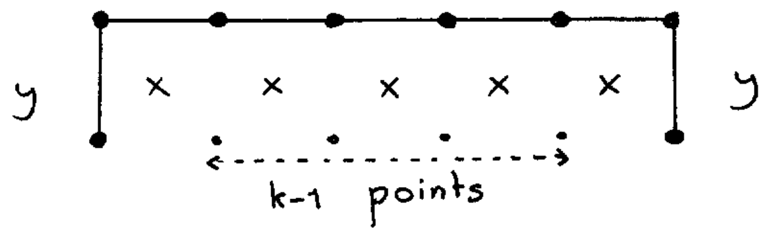
Fa and Fb : # free points in a-region(s)

and b-region(s) respectively
Sa and Sb : # unit squares in a-region(s) and b-region(s) respectively,
we can now formulate

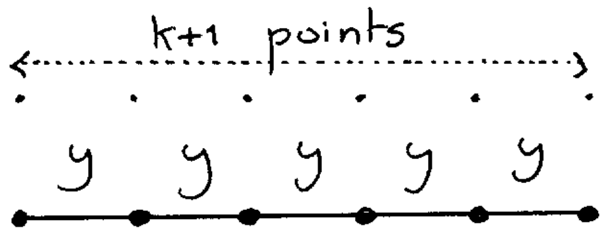
<u>Lemma 0</u> In the case of a shortest path
$Fa = Sa \wedge Fb = Sb$.

The idea of the proof is to start with a given path that visits each point exactly once — i.e. $Fa = 0 \wedge Fb = 0$ —, and to transform it in a finite number of moves into a shortest path for which $Fa = Sa \wedge Fb = Sb$ holds. Initial and final state are then to be linked via an invariant that is maintained by each move.

For the move we consider that, as said, a path, along which turns to the left and turns to the right alternate, is a shortest path. Consequently, if the path is not of minimal length, it contains at least one pair of <u>successive</u> turns in the <u>same</u> direction. With those turns k edges apart it means that there exists a rectangle — a "bar" — of k ($k \geq 1$) unit squares such that 3 of its sides are formed by 1, k, 1 edges of the path, e.g. for k = 5:



We can -and will- always choose a rectangle such that the k-1 interior points of the 4th side are free. The move shortens the path by 2 edges in the obvious way; in the same example, the above configuration is transformed into



With $x,y = a,b$ or $x,y = b,a$, we observe for the transformation by the move:

$$\begin{array}{ll} \Delta\, Sx = -k & \qquad \Delta\, Sy = +k \\ \Delta\, Fx = -(k-1) & \qquad \Delta\, Fy = +k+1 \\ \hline \Delta(Sx - Fx) = -1 & \qquad \Delta(Sy - Fy) = -1 \end{array}$$

The above transformation reduces at both sides of the path the difference S-F by 1, and hence

$$(Sa - Fa) - (Sb - Fb) = q \quad .$$

if initially true, is an invariant of our total transformation process (which obviously terminates since each move reduces the path by 2 edges).

From Lemma 0, which is applicable in the final state, we conclude $q = 0$. Therefore initially, with $Fa = 0 \wedge Fb = 0$, we have $Sa - Sb = 0$ or $Sa = Sb$. Quod erat demonstrandum.

* * *

It took me several hours to design the move of the previous page; for quite a while I considered 2 unparameterized moves: the current one with $k = 1$ and "  ", which changes the shape of the path without shortening it. The mere fact that the second move complicates the termination argument should have made me reject it much faster.

I liked the problem.

Austin, 17 October 1994

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