The arithmetic and geometric means once more

In the following, x, y, c are positive.

In EWD1140, I used

(0) $$(x+y)^2 = (x-y)^2 + 4\cdot x\cdot y$$

to argue that

(1) $$x, y := c,\ x+y-c \quad ,$$

which does not change $x+y$, increases $x\cdot y$ provided c lies between the initial values of x and y.

In EWD1171, I used (0) to argue that

(2) $$x, y := c,\ x\cdot y/c \quad ,$$

which does not change $x\cdot y$, decreases $x+y$ provided c lies between the initial values of x and y.

In both cases the use of (0) came a little bit as a rabbit and the link between the condition on c and the decrease of the distance between x and y remained informal. Last Thursday, when I asked for an expression that contained both $x+y$ and $x\cdot y$, my

An Thai Nguyen suggested that we look at $(c-x)\cdot(c-y)$, and this expression indeed plays a central role in the derivations from which all rabbits have been removed.

* * *

We want to change x, y such that (i) their sum is not changed, and (ii) their product is increased. Any assignment satisfying (i) can be written like (1); in order to satisfy (ii) we now observe for any c

$$
\begin{aligned}
& \text{"(1) increases } x\cdot y\text{"} \\
= & \quad \{\text{program semantics, (1)}\} \\
& x\cdot y < c\cdot(x+y-c) \\
= & \quad \{\text{algebra}\} \\
& (c-x)\cdot(c-y) < 0 \quad .
\end{aligned}
$$

We now consider the change of x, y such that (iii) their product is not changed, and (iv) their sum is decreased. Any assignment satisfying (iii) can be written like (2); in order to satisfy (iv) as well, we observe for any positive c

$$
\begin{aligned}
& \text{"(2) decreases } x+y\text{"} \\
= & \quad \{\text{program semantics, (2)}\} \\
& c + x\cdot y/c < x+y
\end{aligned}
$$

$$
\begin{array}{ll}
= & \{c > 0\} \\
 & c^2 + x \cdot y < c \cdot (x+y) \\
= & \{\text{algebra}\} \\
 & (c-x) \cdot (c-y) < 0
\end{array}
$$

So, in both cases, the completely forced calculations lead in exactly the same form to the conclusion that c should lie between the initial values of x and y. The secret is that Nguyen's expression can be rewritten as

$$x \cdot y - c \cdot (x+y-c)$$

and as

$$(c^2 + x \cdot y) - (c \cdot x + c \cdot y) \quad ,$$

i.e. the difference of two products with equal sums of their factors, and the difference of two sums with equal products of their addenda. I was surprised.

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