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On Equivalent Transformations of Infinitary Formulas under the Stable Model Semantics (Preliminary Report)

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Abstract. It has been known for a long time that intuitionistically equivalent formulas have the same stable models. We extend this theorem to propositional formulas with infinitely long conjunctions and disjunctions and show how to apply this generalization to proving properties of aggregates in answer set programming.

1 Introduction

This note is about the extension of the stable model semantics to infinitary propositional formulas defined in [6]. One of the reasons why stable models of infinitary formulas are important is that they are closely related to aggregates in answer set programming (ASP). The semantics of aggregates proposed in [1, Section 4.1] treats a ground aggregate as shorthand for a propositional formula. An aggregate with variables has to be grounded before that semantics can be applied to it. For instance, to explain the precise meaning of the expression $1\{p(X)\}$ ("there exists at least one object with the property p") in the body of an ASP rule we first rewrite it as

$$1\{p(t_1), \ldots, p(t_n)\},$$

where $t_1, \ldots, t_n$ are all ground terms in the language of the program, and then turn it into the propositional formula $p(t_1) \vee \cdots \vee p(t_n)$. But this description of the meaning of $1\{p(X)\}$ implicitly assumes that the Herbrand universe of the program is finite. If the program contains function symbols then an infinite disjunction has to be used.[3] [4]

[3] There is nothing exotic or noncomputable about ASP programs containing both aggregates and function symbols. For instance, the program

$$\begin{array}{l} p(f(a)) \\ q \leftarrow 1\{p(X)\} \end{array}$$

has simple intuitive meaning, and its stable model $\{p(f(a)), q\}$ can be computed by existing solvers.

[4] References to grounding in other theories of aggregates suffer from the same problem. For instance, the definition of a ground instance in Section 2.2 of the ASP Core doc-

Our goal here is to develop methods for proving that pairs F, G of infinitary formulas have the same stable models. From the results of [5] and [1] we know that in the case of finite propositional formulas it is sufficient to check that the equivalence $F \leftrightarrow G$ is provable intuitionistically. Some extensions of intuitionistic propositional logic, including the logic of here-and-there, can be used as well. In this note we extend these results to deductive systems of infinitary propositional logic.

This goal is closely related to the idea of strong equivalence [4]. The provability of $F \leftrightarrow G$ in the deductive systems of infinitary logic described below guarantees not only that F and G have the same stable models, but also that for any set $\mathcal{H}$ of infinitary formulas, $\mathcal{H} \cup \{F\}$ and $\mathcal{H} \cup \{G\}$ have the same stable models.

We review the stable model semantics of infinitary propositional formulas in Section 2. An infinitary system of natural deduction, similar to propositional intuitionistic logic, is defined in Section 3. Then we discuss the main theorem, which relates this system to stable models (Section 4), and state a few other useful facts (Section 5). In Section 6 this theory is applied to examples involving aggregates.

2 Stable Models of Infinitary Propositional Formulas

The definitions of infinitary formulas and their stable models given below are equivalent to the definitions proposed in [6].

Let σ be a propositional signature, that is, a set of propositional atoms. The sets $\mathcal{F}_0^\sigma$, $\mathcal{F}_1^\sigma$, ... are defined as follows:

- $\mathcal{F}_0^\sigma = \sigma \cup \{\bot\}$,
- $\mathcal{F}_{i+1}^\sigma$ is obtained from $\mathcal{F}_i^\sigma$ by adding expressions $\mathcal{H}^\wedge$ and $\mathcal{H}^\vee$ for all subsets $\mathcal{H}$ of $\mathcal{F}_i^\sigma$, and expressions $F \rightarrow G$ for all $F, G \in \mathcal{F}_i^\sigma$.

The elements of $\bigcup_{i=0}^{\infty} \mathcal{F}_i^\sigma$ are called *(infinitary) formulas* over σ.

Negation and equivalence will be understood as abbreviations: $\neg F$ stands for $F \rightarrow \bot$, and $F \leftrightarrow G$ stands for $(F \rightarrow G) \wedge (G \rightarrow F)$.

We will write $\{F, G\}^\wedge$ as $F \wedge G$, and $\{F, G\}^\vee$ as $F \vee G$. Thus finite propositional formulas over σ can be viewed as a special case of infinitary formulas.

Subsets of a signature σ will be also called its *interpretations*. The satisfaction relation between an interpretation I and a formula F is defined as follows:

- $I \not\models \bot$.
- For every $p \in \sigma$, $I \models p$ if $p \in I$.
- $I \models \mathcal{H}^\vee$ if there is a formula $F \in \mathcal{H}$ such that $I \models F$.
- $I \models \mathcal{H}^\wedge$ if for every formula $F \in \mathcal{H}$, $I \models F$.
- $I \models F \rightarrow G$ if $I \not\models F$ or $I \models G$.

ument (`https://www.mat.unical.it/aspcomp2013/files/ASP-CORE-2.0.pdf`, Version 2.02) talks about replacing the expression $\{e_1; \ldots; e_n\}$ in a rule with a set denoted by $\mathrm{inst}(\{e_1; ...; e_n\})$. But that set can be infinite.

We say that I satisfies a set $\mathcal{H}$ of formulas if I satisfies all elements of $\mathcal{H}$.

The *reduct* F^I of a formula F with respect to an interpretation I is defined as follows:

- $\bot^I = \bot$.
- For $p \in \sigma$, $p^I = \bot$ if $I \not\models p$; otherwise $p^I = p$.
- $(\mathcal{H}^\wedge)^I = \{G^I \mid G \in \mathcal{H}\}^\wedge$.
- $(\mathcal{H}^\vee)^I = \{G^I \mid G \in \mathcal{H}\}^\vee$.
- $(G \to H)^I = \bot$ if $I \not\models G \to H$; otherwise $(G \to H)^I = G^I \to H^I$.

The *reduct* $\mathcal{H}^I$ of a set $\mathcal{H}$ of formulas is the set consisting of the reducts of the elements of $\mathcal{H}$. An interpretation I is a *stable model* of a set $\mathcal{H}$ of formulas if it is minimal w.r.t. set inclusion among the interpretations satisfying $\mathcal{H}^I$; a stable model of a formula F is a stable model of singleton $\{F\}$. This is a straightforward extension of the definition of a stable model due to Ferraris [1] to infinitary formulas.

3 Basic Infinitary System of Natural Deduction

Inference rules of the deductive system described below are similar to the standard natural deduction rules of propositional logic (see, for instance, [3, Section 1.2.1]). In this system, derivable objects are *(infinitary) sequents*—expressions of the form $\Gamma \Rightarrow F$, where F is an infinitary formula, and Γ is a *finite* set of infinitary formulas ("F under assumptions Γ"). To simplify notation, we will write Γ as a list. We will identify a sequent of the form $\Rightarrow F$ with the formula F.

There is one axiom schema $F \Rightarrow F$. The inference rules are the introduction and elimination rules for the propositional connectives

$$(\wedge I)\ \frac{\Gamma \Rightarrow H \quad \text{for all } H \in \mathcal{H}}{\Gamma \Rightarrow \mathcal{H}^\wedge} \qquad (\wedge E)\ \frac{\Gamma \Rightarrow \mathcal{H}^\wedge}{\Gamma \Rightarrow H} \quad (H \in \mathcal{H})$$

$$(\vee I)\ \frac{\Gamma \Rightarrow H}{\Gamma \Rightarrow \mathcal{H}^\vee} \quad (H \in \mathcal{H}) \qquad (\vee E)\ \frac{\Gamma \Rightarrow \mathcal{H}^\vee \qquad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathcal{H}}{\Gamma, \Delta \Rightarrow F}$$

$$(\to I)\ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G} \qquad (\to E)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G}$$

and the contradiction and weakening rules

$$(C)\ \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow F} \qquad (W)\ \frac{\Gamma \Rightarrow F}{\Gamma, \Delta \Rightarrow F}\ .$$

(Note that we did not include the law of the excluded middle in the set of axioms, so that this deductive system is similar to intuitionistic, rather than classical, propositional logic.)

The set of *theorems of the basic system* is the smallest set of sequents that includes the axioms of the system and is closed under the application of its inference rules. We say that formulas F and G are *equivalent in the basic system* if $F \leftrightarrow G$ is a theorem of the basic system. The reason why we are interested in this relation is that formulas equivalent in the basic system have the same stable models, as discussed in Section 4 below.

Example 1. Consider a formula of the form

$$F_0 \wedge \{F_i \rightarrow F_{i+1} \mid i \geq 0\}^{\wedge}$$

or, in more compact notation,

$$F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}). \tag{1}$$

Let us check that it is equivalent in the basic system to the formula $\bigwedge_{i \geq 0} F_i$. The sequent

$$F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}) \Rightarrow F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1})$$

belongs to the set of theorems of the basic system. Consequently so do the sequents

$$F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}) \ \Rightarrow \ F_0$$

and

$$F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}) \Rightarrow \ F_j \rightarrow F_{j+1}$$

for all $j \geq 0$. Consequently the sequents

$$F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}) \Rightarrow \ F_j$$

for all $j \geq 0$ belong to the set of theorems as well (by induction on j). Consequently so does the sequent

$$F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}) \ \Rightarrow \ \bigwedge_{i \geq 0} F_i.$$

A similar argument (except that induction is not needed) shows that the sequent

$$\bigwedge_{i \geq 0} F_i \ \Rightarrow \ F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1})$$

is a theorem of the basic system also. Consequently so is the sequent

$$\Rightarrow \ F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}) \ \leftrightarrow \ \bigwedge_{i \geq 0} F_i.$$

This argument could be expressed more concisely, without explicit references to the set of theorems of the basic system, as follows. Assume (1). Then F_0 and, for every $i \geq 0$, $F_i \rightarrow F_{i+1}$. Then, by induction, F_i for every i. And so forth. This style of presentation is used in the next example.

Example 2. Let $\{F_\alpha\}_{\alpha \in A}$ be a family of formulas from some $\mathcal{F}_i^\sigma$, and let G be a formula. We show that

$$\left(\bigvee_{\alpha \in A} F_\alpha\right) \rightarrow G \tag{2}$$

is equivalent in the basic system to the formula

$$\bigwedge_{\alpha \in A} (F_\alpha \rightarrow G). \tag{3}$$

Left-to-right: assume (2) and F_α. Then $\bigvee_{\alpha \in A} F_\alpha$, and consequently G. Thus we established $F_\alpha \rightarrow G$ under assumption (2) alone for every α, and consequently established (3) under this assumption as well. Right-to-left: assume (3) and $\bigvee_{\alpha \in A} F_\alpha$, and consider the cases corresponding to the disjunctive terms of this disjunction. Assume F_α. From (3), $F_\alpha \rightarrow G$, and consequently G. Thus we established G in each case, so that (2) follows from (3) alone.

4 Main Theorem

Main Theorem. *For any set $\mathcal{H}$ of formulas,*

(a) if a formula F is a theorem of the basic system then $\mathcal{H} \cup \{F\}$ has the same stable models as $\mathcal{H}$;
(b) if F is equivalent to G in the basic system then $\mathcal{H} \cup \{F\}$ and $\mathcal{H} \cup \{G\}$ have the same stable models.

The proof of the main theorem relies on the following lemma: *For any theorem $\Gamma \Rightarrow F$ of the basic system and any interpretation I, the sequent $\{G^I \mid G \in \Gamma\} \Rightarrow F^I$ is a theorem of the basic system as well.* To prove the lemma, we show that the set of sequents $\Gamma \Rightarrow F$ such that $\{G^I \mid G \in \Gamma\} \Rightarrow F^I$ is a theorem of the basic system includes the axioms of the basic system and is closed under its inference rules.

The assertion of the theorem will remain true if we add an axiom schema corresponding to an infinitary version of the weak law of the excluded middle $\neg F \vee \neg\neg F$:

$$\bigvee_{\mathcal{I} \subseteq \mathcal{H}} \left(\neg \bigvee_{F \in \mathcal{H} \setminus \mathcal{I}} F \wedge \neg\neg \bigwedge_{F \in \mathcal{I}} F \right), \tag{4}$$

where $\mathcal{H}$ is an arbitrary subset of one of the sets $\mathcal{F}_i$.

5 Some Useful Properties of the Basic System

Let σ and σ' be disjoint signatures. A *substitution* is an arbitrary function from σ' to $\mathcal{F}_i^\sigma$, where i is a nonnegative integer. For any substitution α and any formula F over the signature $\sigma \cup \sigma'$, F^α stands for the formula over σ formed as follows:

- If $F \in \sigma$ or $F = \bot$ then $F^\alpha = F$.
- If $F \in \sigma'$ then $F^\alpha = \alpha(F)$.
- If F is $\mathcal{H}^\wedge$ then $F^\alpha = \{G^\alpha | G \in \mathcal{H}\}^\wedge$.
- If F is $\mathcal{H}^\vee$ then $F^\alpha = \{G^\alpha | G \in \mathcal{H}\}^\vee$.
- If F is $G \to H$ then $F^\alpha = G^\alpha \to H^\alpha$.

Formulas of the form F^α will be called *instances* of F.

Proposition 1. *If F is a theorem of the basic system then every instance of F is a theorem of the basic system also.*

Corollary. *If F is a finite formula provable in intuitionistic propositional logic then every instance of F is a theorem of the basic system.*

Proposition 2. *If for every atom p, $\alpha(p)$ is equivalent to $\beta(p)$ in the basic system then F^α is equivalent to F^β in the basic system.*

6 Examples Involving Aggregates

As discussed in the introduction, infinitary formulas can be used to precisely define the semantics of aggregates in ASP when the Herbrand universe is infinite. In this section, we give three examples demonstrating how the theory described above can be applied to prove equivalences between programs involving aggregates.

Example 3. Intuitively, the rule

$$q(X) \leftarrow 1\{p(X, Y)\} \tag{5}$$

has the same meaning as the rule

$$q(X) \leftarrow p(X, Y). \tag{6}$$

To make this claim precise, consider first the result of grounding rule (5) under the assumption that the Herbrand universe C is finite. In accordance with standard practice in ASP, we treat variable X as global and Y as local. Then the result of grounding (5) is the set of ground rules

$$q(a) \leftarrow 1\{p(a, b) \mid b \in C\}$$

for all $a \in C$. In the spirit of the semantics for aggregates proposed in [1, Section 4.1] these rules have the same meaning as the propositional formulas

$$\left(\bigvee_{b \in C} p(a, b) \right) \to q(a). \tag{7}$$

Likewise, rule (6) can be viewed as shorthand for the set of formulas

$$p(a, b) \to q(a) \tag{8}$$

for all $a, b \in C$. It easy to see that these sets of formulas are intuitionistically equivalent.

How can we lift the assumption that the Herbrand universe is finite? We can treat (7) as an infinitary formula, and show that the conjunction of formulas (7) is equivalent to the conjunction of formulas (8) in the basic system. The fact that the conjunction of formulas (8) for all $b \in C$ is equivalent to (7) in the basic system follows from Example 2 (Section 3).

Example 4. Intuitively,

$$q(X) \leftarrow 2\{p(X,Y)\} \tag{9}$$

has the same meaning as the rule

$$q(X) \leftarrow p(X,Y1),\ p(X,Y2),\ Y1 \neq Y2. \tag{10}$$

To make this claim precise, consider the infinitary formulas corresponding to (9):

$$\left(\bigvee_{b \in C} p(a,b) \wedge \bigwedge_{b \in C}\left(p(a,b) \rightarrow \bigvee_{\substack{c \in C \\ c \neq b}} p(a,c)\right)\right) \rightarrow q(a) \tag{11}$$

$(a \in C)$; see [1, Section 4.1] for details on representing aggregates with propositional formulas. The formulas corresponding to (10) are

$$(p(a,b) \wedge p(a,c)) \rightarrow q(a) \tag{12}$$

$(a, b, c \in C,\ b \neq c)$. Using the propositions stated above, we can show that the conjunction of formulas (11) is equivalent to the conjunction of formulas (12) in the basic system.

Example 5. Intuitively, the cardinality constraint $\{p(X)\}0$ ("the set of true atoms with form $p(X)$ has cardinality at most 0") has the same meaning as the conditional literal $\bot : p(X)$ ("for all X, $p(X)$ is false"). If we represent this conditional literal by the infinitary formula

$$\bigwedge_{a \in C} \neg p(a) \tag{13}$$

then this claim can be made precise by showing that (13) is equivalent to the formula

$$\bigwedge_{\substack{A \subseteq C \\ A \neq \emptyset}}\left(\bigwedge_{a \in A} p(a) \rightarrow \bigvee_{a \in C \setminus A} p(a)\right), \tag{14}$$

which corresponds to $\{p(X)\}0$ in the sense of [1], in the extended system described at the end of Section 4. It is easy to derive (14) from (13) in the basic system. The derivation of (13) from (14) uses the following instance of axiom schema (4):

$$\bigvee_{A \subseteq C}\left(\neg \bigvee_{a \in C \setminus A} p(a) \wedge \neg\neg \bigwedge_{a \in A} p(a)\right). \tag{15}$$

7 Future Work

Two finite propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there [1, Proposition 2]. The results of this note are similar to the if part of that theorem; we don't know how to extend the only if part to infinitary formulas. Axioms that are stronger than (4) are apparently required (perhaps a generalization of the axiom $F \vee (F \rightarrow G) \vee \neg G$ that is known to characterize the logic of here-and-there [2]). Identifying such axioms is a topic for future work.

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