GENERALIZED UNFOLDINGS FOR SHORTEST PATHS IN EUCLIDEAN 3-SPACE

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ABSTRACT

The problem of determining shortest paths in the presence of polyhedral obstacles between two points in Euclidean 3-space stems from the general problem of obtaining optimal collision free paths in robot systems. For the special case when paths are constrained to the surfaces of 3-dimensional objects, simple planar unfoldings are used to obtain the shortest path. For the general case when paths are not constrained to lie on any surface, we show the existence of generalized unfoldings wherein the shortest path in 3-space again becomes a straight line. These unfoldings consist of multiple rotations about the edges of the polyhedral obstacles.

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1. Introduction

The problem of finding the shortest path between two points in Euclidean 3-space, bounded by a finite collection of polyhedral obstacles is a special case of the more general problem of planning optimal collision-free paths for a given robot system. In Euclidean 2-space (the Euclidean plane) the problem is easy to solve and the shortest path is polynomial time computable, Lozano-Perez, Wesley [2]. The shortest path is still polynomial time computable if the allowed paths are constrained to lie on surfaces of polyhedral objects, Sharir, Schorr [5], and O'Rourke, Suri, Booth [4]. This is generally considered to be a problem in 2-1/2 dimensional space as it has aspects of 3-dimensionality while still confining the path to a two-dimensional surface. To compute the surface constrained shortest path the polyhedral surfaces are unfolded onto a common plane wherein the shortest path becomes a straight line. Such planar unfoldings have also been studied in the past to obtain surface constrained shortest paths for a variety of solid objects, Lyusternik [3].

The problem for Euclidean 3-space is much harder and known shortest path computations require doubly exponential time, Sharir, Schorr [5]. In Euclidean 3-space the shortest path between two given points, in the presence of polyhedral obstacles, can again be shown to be *polygonal* lines (piecewise straight lines, as for the planar 2-dimensional problem), with break points that lie on the edges of the given polyhedral obstacles. In this paper, we show the existence of generalized unfoldings wherein again the piecewise polygonal shortest path in 3-space becomes a straight line. These unfoldings consist of multiple rotations about the edges of the polyhedral obstacles.

Since the edges of the polyhedral obstacles are arbitrary lines in Euclidean 3-space, the problem of determining the points of contact of the shortest path with these edges can without loss of generality be versed also as follows.

Shortest Path Problem

Given a sequence $L=(l_1,l_2, \cdots ,l_n)$ of lines in 3-dimensional space , find the shortest path from a source point X to a destination point Y constrained to pass through interior points of each of the lines $l_1,l_2, \cdots ,l_n$ in this order.

We identify three different cases of the relative positions of the lines. All the various configurations of the n lines in 3-space consist of combinations of these basic orientations between pairs of lines.

(a) Lines are parallel to each other.

(b) Lines are not parallel but intersect.

(c) Lines are skew and do not intersect.

In § 2, we show that when the lines are oriented as a combination of the cases (a) and (b), then the shortest path problem in Euclidean 3-space reduces to a 2-1/2 dimensional space problem where paths are constrained to a sequence of planar surfaces. Hence planar unfoldings suffice, that is unfoldings onto a common plane where the shortest path becomes a straight line. Simple polynomial time exact algorithms which use these planar unfoldings have been known and used in the the past to determine surface constrained shortest paths, Lyusternik [3], O'Rourke, Suri, Booth [4] and Sharir, Schorr[5]. Furthermore for these cases (a) and (b), the shortest path solution has also been shown to be *constructible*†, Bajaj [1].

Next, in § 3, for the case (c) of non-intersecting skew lines, where the above planar unfoldings fail, we show the existence of generalized unfoldings wherein the shortest path in 3-space again becomes a straight line. These unfoldings consist of multiple rotations about the skew lines. For this general case however, the shortest path solution has been shown to be *not* constructible and furthermore *not* solvable by *radicals*‡, Bajaj [1]. This proves there exists *no exact* algorithm for this shortest path problem in general, under models of computation where the root of an algebraic equation is obtained using arithmetic operations and the extraction of k^{th} roots. This also rules out any apriori calculation of the amount the skew lines need to be rotated via the generalized unfolding scheme, such that the piecewise linear

† By constructible we mean straight-edge and compass constructible. The complexity of straight-edge and compass constructions has been known to be equivalent to the geometric solution being expressible in terms of $(+,-,*,/,\sqrt{})$ over Q, the field of rationals.

‡ A real number α is expressible in terms of *radicals* if there is a sequence of expressions $\beta_1, \ldots, \beta_n$, where $\beta_1 \in Q$, and each β_i is either a rational or the sum, difference, product, quotient or the k^{th} *root* of preceding β's and the last β_n is α.

path becomes an approximate straight line. (This compared to the planar unfoldings of § 2 where such apriori calculations are possible). Hence this only leaves numeric or symbolic approximation methods to obtain the shortest path solution.

In § 4, we elaborate on a numerical procedure of Sharir, Schorr [5] and illustrate the iterative approximations to the solution for skew lines by using generalized unfoldings. We show how we could iteratively rotate the lines under the generalized unfolding scheme till the piecewise linear path becomes a straight line. Furthermore we see that simultaneous iterative improvements of segments of the piecewise linear path are possible, corresponding to simultaneous rotations of lines in the generalized unfoldings.

2. Planar unfoldings

Theorem 1: When the n lines are oriented as a combination of the cases (a) and (b), then the problem can be solved by planar unfoldings.

Proof : Between pairs of lines in 3-space which are parallel to each other there exists a unique plane which contains both of them. The same applies to pairs of lines in 3-space which intersect. Also a point and a line in 3-space define a unique plane between them. The problem of finding the shortest path between X and Y in 3-space for cases (a) and (b), then reduces to a constrained 2-1/2 dimensional space problem as follows. Let the point X and line l_1 define the plane P_1, the lines l_i and l_{i+1} define the planes P_{i+1}, $i=1..n-1$, and the line l_n and the point Y define the plane P_{n+1}. The original problem is now reduced to finding the shortest path between two points X and Y in 3-space with the path constrained to the planes P_i, $i=1..n+1$, (Figure 1). Then the points of contact of the shortest path with the lines l_i, the edges of the planes, are determined by first *unfolding* all the planes P_i so that they all lie on the common plane defined by say, plane P_1 containing point X. This can be done iteratively by first unfolding P_2 to be coplanar with P_1, followed by unfolding P_3 till its coplanar with P_1 and P_2 and so on. The shortest path joining X and Y now becomes the shortest plane path that is the straight line, connecting X and Y', (the transformed point Y now on the common plane P_1 and thus coplanar with X). The points of intersection of this straight line with the transformed lines l_i', when transformed back, give the points of contact with the lines l_i of our original problem, (Figure 1). To prove correctness we note that the length of the the shortest path is kept invariant under such simple planar unfoldings and thus these unfoldings give the unique shortest path. □

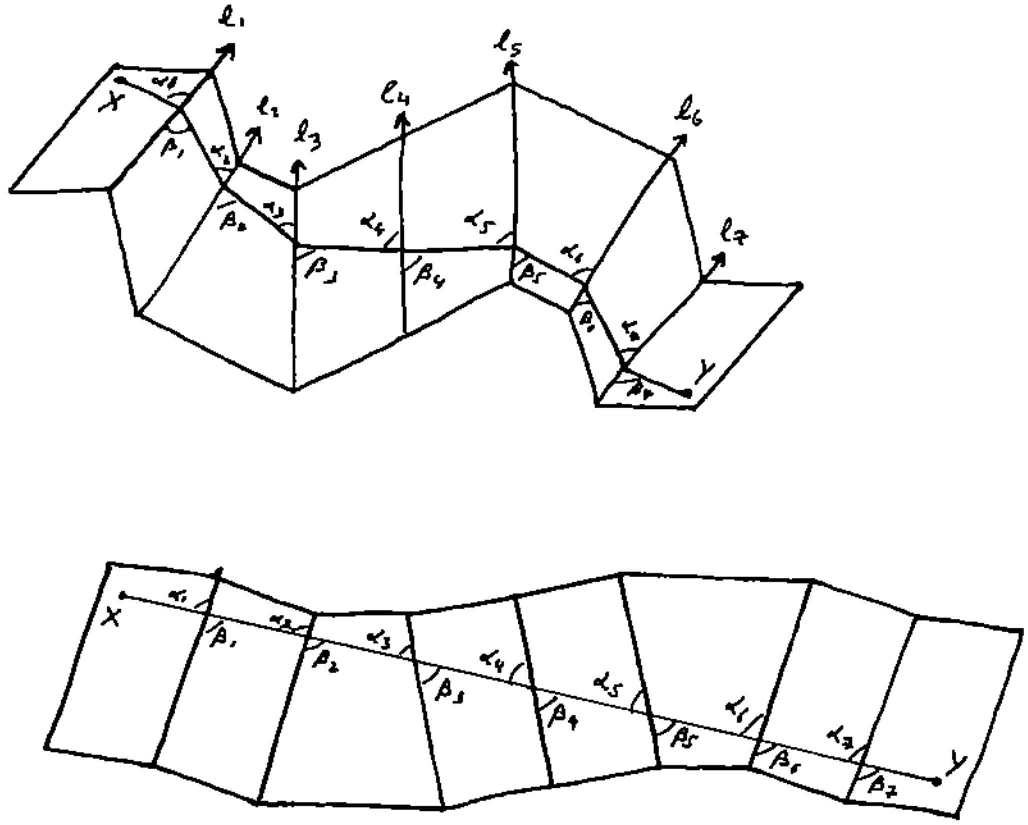

Figure 1 : Planar Unfoldings

The unfolding procedure sketched in the proof of Theorem 1 above is essentially an efficient algorithm to precisely determine the points of contact of the shortest path with the obstacle edges l_i. The time complexity being a polynomial in n, the number of obstacle edges. This polynomial time planar unfoldings algorithm has been known and used in the the past to determine surface constrained shortest paths, Lyusternik [3], O'Rourke, Suri, Booth [4] and Sharir, Schorr [5].

The above unfolding also links shortest paths to what are known as *geodesic* paths on surfaces. In very general terms a path q is called *geodesic* on a surface if at each point of q the principal normal coincides with the normal to the surface. For our shortest path problem, a path q from X to Y which passes through the lines $l_1, l_2, \cdots, l_n$ is *geodesic* if for each $i=1..n$, the path q enters and leaves l_i at equal angles. On unfolding all the planes P_i to a common plane P_1, the straight line connecting X and Y' clearly subtends equal angles at each of the lines l_i'. Such angles remain invariant under the above unfoldings and thus the shortest path from X to Y which passes through the given sequence of lines l_i is geodesic and furthermore unique. For cylindrical and conic surfaces the shortest path between two points on the surface is a geodesic curve which subtends equal angles with the generators of the curved surface. On unfolding the surface to a plane this geodesic curve becomes a straight line, Lyusternik [3].

3. Generalized Unfoldings

The notion of geodesic paths extends to the case of skew lines as well. Hence for shortest paths in Euclidean 3-space the points of contacts on the lines $l_1, \ldots, l_n$ are such that the piecewise straight line shortest path enters and leaves l_i at equal angles. To see this consider the straight line segments $IN(i)$ of the shortest path incident on line l_i. The line segment $IN(i)$ together with the line l_i define a plane. The same applies to the line l_i and its outgoing straight line segment $OUT(i)$ of the shortest path. On unfolding these planes about their common edge l_i the two segments $IN(i)$ and $OUT(i)$ must be collinear (the straight line being the shortest path in the plane) and thus subtend equal angles at the line l_i. Suffice it is to note that the length of the segments of the path as well as the subtended angles are invariant under the planar unfolding. A similar argument applies to the ingoing and outgoing straight line segment at each of the lines l_i.

However whenever any two adjacent lines l_i and l_{i+1} are skew to one another, there exists no common plane containing both of them. Hence the straight line segment $OUT(i) = IN(i+1)$ is no longer constrained to a planar surface and the planar unfolding fails. In fact the line segment $OUT(i) = IN(i+1)$ is the intersection of two planes, one containing l_i and $OUT(i)$ and the other containing $IN(i+1)$ and l_{i+1}. Looking at it differently, the locus (or envelope) of all possible straight line segments connecting skew lines l_i and l_{i+1} is no longer a planar surface but a 3-dimensional volume.

Nevertheless there still exists an unfolding of planes about lines l_i, $i=1..n$ wherein the piecewise straight line segments of the shortest path all become collinear.

Theorem 2: If the lines are skew there exists an unfolding where the shortest path becomes a straight line.

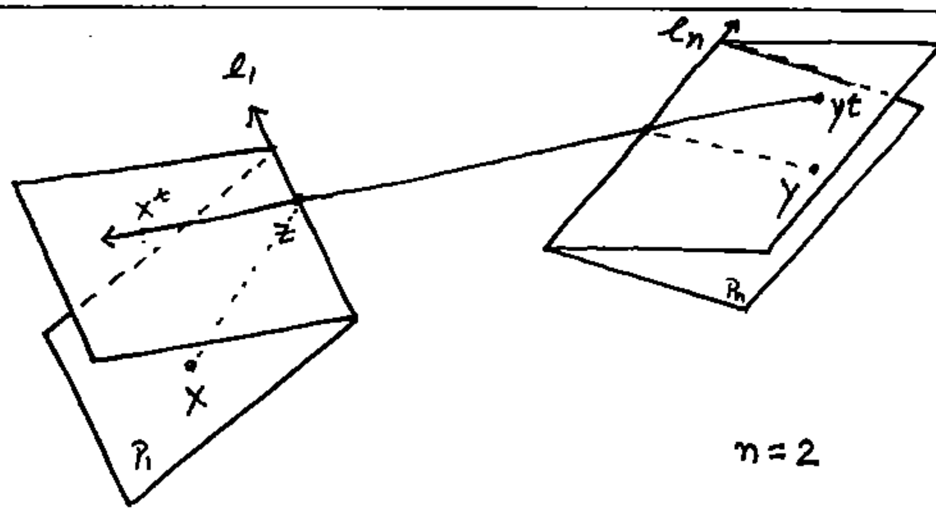

Figure 2: Case of 2 skew lines

Proof : Consider first the case of two non-intersecting skew lines l_1 and l_n and the two points X and Y in 3-space, Figure 2. Line l_1 and point X define a unique plane P_1. Similarly line l_n and point Y define the unique plane P_n. Also let X' and Y' be respectively the transformed points when planes P_1 and P_n are rotated about their corresponding lines l_1 and l_n. These two independent rotations are the generalized unfoldings.

Choose a point Z on line l_1. Then line l_n and point Z define a plane P_3. By rotating the plane P_n about line l_n we can make it coplanar with P_3. Thus there is a straight line L connecting Y' and Z and passing through a point in l_n and through the point Z on l_1. By rotating the plane P_1 about the line l_1 the line L can be brought onto the plane P_1. By choosing points Z on line l_1 appropriately the line L can be made to span all the points on the plane of P_1 via the double rotation about the lines l_1 and l_n. Hence there exists a point Z on l_1 wherein the straight line L passes through the point X'. That is there is a straight line connecting points X' and Y' passing through interior points of lines l_1 and l_n. Since under that unfolding the straight line is the shortest distance between X' and Y' and the rotation of the planes keeps the length of the lines invariant such a polygonal path with break points on the lines l_1, l_n must be the shortest path connecting the original points X and Y.

For the general case of n lines we prove the theorem by induction on n. The generalised unfolding consists of rotations of planes P_1 and P_n (as defined above) about the lines l_1 and l_n respectively and rotations of each line l_i about line l_{i-1},

$i=3..n$. In total n rotations. The base case of $n=2$ is as above. For the inductive step choose again a point Z on line l_1. This point Z and l_2 define a plane P_2. Then by invoking the inductive hypothesis on the $n-1$ lines $l_2, ..., l_n$, there exists a piecewise straight line connecting Y_t to Z and passing through the interior points of lines l_2, ..., l_n which becomes a straight line on $n-1$ rotations about these lines. The rotations consist of the planes P_2 and P_n about the lines l_2 and l_n respectively and the rotations of l_i about l_{i-1}, $i=4..n$. Also such a line exists for all points Z on l_1. By a rotation of plane P_1 about line l_1 the line L can be brought to the plane P_1 and thus span all points on A, in particular point X'. Since the rotations again keeps the length of the paths invariant, our theorem follows. □

4. Iterative Approximations

For the general case of skew lines, the shortest path solution has been shown to be *not* constructible and furthermore *not* solvable by *radicals*, Bajaj [1]. This proves there exists *no exact* algorithm for this shortest path problem in general, under models of computation where the root of an algebraic equation is obtained using arithmetic operations and the extraction of k^{th} roots. This also rules out any apriori calculation of the amount the skew lines need to be rotated via the generalized unfolding scheme, such that the piecewise linear path becomes a straight line. (This compared to the planar unfoldings of § 2 where such apriori calculations are possible). Hence this only leaves numeric or symbolic approximation methods to obtain the shortest path solution.

A general numerical procedure is given in Sharir, Schorr [5]. Initially a piecewise linear path is passed through an arbitrary sequence of points one on each of the given lines. Then this path is iteratively improved by replacing each contact point at which the incoming and outgoing angles are not equal by another point on the same line at which these angles become equal, (without changing the other contact points). Each such iterative step shortens the length of the path and the sequence of paths thus obtained will converge to a path of locally minimal length and hence to the desired shortest path. This because the shortest path from X to Y is unique, the length of the shortest path as a function of the contact points has one global minimum and no other local extremum.

We elaborate on this numerical procedure and illustrate the iterative approximations to the solution for skew lines by using generalized unfoldings. We show how we could iteratively rotate the lines under the generalized unfolding scheme till the piecewise linear path becomes an approximate straight line. Furthermore we see that simultaneous iterative improvements of segments of the piecewise linear path are possible, corresponding to simultaneous rotations of lines in the generalized unfoldings.

Case of 2 skew lines

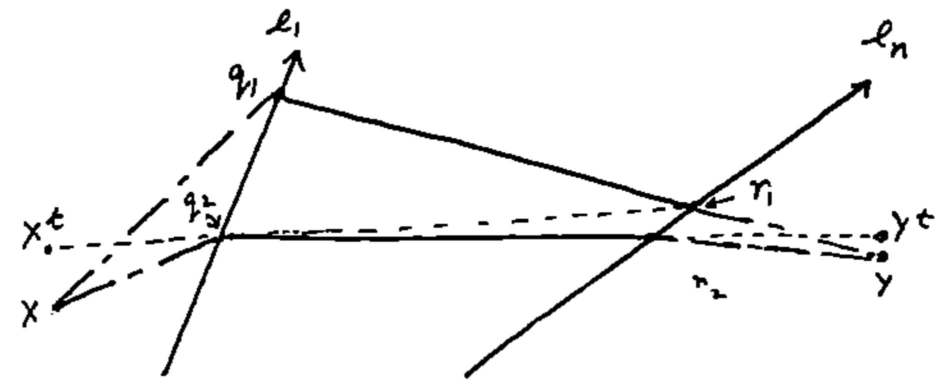

Figure 3: Iterative Approximations for 2 skew lines

Consider first the case of $n=2$ skew lines l_1 and l_n and the two points X and Y in 3-space, Figure 3. Line l_1 and point X define a unique plane P_1. Similarly line l_n and point Y define the unique plane P_n. Let the line l_1 intersects the plane P_n at the unique point q_1. Since q_1 and Y lie on the same plane P_n, there exists the straight line q_1Y which intersects l_n at a point we call r_1. The points q_1 and r_1 are the initial approximations to the points of contact on the lines l_1 and l_n respectively, of the shortest path connecting the points X and Y. The initial approximation to the shortest path is thus the piecewise linear path consisting of the segments Xq_1, q_1r_1 and r_1Y. In subsequent iterations we refine the approximations by using the above generalised unfoldings till the time that this piecewise linear path becomes a straight line. From Theorem 2 we know that under these generalised unfoldings this straight line path gives us the unique shortest path solution connecting X and Y.

For the case of two skew lines we recall that the independent rotations of the planes P_1 and P_n about their corresponding lines l_1 and l_n are the generalized unfoldings. The iterative improvement of the piecewise linear path is as follows. Rotate plane P_1 about line l_1 till the point r_1 becomes the new intersection of the rotated plane P_1' with the line l_n. Now since points r_1 and X' lie on the same plane P_1', there exists the straight line $X'r_1$ which intersects l_1 at the point we call q_2. This point is now a refinement of the contact point q_1 on line l_1 since the length $|X'q_2| + |q_2r_2| \leq$ the length $|X'q_1| + |q_1r_2|$, (Euclidean triangle inequality). Now rotate plane P_n about line l_n till the point q_2 becomes the new intersection of the rotated plane P_n' with the line l_1. Next since points q_2 and Y' lie on the same plane P_n', there exists the straight line q_2Y' which intersects l_n at the point we call r_2. This point is a refinement of the contact point r_1 on line l_n since the length $|q_1r_2| + |r_2Y'| \leq$ the length $|q_1r_1| + |r_1Y'|$, (triangle inequality). The updated contact points q_2 and r_2 are thus the new approximations after the first iteration. Repeating the independent rotations of the planes P_1 and P_n about their corresponding lines l_1 and l_n we iteratively improve the piecewise linear path till it eventually becomes a straight line.

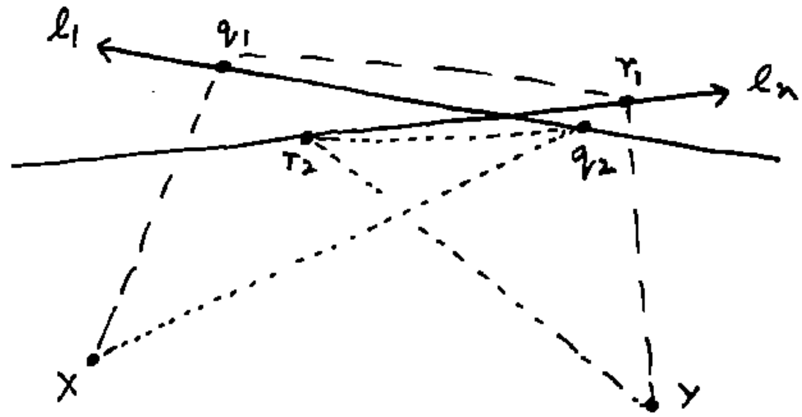

Figure 4: Simultaneous rotations for 2 skew lines

We point out however an important distinction for the above case. We could consider simultaneous rotations of the two planes P_1 and P_n about their respective lines. We have the unique point q_1 which is the intersection of the line l_1 with the plane P_n and the unique point r_1 which is the intersection of the line l_2 with the plane P_1. Since q_1 and Y lie on the same plane P_n, there exists the straight line q_1Y which intersects l_n at a point we call r_2. Similarly since r_1 and X lie on the same plane P_1, there exists the straight line Xr_1 which intersects l_1 at a point we call q_2. Changing r_1 to r_2 is achieved by rotating plane P_1 about line l_1 till the point r_2 becomes the new intersection of the rotated plane P_1' with the line l_n.

Simultaneously changing q_1 to q_2 is achieved by rotating plane P_n about line l_n till the point q_2 becomes the new intersection of the rotated plane P_n' with the line l_1. However in general such simultaneous rotations of the two planes P_1 and P_n about thier respective lines does not achieve an improvement of the piecewise linear path. An example of this is shown in Figure 4. Since the length $|q_2r_2|$ can be made arbitrarily close to the length $|q_1r_1|$ and thereby the length $|Xq_2 + q_2r_2 + r_2Y| \geq |Xq_1 + q_1r_1 + r_1Y|$.

The general case of n skew lines

In the general case, initial approximations q_i^1 are taken on each of the lines l_i $i=1..n$, respectively. Alternatively they may be taken to be the intersection of the lines l_i with the planes P_1 or P_n. As before line l_1 and point X define the unique plane P_1 and the line l_n and point Y define the unique plane P_n.

An iterative improvement of the contact points consist of two phases. Let $i=1$ *mod* 2 and $j=0$ *mod* 2, $1 \leq i,j \leq n$ in the following. In the first phase each of the points q_i^1 on the odd numbered lines l_i are replaced by new points q_i^2. These new points are obtained from the points q_j^1 on the even numbered lines l_j, as follows. Each point q_j^1 defines a unique plane P_j with the line l_{j-1}. Further the point q_j^1 defines a unique plane P_{j+1} with the line l_{j+1}, (except for $j=n-1$ for $n=odd$ and $j=n$ for $n=even$). The planes P_i are unfolded so as to become coplanar with the planes P_{i+1}. This is achieved by rotating planes P_1 and P_n about their respective lines l_1 and l_n and further rotating lines l_j about the lines l_{j+1}, (except for $j=n-1$ for $n=odd$ and $j=n$ for $n=even$). The points q_i^2 are the intersections of the straight lines $q_j^1q_{j+2}^1$ with the lines l_i after the unfoldings. In the second phase each of the points q_j^1 on the even numbered lines l_j are replaced by new points q_j^2. These new points are obtained from the points q_i^2 on the odd numbered lines l_i, by use of similar unfoldings as above. We note that the computations involving all the unfoldings (rotations) in each phase can be performed simultaneously and hence the new points q_i^2 can be computed simultaneously in the first phase as can the new points q_j^2 in the second phase. The new contact points q_2^k, $k=1..n$ at the end of an iteration are improved approximations to the solution since each of the above unfoldings shortens the length of the entire path (straightforward triangle inequality). Repeating the above two phases of unfoldings we iteratively improve the piecewise linear path till it eventually becomes a straight line (approximately).

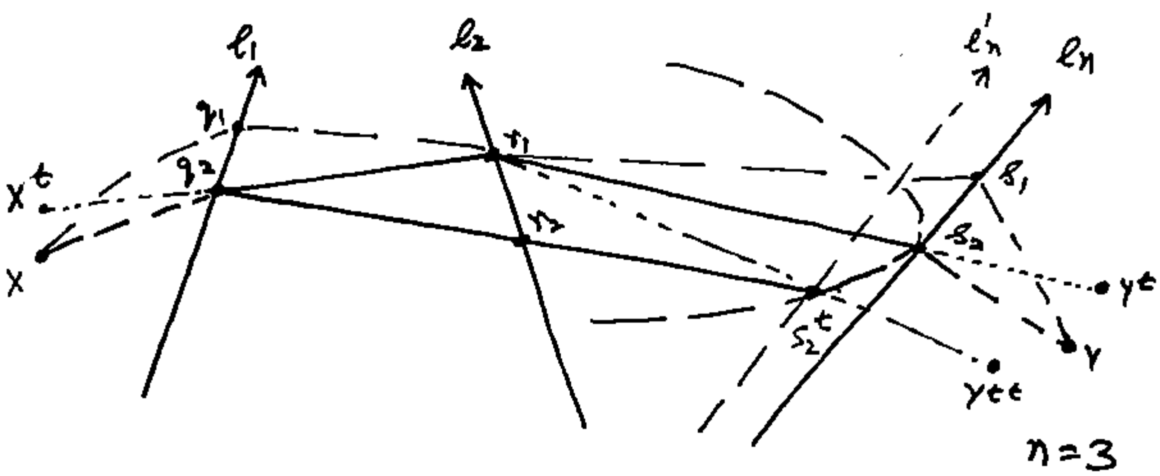

Figure 5: Iterative Approximations and Generalized Unfoldings

We now further illustrate the generalized unfoldings and in particular the rotation of a line l_i about the line l_{i-1} in the general case procedure with the case of $n=3$ skew lines. Consider the $n=3$ skew lines l_1, l_2 and l_n and the two points X and Y in 3-space, Figure 5. For the case of three skew lines we recall that the independent rotations of the two planes P_1 and P_n about their corresponding lines l_1 and l_n respectively and the line l_n about line l_2, comprise of the generalized unfoldings. As before let the line l_1 intersect the plane P_n at the unique point q_1 and the line l_n intersect the plane P_1 at the unique point s_1. We assume an initial contact point r_1 on the line l_2. Alternatively we could take it to be the intersection of the plane P_1 or P_n with the line l_2. The points q_1, r_1 and s_1 are then the initial approximations to the points of contact on the lines l_1, l_2 and l_n respectively, of the shortest path connecting the points X and Y. The initial approximation to the shortest path is thus the piecewise linear path consisting of the segments Xq_1, q_1r_1, r_1s_1 and s_1Y.

The iterative improvement of the piecewise linear path is as follows. Rotate plane P_1 about line l_1 till the point r_1 becomes the intersection of the rotated plane P_1^t with the line l_2. Now since points r_1 and X^t lie on the same plane P_1^t, there exists the straight line X^tr_1 which intersects l_1 at the point we call q_2. This point is now a refinement of the contact point q_1 on line l_1 since the length $|X^tq_2| + |q_2r_1|$ $\leq$ the length $|X^tq_1| + |q_1r_1|$, (triangle inequality). Also rotate plane P_n about line l_n, (this could be done simultaneously with the earlier rotation) till the point r_1 becomes the intersection of the rotated plane P_n^t with the line l_2. Now since points

r_1 and Y' lie on the same plane $P_n^{'}$, there exists the straight line $Y'r_1$ which intersects l_n at the point we call s_2. This point is now a refinement of the contact point s_1 on line l_n since the length $|Y's_2| + |s_2r_1| \leq$ the length $|Y's_1| + |s_1r_1|$, (triangle inequality). Now, the rotation of the line l_n about the line l_2 describes a circle C having center r_1 and radius vector r_1s_2. The the unique plane, call it P_2, defined by line l_2 and point q_2, intersects with the circle C at a point we call $s_2^{'}$. Since points q_2 and $s_2^{'}$ lie on the same plane P_2, there exists the straight line $q_2s_2^{'}$ which intersects l_2 at the point we call r_2. This point r_2 is now a refinement of the contact point r_1 on line l_2 because $|r_1s_2| = |r_1s_2^{'}|$ and the length $|q_2r_1| + |r_1s_2^{'}| \leq$ the length $|q_2r_2| + |r_2s_2^{'}|$ (triangle inequality). Hence the length $|q_2r_1| + |r_1s_2| \leq$ the length $|q_2r_2| + |r_2s_2|$. Note that $s_2^{'}$ and Y'^t are the new transformed points of s_2 and Y^t under the above rotation, wherein $|s_2Y^t| = |s_2^{'}Y'^t|$ is maintained. The updated contact points q_2, r_2 and s_2 are thus the new approximations after the first iteration. Repeating the above generalised unfoldings we iteratively improve the piecewise linear path till it eventually becomes a straight line.

5. References

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