

EFFICIENT ALGORITHMS FOR  
COMMON TRANSVERSALS

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## Efficient Algorithms For Common Transversals

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### ABSTRACT

Suppose we are given a set  $S$  of  $n$  (possibly intersecting) simple objects in the plane, such that for every pair of objects in  $S$ , the intersection of the boundaries of these two objects has at most  $\alpha$  connected components. The integer  $\alpha$  is independent of  $n$ , i.e.  $\alpha = O(1)$ . We consider the problem of determining whether there exists a straight line that goes through every object in  $S$ . We give an  $O(n \log n \gamma(n))$  time algorithm for this problem, where  $\gamma(n)$  is a very slowly growing function of  $n$ . If  $\alpha < 3$  then our algorithm runs in  $O(n \log n)$  time. Previously, only special cases of this problem were considered: In [6] the case when every object is a straight-line segment, in [2] the case when the objects are equal-radius circles and in [5] the case when objects all maintain the same orientation. All these cases follow from our general approach, which places no constraints on the size and/or configuration of the objects in  $S$ .

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## 1. Introduction

Consider being given a set  $S$  of  $n$  simple objects in the plane. By simple objects we mean those that have an  $O(1)$  storage description each, and that are such that, for every pair of such objects, constant time suffices to compute their intersection, common tangents, etc. Typical examples of such objects are polygons with a constant number of edges, discs, ellipses, sectors of discs, etc. We seek straight lines, if they exist, that intersect all members of  $S$ . Such straight lines are called common transversals or stabbing lines of the set  $S$ . Since there exists a common transversal for  $n$  possibly non-convex objects iff there exists a common transversal for the  $n$  convex hulls of these objects, we can replace every input object by its convex hull (this takes  $O(1)$  time per object since we are considering simple objects). We assume that this has already been done, i.e. from now on we assume that each of the  $n$  objects in  $S$  is convex.

Throughout the paper, we use  $\alpha$  to denote the largest number of connected components that the intersection of two object boundaries can have, and we assume that  $\alpha$  is a constant independent of  $n$  (i.e.  $\alpha = O(1)$ ).

Algorithms for determining transversals are known, however in special cases only. Straightforward solutions arise from results in combinatorial geometry, Danzer, Grunbaum, Klee [3] and Hadwiger, Debrunner [9], which give rise to worst case time bounds of  $O(n^k)$ ,  $k \geq 3$ . Edelsbrunner, Overmars, Wood [7] have a general method for visibility problems in the plane which can be used to determine transversals, however in time  $O(n^2 \log n)$ .  $O(n \log n)$  time algorithms were given for the special cases of line segments [6] and for circles of equal radius [2]. Efficient algorithms were then given by Edelsbrunner [5], who reduced transversal problems for a set of homothets of a simple planar object to convex hull problems. Though this gives  $O(n \log n)$  time algorithms to determine transversals for a wide class of objects, it applies to only special constrained configurations of the set  $S$  of objects. In particular, homothety which involves only scaling and translation, forces all objects to maintain the same orientation.

In this paper we give efficient  $O(n \log n \gamma(n))$ , (and, if  $\alpha < 3$ ,  $O(n \log n)$ ), time algorithms to determine transversals of simple planar objects without any constraints on the

size of the objects or constraints on the configuration of the set of objects,  $S$ . Our algorithm actually computes a description of all transversals of  $S$ .

### 1.1. Some Preliminaries

Let the functions  $f_1, \dots, f_n$  be real-valued, continuous functions of a parameter  $t$ , where each  $f_i$  has an  $O(1)$  storage description. Suppose we want to compute the pointwise *Min* of these functions, defined by  $h(t) = \underset{1 \leq i \leq n}{\text{Min}} f_i(t)$ . Note that  $h$  itself is continuous and is typically made up of "pieces" each of which is a section of one of the  $f_i$ 's. More formally, a piece of  $h$  is the portion of a function  $f_i$  over an interval  $[t_1, t_2]$  such that (i)  $h$  is identical to  $f_i$  over that interval, (ii)  $h$  is not identical to any  $f_j$  over an interval which properly contains  $[t_1, t_2]$ . The storage representation of such a piece consists of the index  $i$  together with the interval  $[t_1, t_2]$  (so a piece has an  $O(1)$  storage description). (Detail: If  $f_i$  and  $f_j$  are identical over the interval  $[t_1, t_2]$  then we break the tie arbitrarily, e.g. by taking  $\min(i, j)$ .) The desired description of  $h$  is a list of the descriptions of the successive pieces that make it up. The next lemma bounds the number of pieces that make up  $h$  if no two distinct functions  $f_i$  and  $f_j$  intersect more than  $s$  times ( $f_i$  and  $f_j$  intersect  $p$  times iff the set of real values of  $t$  for which  $f_i(t) = f_j(t)$  consists of  $p$  disjoint intervals on the real line).

**Lemma 1.** Let  $f_1, \dots, f_n$  be continuous, real-valued functions of variable  $t$ . Every  $f_i$  has an  $O(1)$  storage description and can be evaluated at any  $t$  in  $O(1)$  time. Every two distinct functions  $f_i$  and  $f_j$  intersect at most  $s$  times where  $s = O(1)$ ; furthermore, these (at most  $s$ ) intersections can all be computed in  $O(1)$  time. Let  $h$  be the pointwise *Min* of the  $f_i$ 's; i.e.  $h(t) = \underset{1 \leq i \leq n}{\text{Min}} f_i(t)$ . Then the description of  $h$  can be computed in  $O(n \log n)$  time if  $s < 3$ , in  $O(n \log n \gamma(n))$  if  $s \geq 3$ , where  $\gamma(n)$  is an extremely slowly growing function of  $n$ .

**Proof:** Recursively compute the description of the pointwise *Min* of  $f_1, \dots, f_{n/2}$ , and that of the pointwise *Min* of  $f_{n/2+1}, \dots, f_n$ . Each of these two descriptions, as well as the description of the desired  $h$ , has  $O(n)$  pieces if  $s < 3$  [4],  $O(n \gamma(n))$  pieces if  $s \geq 3$  (for details about  $\gamma(n)$ , see the note that follows). These two descriptions are then combined

to obtain that of  $h$ , giving the following recurrence for the time complexity  $T(n)$ :  
 $T(n)=2T(n/2)+(\text{number of pieces})$ . Thus  $T(n)=\log n \cdot (\text{number of pieces})$ .  $\square$

*Note:* Let  $\log^* n$  denote the smallest integer  $i$  for which  $\text{exp}_i(1) > n$ , where  $\text{exp}_1(x) = e^x$  and  $\text{exp}_i(x) = e^{\text{exp}_{i-1}(x)}$ . The function  $\log^* n$  grows extremely slowly with  $n$  and is "almost" a constant for all practical values of  $n$ , e.g.  $\log^*(10^{100}) = 4$ . Szemerédi proved an upper bound of  $O(\log^* n)$  for  $\gamma(n)$  [15], and sharper upper bounds were later given by Hart and Sharir [10] and Sharir and Livne [14].

## 2. Common Transversals

Consider the 1-1 geometric transformation which transforms a line  $l_0$  in the  $x - y$  plane into a pair  $(\rho_0, \theta_0)$ , a point in the  $\rho - \theta$  parameter space, [Figure 1].

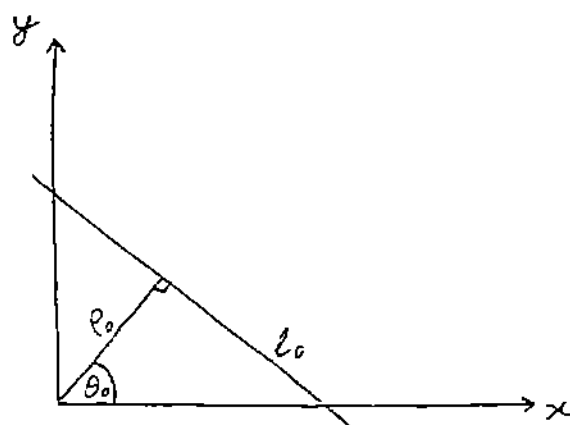


Figure 1 Geometric Transformation

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We illustrate our method by first giving an  $O(n \log n)$  time algorithm for the case of a set  $S$  consisting of  $n$  arbitrary circles in the plane (in [2] only the case where all the radii are equal was considered). To determine whether they permit a common transversal or stabbing line we use the above geometric transformation as follows. Each circle  $C_i$  is defined by a radius  $r_i$  and a center whose polar coordinates are  $(\rho_i, \theta_i)$ . To obtain all possible stabbing lines for  $C_i$  consider a general line defined by the pair  $(\rho, \theta)$ . As shown in Figure 2, this line stabs  $C_i$  iff  $\rho_i \cos(\theta - \theta_i) - r_i \leq \rho \leq \rho_i \cos(\theta - \theta_i) + r_i$ .

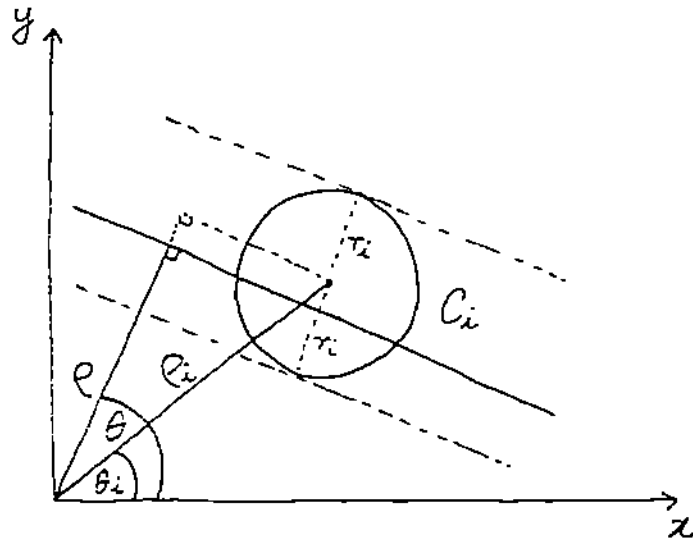


Figure 2 Stabbing Line

Furthermore, the line defined by  $(\rho, \theta)$  is a stabbing line to all  $n$  circles or a common transversal iff

$$\rho_1 \cos(\theta - \theta_1) - r_1 \leq \rho \leq \rho_1 \cos(\theta - \theta_1) + r_1$$

$$\rho_2 \cos(\theta - \theta_2) - r_2 \leq \rho \leq \rho_2 \cos(\theta - \theta_2) + r_2$$

...

$$\rho_n \cos(\theta - \theta_n) - r_n \leq \rho \leq \rho_n \cos(\theta - \theta_n) + r_n$$

which implies that  $(\rho, \theta)$  is a common transversal iff

$$\max_{1 \leq i \leq n} f_i(\theta) \leq \rho \leq \min_{1 \leq i \leq n} g_i(\theta)$$

where

$$f_i(\theta) = \rho_i \cos(\theta - \theta_i) - r_i$$

$$g_i(\theta) = \rho_i \cos(\theta - \theta_i) + r_i$$

Now, observe that every point  $(\rho, \theta)$  in the intersection of  $f_i$  and  $f_j$  (i.e.  $\rho = f_i(\theta) = f_j(\theta)$ ) defines a line which is tangent to both  $C_i$  and  $C_j$ , and is such that  $C_i$  and  $C_j$  are on the same side of that common tangent. If  $C_i$  and  $C_j$  are distinct circles, then there are at most two such common tangents, and hence  $f_i$  and  $f_j$  intersect at most twice. If  $C_i$  and  $C_j$  coincide, then  $f_i = f_j$  and hence  $f_i$  and  $f_j$  intersect once. Hence by Lemma 1, the description of the pointwise *Max* of the  $f_i$ 's (call it  $\hat{f}$ ) can be computed in  $O(n \log n)$  time. Similar remarks holds for  $g_i$  and  $g_j$ , and the pointwise *Min* of the  $g_i$ 's (call it  $\hat{g}$ ). Once  $\hat{f}$  and  $\hat{g}$  are known, we have a complete description of all the stabbing lines of the  $C_i$ 's, viz., every point  $(\rho, \theta)$  in the region below the graph of  $\hat{g}$  and above that of  $\hat{f}$  defines a stabbing line of the  $C_i$ 's (if that region is empty then there is no stabbing line).

The above method generalizes for planar objects such as ellipses, ovals, etc., whose boundaries consist of a single smooth closed curve. The method also generalizes for a larger variety of planar objects whose boundary consists of piecewise smooth curves, such as sectors of discs,  $k$ -gons etc. The only restriction is that the intersection of any pair of object boundaries must have no more than  $\alpha$  connected components, where  $\alpha = O(1)$ . When  $\alpha \geq 3$ , we obtain  $O(n \log n \gamma(n))$  time performance (rather than  $O(n \log n)$ ). The rest of this section sketches this generalization when each object is a convex  $k$ -gon, where  $k = O(1)$ .

For a set  $S$  of  $n$  convex  $k$ -gons consider again the  $i^{\text{th}}$  object of the set,  $O_i$ . We need to obtain the functions  $f_i$  and  $g_i$  for every object  $O_i$  (as for the circles before). These functions are still continuous, but they are no longer smooth everywhere; instead they are piecewise smooth, with angular points separating the smooth pieces. The descriptions of  $f_i$  and  $g_i$  are computed as follows. We first compute, for every  $O_i$ , the set  $P_i$  of all antipodal pairs of vertices [13]. This takes  $O(1)$  time per object. Corresponding to each antipodal pair  $(p, q) \in P_i$  there exists a range of angles  $[\theta_1, \theta_2]$  such that any line  $L = (\rho, \theta)$  for which  $\theta_1 \leq \theta \leq \theta_2$  stabs  $O_i$  iff it stabs the straight-line segment  $pq$ . Therefore within each such range  $[\theta_1, \theta_2]$  the functions  $f_i$  and  $g_i$  are smooth and easily defined. Since  $O_i$  has  $O(k)$  antipodal pairs, each of  $f_i$  and  $g_i$  consists of  $O(k)$  such

smooth pieces.

As before, a straight line defined by  $(\rho, \theta)$  in parameter space is a stabbing line for the object  $O_i$  iff  $f_i(\theta) \leq \rho \leq g_i(\theta)$ . Further the line  $(\rho, \theta)$  intersects all  $n$  objects iff  $\forall i, i = 1, \dots, n$ , we have  $f_i(\theta) \leq \rho \leq g_i(\theta)$ . Again, this implies that line  $(\rho, \theta)$  is a transversal of the  $n$  objects iff

$$\text{Max}_{1 \leq i \leq n} f_i(\theta) \leq \rho \leq \text{Min}_{1 \leq i \leq n} g_i(\theta)$$

The piecewise smooth envelope  $\text{Max}_{1 \leq i \leq n} f_i(\theta)$  is computed using Lemma 1. However, in order to be able to use this lemma, we must first show that  $f_i$  and  $f_j$  intersect  $O(1)$  times. Actually, they intersect at most  $2k$  times. To see this, note that there are as many such intersections as there are common tangents between  $O_i$  and  $O_j$ , and that there are at most  $2k$  such common tangents (where by common tangent we mean one, such that both objects are on the same side of it).

The other piecewise continuous envelope  $\text{Min}_{1 \leq i \leq n} g_i(\theta)$  is computed analogously. The region below the *Min* envelope and above the *Max* envelope describes all the transversals of  $S$ .

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