A FACE AREA EVALUATION ALGORITHM FOR SOLIDS IN CSG REPRESENTATION

Muh-Cherng Wu
Chanderjit Bajaj
C. R. Liu

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Muh-Cherng Wu†, Chanderjit Bajaj* and C. R. Liu†

† School of Industrial Engineering,

* Department of Computer Science,

Purdue University,

West Lafayette, IN 47907.

Abstract

This paper presents a scan curve algorithm for evaluating the face area of solids in constructive solid geometry (CSG) representation. Compared to previous methods, the algorithm is more accurate and computationally faster. The applicable domain is limited to solids bounded by three classes of surfaces: all quadric surfaces, cylindrical surfaces, and surfaces of revolution which are algebraic surfaces with rational parametric equations for their generating curves. The algorithm has been implemented in FORTRAN 77 on a VAX 11/780 machine. The extensions of this algorithm may also be applied to the solution of the following three problems: (1) Boundary representation (BREP) evaluation from CSG; (2) Face area evaluation for solids in BREP; and (3) Triangulation of the faces of solids in CSG or BREP.

1. Introduction

Solid modeling is concerned with the representation of physical objects on a computer. Particular to a certain solid model representation, algorithms are developed to analyze the property and plan the behavior of the physical objects. Solid models thus play a key role in computer aided design and manufacturing. Several solid model representations have been developed and have been surveyed [1, 2]. Among these representations, constructive solid geometry (CSG) is very popular because of its simple and efficient user interface.

In a CSG scheme, a workpiece is represented as a tree of regularized Boolean operations [3] on several primitive objects which may be solids or half-spaces. The regularized Boolean operators are modified set operations which include union, intersection and subtraction. The primitive solids are simple solids such as cuboids, spheres, cylinders, etc., and are also constructed from the intersection of several half-spaces.

The problem discussed in this paper can be briefly stated as follows: Given a solid in CSG representation, compute the area of all its bounding faces. The domain of associated surfaces of the boundary faces of the solids are limited to three classes: quadric surfaces, cylindrical surfaces, surfaces of revolution, where the last two classes are algebraic surfaces with rational parametric equations for their generating curves [4].

Several studies in face area evaluation have been done previously. These works can be classified into two groups. One group studies face area evaluation for solids given by an explicit boundary representation [5, 6, 7, 8]. These papers all apply Green's Theorem [9] to transform a surface integral into a line integral and make appropriate approximations respectively in evaluating the integral. Requiring explicit boundary information, these methods cannot be directly applied to solids with a CSG representation.

The other group including this work considers the face area evaluation of solids in CSG representation [10, 11]. These methods are of a divide-and-conquer paradigm [12] and vary in the different ways of face patch decomposition. Each boundary face of the primitive solids is decomposed into a set of face patches. From each face patch, a point is randomly chosen and checked whether it is on, inside or outside the final resulting solid described by the CSG representation. If the point is on the resulting solid, its associated face patch is taken as a qualified patch. Each patch is individually tested and the face area of the qualified patches are summed up to get the face area of the final solid.

The patch decomposition methods are good but deficient in accuracy and speed. In summing up face patches, the overall area of a face may be either overcounted or undercounted. For example, a face patch lying partially on the resulting solid may either be considered as lying fully or not at all on the resulting solid. To mprove accuracy, the number of decomposed face patches would have to be increased which correspondingly increases the computation time. On the other hand, if the decomposed patches are made exceedingly small, their areas would have to be summed up in multiple precision to reduce the effect of round-off error propagation.

The objective of this paper is to present an algorithm which improves the accuracy and reduces the computation time in evaluating the face area of solids in CSG representation. We also briefly describe the extensions of this algorithm for the solution of three other problems: (1) CSG to BREP evaluation; (2) BREP face area evaluation; (3) face triangulation for BREP or CSG.

The paper is organized as follows. Section 2 develops the basic idea of this algorithm and its components. Sections 3, 4, and 5 discuss each component of the algorithm. The complexity analysis and the implementation results of this algorithm are presented in Section 6. Section 7 reviews the advantages of this approach and a certain embellishment. Section 8 discusses the extensions of this algorithm.

2. Basic Idea of This Algorithm

The basic idea of this algorithm is to use "strip decomposition" rather than "patch decomposition" as in previous approaches. As shown in Figure 1, our method decomposes the face of each primitive solid into a set of strips bounded by "scan curves". These curves may also intersect with faces of other primitive solids and thereby decompose their associated strips into several sub-strips. Each sub-strip is either fully on the solid or not at all. Thus, the face area of the solid can be calculated by summing up the face area of all the sub-strips on the solid.

To calculate the face area of a sub-strip, we need to classify the scan curves with respect to the solid. In particular, each portion of the scan curve should be classed as to whether it is "on", "inside", or "outside" the solid. The sub-strips bounded by "on" portions of the scan curves constitute the faces of the solid. Thus, they should be identified and their areas evaluated.

In summary, this algorithm consists of three different components: (1) scan curve generation; (2) scan curve/solid classification; and (3) contributing sub-strip identification and area evaluation.

This approach has two major distinctions from prior approaches to face area evaluation. One distinction stems from using elongated strips rather than patch as in decomposing the primitive solid. A strip is based on the scanning curve for the surfaces. Given the same resolution, the number of decomposed strips is less than that of decomposed patches and thus often requires less computation time in evaluating the face area of the solid.

The other distinction is classifying the strips analytically. Thus, each contributing sub-strips is fully on the solid, rather than for patches which may be partially on the solid. This characteristic makes the face area evaluation more accurate.

3. Scan Curve Generation on Surface of Primitives

This section discusses the first component of this algorithm: generating scan curves for the surfaces on each primitive solid. The scan curves generation methods are applicable to all quadric surfaces. Yet, cylindrical surface and surfaces of revolution are restricted to those which are algebraic $f(x,y,z)=0$, and have rational parametric equations for their generating curves. A rational parametric equation for a generating curve is given by $x=h(t)/g(t)$, $y=p(t)/q(t)$, $z=s(t)/r(t)$.

The determination of scan curves is by the following criteria. First, the area of each decomposed strip should be easy to evaluate. Second, the scan curves should be as simple as possible so as to be easily classified with respect to the solids. Third, the decomposed strips should not overlapping.

3.1. Scan Curve Determination of Cylindrical Surfaces

A cylindrical surface is a surface generated by moving a straight line along a fixed plane curve so that the direction of the moving line remains parallel to the original straight line. The fixed curve is called the generating curve and the moving line is called the generator of the cylindrical surface.

By its moving characteristic, the scan curves of the cylindrical surfaces are determined as the lines parallel the generator shown in Figure 2. Each scan line passes through a point on the generating curve. These passed points can be sequentially traced from the parametric equation of the generating curve by increasing its t value with a fixed step.

Tracing points sequentially for an algebraic curve in implicit form is somewhat more complicated [13]. Thus, the domain of cylindrical surfaces is restricted to those with generating curves having rational parametric form. This also simplifies computing the curve/surface intersections during curve/solid classification. However, some algebraic curves can be rationally parameteized [14, 15].

3.2. Surfaces of Revolution

A surface of revolution is a surface generated by revolving a plane curve about a fixed line on the same plane. The fixed line is called the axis of revolution and the given curve is called the generating curve. For example, torus, a popular primitive solid in most solid modelers, is a special case of the surfaces of revolution, whose generating curve is a circle.

According to the revolution nature of the surface, the scan curves are determined as the set of circles with the center on the axis as in Figure 3. Each circle passes through a point on the the generating curve, where each point can be sequentially traced from its rational parametric form of the curve. Like cylindrical surfaces, the domain of surfaces of revolution is thus restricted to those with generating curves having rational parametric form. However, a surface of revolution may become a cylindrical surface if the generating curve is a line. In this case, the modeling scan curves become lines rather than circles.

3.3. Quadric Surfaces

A quadric surface is a surface which has a second degree equation of the following form.

$$Ax^2 + By^2 + Cz^2 + 2Dxy + 2Eyz + 2Fxz + GX + HY + IZ + J = 0 \tag{1}$$

Given its implicit equation, the quadric surface can be classified into one of the following eight groups [4]: (1) degenerate cases which may be an empty set, a point or a line; (2) second degree cylinders; (3) elliptic paraboloids; (4) hyperbolic paraboloida;s (5) elliptic cones; (6) ellipsoid; (7) hyperboloids of one sheet; and (8) hyperboloids of two sheets. By this grouping, all the surfaces in a given group are similar in shape and require the same type of scan curves.

By choosing appropriate sectioning planes for the quadric surfaces, we obtain sectioning curves of degree one or two, which may be lines, parabolas, circles, ellipses and hyperbolas. Each group of quadric surfaces usually has two types of sectioning curves. For example, the sectioning curves of elliptic paraboloids may be parabolas or ellipses.

From the two types of sectioning curves on a quadric surface, the one which makes non-overlapping strips is chosen as scan curves. See the elliptic paraboloid in Figure 4, choosing parabola as scan curves makes overlapping strips; while choosing ellipses as scan curves makes non-overlapping strips. Among all scan curves which yield non-overlapping strips, the choice is also dictated by the scan curves with simpler parametric form. For each group of quadric surface, the possible sectioning curves and the selected scan curves are summarized in Table 1.

4. Scan Curve/Solid Classification

This section discusses the second component of this algorithm, scan curve/solid classification. The problem can be stated as follows: Given a scan curve C and the solid S in CSG representation, determine which segment of C is "on" "inside", or "outside" the solid S. This problem is solved in two phases: (1) classifying scan curves with respect to each primitive solid; and (2) merging classification results of scan curve/primitive solids to classify scan curves with respect to the solid.

4.1. Scan Curve/Primitive Solid Classification

Classifying a scan curve/primitive solid means that the curve is decomposed into a set of segments, where each segment has a unique classification property of "on", "inside", or "outside". In this approach, the first step is determining the intersection points of the scan curve and the surfaces of the primitive solid. As shown in Figure 1, the intersection points divide the scan curve into several curve segments. On each curve segment, all points except end points thus have the same classification property. Further, neighboring curve segments which have the same classifications are merged into a larger segment.

In computing the intersection points of scan curves and surfaces, surfaces are represented in implicit polynomial form, $F(x, y, z) = 0$. and curves are represented in rational parametric form, $x = f(t)/g(t)$, $y = p(t)/q(t)$, $z = h(t)/s(t)$. By inserting the parametric form into the implicit form, the implicit form becomes a single variable polynomial, $F(x(t), y(t), z(t)) = 0$. The intersection points of the curve and the surface can be obtained by solving this single variable polynomial. Each real root of the polynomial corresponds to a real intersection point of the curve and the surface.

4.2. Scan Curve/Resulting Solid Classification

After classifying scan curve/primitive solids, the results can be merged to classify scan curve/solid. The merging rules for various operators and classification results are listed in Table 2, where "in" means inside, "on" means on the boundary and "out" means outside the combined solid. These merging rules can be proved by set theory [16]. As noted in the table, the merging of classification results may be ambiguous when two classification results of a region are all "on" situations. Figure 5 illustrates a case where merging a "on/on" region may generate "in" or "on" situations.

In the previous patch decomposition approach, the on/on ambiguities also occur in classifying a point with respect to the solid. Sarraga [10] partially solves this problem by including the point neighborhood information developed by Tilove [16]. Yet, this neighborhood method is easy to apply for points on the interior of faces but not for points on edges of solids. In the previous approach [10], the on/on ambiguities still exist for points on edges of solids but are treated as an "edge" case which means "not on" and may cause error classification.

In this scan curve approach, the on/on ambiguities on the interior of faces are resolved by introducing the neighborhood information of scan curves. The curve neighborhood information is represented by splitting the "on" case as two cases, either "on1" or "on2". For a scan curve on a surface of a primitive solid, if the direction of the interior side of the primitive solid is the same as the normal direction of the surface, it is termed as the "on1" case. If these two sides are on opposite sides, it is termed as the "on2" case. The merge rules for "on1" and "on2" are listed in Table 3.

The on/on ambiguities on edges of solids are resolved by scanning virtual scan curves. A virtual scan curve is a newly generated scan curve with an infinitesimal distance with the original scan curve on edges of solids as in Figure 6. The intersection points are determined by the original scan curve. Yet, the classification properties of the original scan curve are determined by the virtual scan curves. In this situation, the on/on ambiguities on edges of solids disappear while preserving the appropriate classification for scan curve/solid.

5. Sub-strip Identification and Area Evaluation

This section discusses the third component of this algorithm: contributing sub-strips identification and area evaluation. The problem can be stated as follows: Given a strip having been segmented as set of sub-strips by the intersection curves of two surfaces, identify which substrips are "on" the solid and evaluate their area.

5.1. Contributing Sub-strips Identification

A contributing sub-strip is bounded by two scan curve segments which are classified as "on" with respect to the solid. As shown in Figure 7, two consecutive scan curves usually have similar classification and intersection patterns, namely, with same number of curve segments, similar classification properties, similar sequences of intersections with surfaces. Taking curve A, B in Figure 7 as an example, there are five curve segments which are sequentially classified as "out", "on", "out", "on" and "out". Further, the surfaces intersected by the curves are sequentially surfaces S and T. In this case, to identify the contributing sub-strips, one only needs to consider the "on" regions on the first scan curves and the corresponding regions on the next scan curve.

However, two consecutive scan curves may have different classification and intersection patterns. As shown in Figure 8, the cases may be: (1) with different number of regions as curves A, B; (2) with different classification properties as curves C and D; and (3) with different sequences of intersections with surfaces as curves E and F. Referring to Figure 8, the change of classification and intersection patterns come from the "singular intersection points" which exist in between two scan curves. The singular intersection point may be either an intersection point of three or more surfaces as point P, or a tangent point of a scan curve with a surface on the primitive solid as point Q.

To identify the contributing sub-strips for the strip including singular points, we further divide the strip into a number of, say, ten smaller strips by scanning more curves within the strip as shown in Figure 9. In these smaller strips only some contain the singular points. The other smaller strips do not include singular points and their contributing sub-strips can be identified as stated above. The strip division procedure is recursively repeated until a desirable resolution is achieved.

Note that the intersection points a, b, c, d in Figure 8c, where the intersection sequences change, are close to the singular point p and may serve as good candidates for starting points in a numerical search.

5.2. Contributing Sub-strip Area Evaluation

The area evaluation of contributing sub-strips may require some suitable approximation. See Figure 10, a contributing sub-strip is bounded by four curves, two scan curves and two intersection curves of the surfaces of two primitive solids. The equations of the intersection curves may be complicated. For example, the intersection curve of degree two and degree three surfaces may be a space curve of degree six and may not be rational [15]. In this case, the sub-strip area is evaluated by calculating two virtual substrips, where one is bigger and the other is smaller as shown in Figure 10, and finally the average is taken. The intersection curve is thus approximated by the curve which evenly divides a small patch into two regions.

The error included by this approximation is the difference between the real intersection curve and the approximated curve as shown in Figure 11a. It is much less than that of previous approaches, in which the error is the difference between the real intersection curve and the patch boundary, as shown by the shaded area in Figure 11b.

The sub-strip area evaluation is basically evaluating a double integral which may or may not be represented by an analytic form. If an analytic form exists, the strip area evaluation takes the same amount of computation time irrespective of the strip length. In this situation, the scan curve approach takes less computation time than the patch decomposition approach. If there is no analytic form, the strip area evaluation requires some numerical computation. The scan curve approach again in general takes less computation time than the patch decomposition approach which requires more sub-divisions during numerical integration [17].

6. Complexity Analysis and Implementation

6.1. Complexity Analysis

The computational complexity of the proposed scan curve algorithm and the previous patch decomposition algorithm are analyzed and listed in Table 4. The scan curve approach is better than the patch decomposition approach in computational complexity. It shows that the complexity of the scan curve algorithm is $O(NF^2M)$ and that of the patch decomposition algorithm is $O(N^2F^2M)$, where N is the number of scan curves in each primitive solid, M is the number of primitive solids, and F is the number of surfaces on each primitive solid.

6.2. Implementation

This program has been implemented in FORTRAN 77 on a VAX 11/780; the program uses two IMSL [18] routines ZRPOLY and DBLIN. ZRPOLY is a routine for calculating the roots of a single variable polynomial, and is used in determining the intersection points of scan curves and surfaces. DBLIN performs evaluating double integration, and is used in calculating the strip area and its resolution parameter, AEER, which is set to 10^{-5}.

There are four main subroutines in this program. Subroutine GENSCAN generates the scan curves for each face of the primitive solids. Subroutine INTCVSD calculates and sorts the intersection points of scan curves and primitive solids to divide each scan curve into a set of curve segments. Subroutine PCLASFY classify each scan curve segment with respect each primitive solid. Subroutine MCLASFY merges the primitive classification results to classify scan curve with respect to the solid. Subroutine ARSTIRP identifies the contributing sub-strips and calculates their area.

The testing solids were constructed by three primitive solids, a cylinder, a sphere and an ellipsoid as in Figure 12. The regularized Boolean combination of these three primitive solids generate nine different kinds of testing solids. For each testing solid, we computed the face area of its bounding faces respectively and summed them up to give the total face area of the testing solid. These results were obtained by scanning 200 curves on the faces of each primitive solid. The calculated face area are listed in Table 5. For the nine testing solids, the average CPU time for each component of the algorithm is listed in Table 6.

From Table 6, we see that evaluating strip areas takes the longest computation time if there is no analytic form for the double integration formula, such as strips on the ellipsoid surface. Scan curve/solid classification is the second major item in computation. Strip area evaluation ranks the third and scan curve generation takes the least time.

7. Conclusion and Discussion

Compared to previous approaches, there are three main advantages of this scan curve approach. First, it needs less computation time. Second, its results are more accurate than the decomposition approach if the resolution of decomposition is the same. Third, the on/on ambiguities occurring on edges of solids are appropriately treated by the proposed virtual scan curves concept, which had been ignored in previous approaches.

The computation time of this algorithm can be improved by certain embellishments. One of these is by detecting if two primitive solids intersected with each other before computing the scan curve/primitive solid classification. For example, if two primitive solids are not intersected and far apart, then there is no need to compute the scan curve/primitive solid classification, where the classification properties are all "out".

8. Extensions of This Algorithm

The above algorithm may also be extended to the solution of the following three problems: (1) Boundary representation evaluation from CSG; (2) Face area evaluation for solids in Boundary representation; and (3) Triangulation of the faces of solids in CSG or BREP, where BREP is a solid modeling scheme [1] which describes a solid by representing its faces, edges and vertices explicitly.

8.1. BREP evaluation from CSG

The first problem can be solved by merging the contributing sub-strips. As stated and shown in Figure 7, two consecutive scan curves have similar classification and intersection patterns if there are no singular points within their bounding strip. Singular points are either tangent points or intersection points of three surfaces which imply that they are on the boundary of two faces.

The BREP evaluation algorithm may be applied as follows: tracing subsequent scan curves sequentially, the corresponding contributing sub-strips of every two consecutive scan curves are on the same face of the solid if they do not meet singular points. After meeting singular points, the corresponding contributing sub-strips jump to another face of the solid.

To explicitly represent the face of the solid, one only requires to sequentially merge the corresponding sub-strips on the face. The bounding edges of the faces can thus be represented by the set of intersection points on the contributing sub-strips. The vertices are the intersection points of three or more surfaces and can be determined numerically.

8.2. Face Area Evaluation for Solids in BREP

The second problem can be solved by scanning curves on each face of solids in BREP. Each scan curve should be classified with respect to its associated face to determine which portions are "in", "out" or "on" the face. The "on" strips bounded by the scan curves contribute the area of the face.

Classifying scan curve/face requires the calculation of the intersection points of scan curve and the bounding edges of the face. The bounding edges are on the intersection curve of two surfaces, one of which fully contains the face and the scan curve, and the other intersects the scan curve. Thus, the intersection points of the scan curve and the bounding edges can be determined by calculating the intersection points of curve/surface as shown earlier.

8.3. Triangulation of the faces of solids in CSG or BREP

The third problem often occurs in applying the finite element analysis method, which usually requires the decomposition of the bounding faces of solids into a set of triangular patches. This problem can be solved by decomposing each contributing sub-strip as two or more triangular patches. As stated, each contributing sub-strip is fully on the solid; thus its decomposed triangular patches are also fully on the solid. There are several methods to decompose a strip into a set of triangular face patches. One of these is by introducing a curve which diagonally connects the two points on the contributing sub-strips. This curve is the intersection of the strip and a plane which is defined by two points and a normal direction. Note that the triangulation process is applicable to solids in CSG or BREP, because the contributing sub-strips can be derived from both schemes.

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Table 1: Concerned Surfaces and Corresponding Scan Curves

Surface Type	Sectioning Curves	Scan Curves
Cylindrical Surface		line
Surface of Revolution		circle
Quadric Surfaces	Sectioning Curves	Scan Curves
2nd Degree Cylinder	line, 2nd degree curve	line
Elliptic Paraboloid	ellipse, parabla	ellipse
Hyperboli Paraboloid	parabola, hyperbola	parabola
Elliptic Cone	ellipse, line	line
Ellipsoid	ellipse	ellipse
Hyperboloids of 2 sheets	ellipse, hyperbola	ellipse
Hyperboloid of 1 sheet	ellipse, hyperbola	ellipse

Table 2: Merge Classification Results

$\cup^*$	on	in	out
on	ambiguous	in	on
in	in	in	in
out	on	in	out

$-^*$	on	in	out
on	ambiguous	out	on
in	on	out	in
out	on	out	out

$\cap^*$	on	in	out
on	ambiguous	on	out
in	on	in	out
out	on	out	out

Table 3: Merge Classification for Two "on" Cases

$\cup^*$	on1	on2
on1	on1	in
on2	in	on2

$\cap^*$	on1	on2
on1	on1	out
on2	out	on2

$-^*$	on1	on2
on1	out	on1
on2	on2	out

Table 4: Computational Complexity of Each Component of This Algorithm

Scan Curve Approach	Complexity	Patch Decomposition Approach	Complexity
Scan Curve Generation	O(MF)	Patch Generation Generation	O(MF)
Curve/Solid Classification	$O(NF^2M)$	Point/Solid Classification	$O(N^2F^2M)$
Strip Area Evaluation	O(NFM)	Patch Area Evaluation	$O(N^2FM)$

N: number of scan curves in each face
F: number of faces in each primitive solid
M: number of primitive solids in the CSG solid

Table 5: Face Area of Each Face of the Testing Solids

Solid	Cylinder	Sphere	Ellipsoid	Plane	Total
$(A \cup^* B) \cup^* C$	59.2192	7.3983	4.4221	6.2831	77.3227
$(A \cap^* B) \cup^* C$	2.9087	4.1851	10.3311	0.0000	17.4249
$(A -^* B) \cup^* C$	59.2192	4.1851	8.8693	6.2831	78.5567
$(A \cup^* B) \cap^* C$	0.3600	0.5035	7.7976	0.0000	8.6611
$(A \cap^* B) \cap^* C$	0.3387	0.4791	1.8887	0.0000	2.7065
$(A -^* B) \cap^* C$	0.3600	0.4791	3.3502	0.0000	4.1893
$(A \cup^* B) -^* C$	59.2192	7.3983	7.7976	6.2831	80.6982
$(A \cap^* B) -^* C$	2.9087	4.1851	1.8887	0.0000	8.9825
$(A -^* B) -^* C$	59.2192	4.1851	3.3508	6.2831	73.0382

Table 6: Average CPU Time in Seconds of Each Component of the Algorithm for Each Face of the Nine Testing Solids

Component of Algorithm	Cylinder	Sphere	Ellipsoid
Scan Curve Generation	0.4	0.4	0.4
Curve/Solid Classify	2.7	11.8	11.0
Strip Area Evaluation	0.6	5.4	180.3

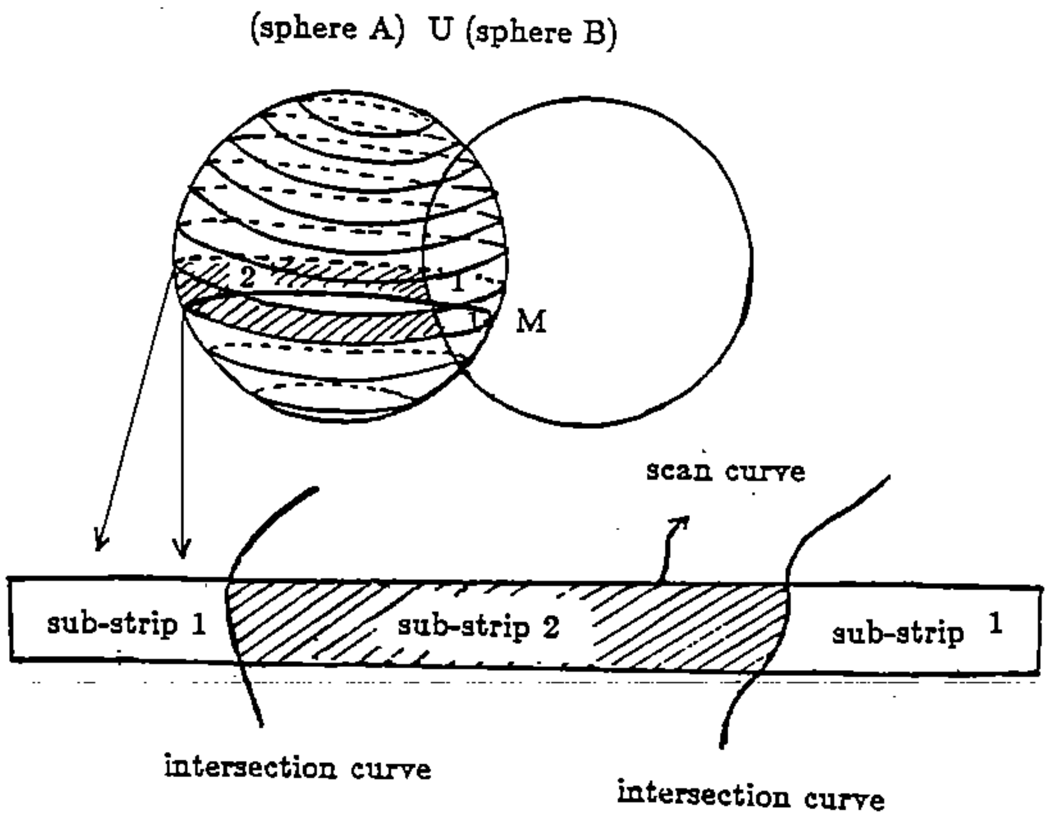

Figure 1: Scan curves of primitive solids and their subtrips, the resulting CSG solid is the union of two spheres

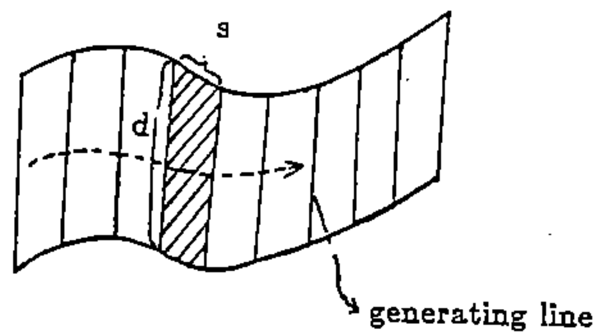

Figure 2 : Cylindrical surface

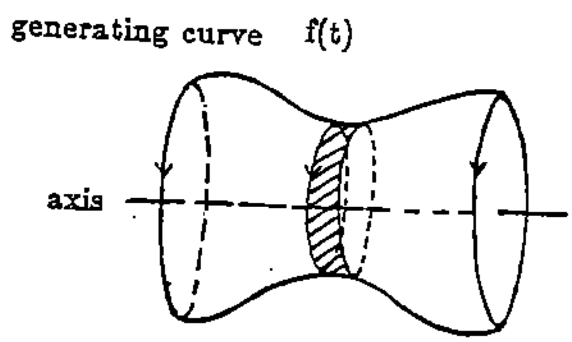

Figure 3 : Surface of revolution,

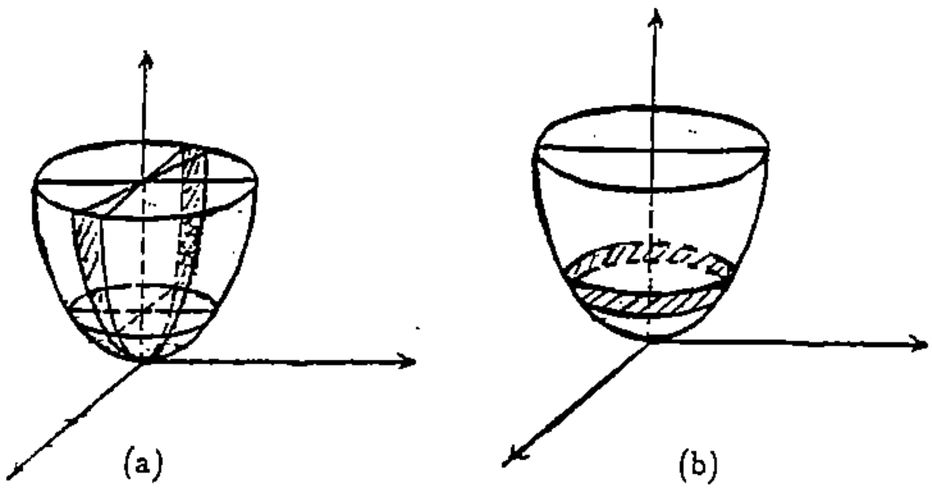

Figure 4: Non-overlap strips by scanning ellipses and overlap strips by scanning parablas for elliptic paraboloid

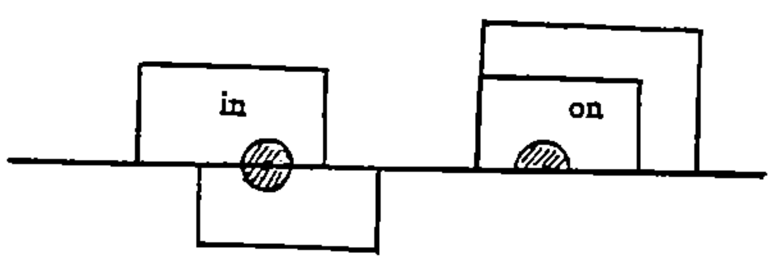

Figure 5: Point neighborhood representation and combination

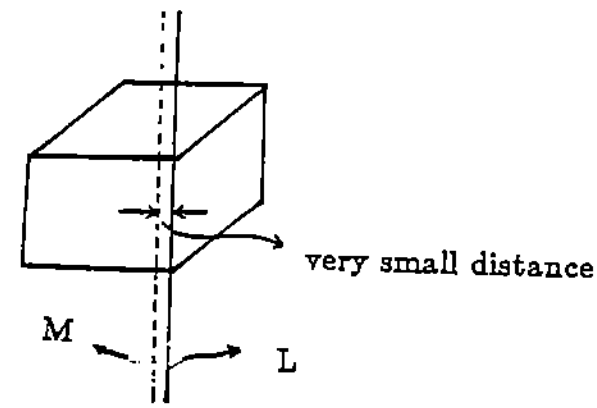

Figure 6 : Virtual scan curve

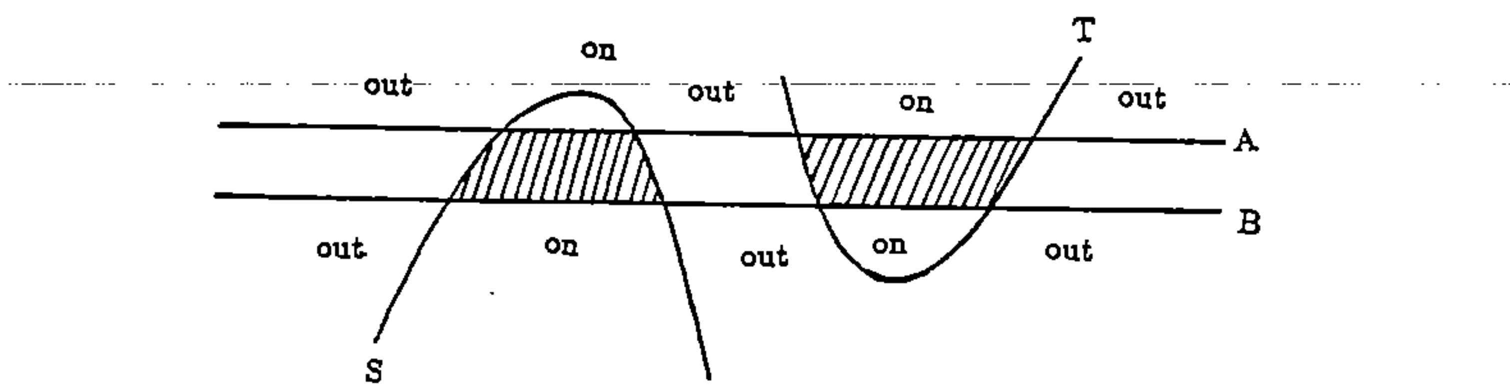

Figure 7 : Strip not including singular points

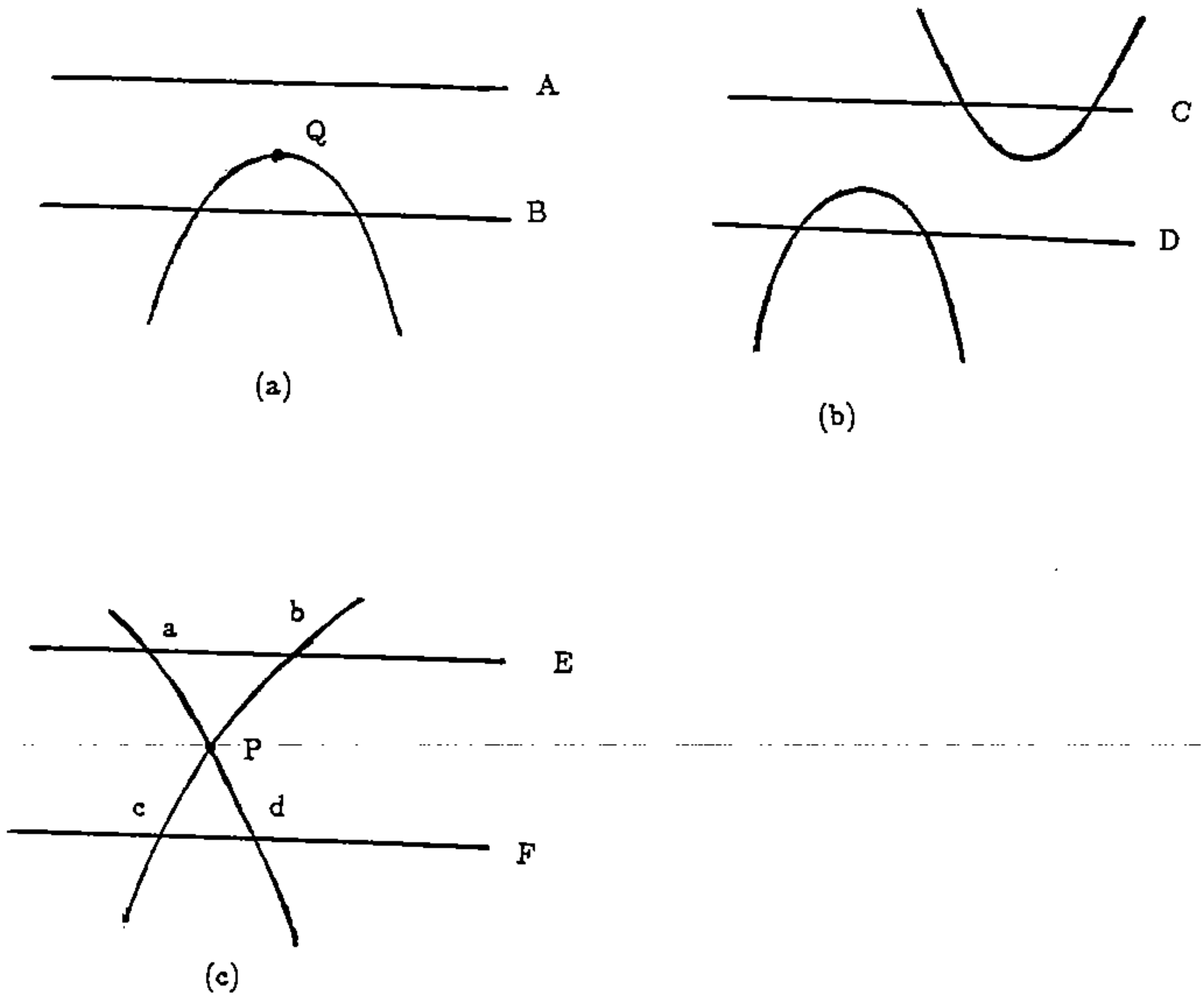

Figure 8: Strip including singular points have different classification and intersection patterns, (a) different numbers of curve segments (b) different classification pattern (c) different intersection pattern

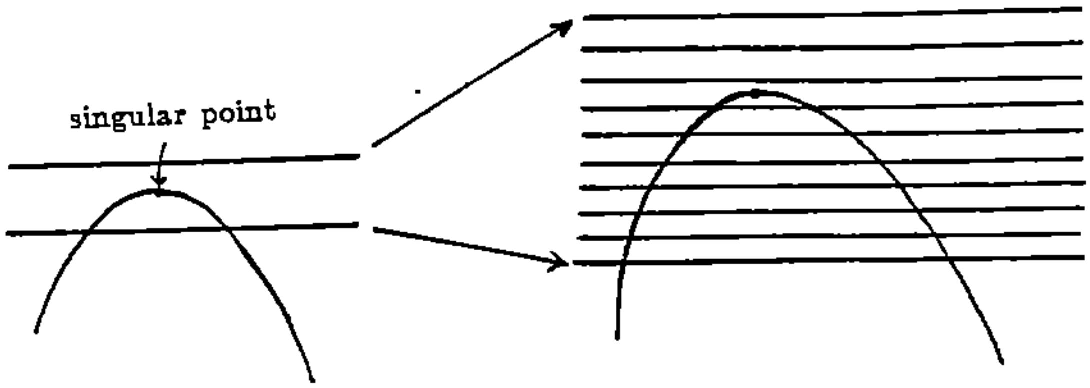

Figure 9: Decomposing a strip which including singular points into a set of smaller strips

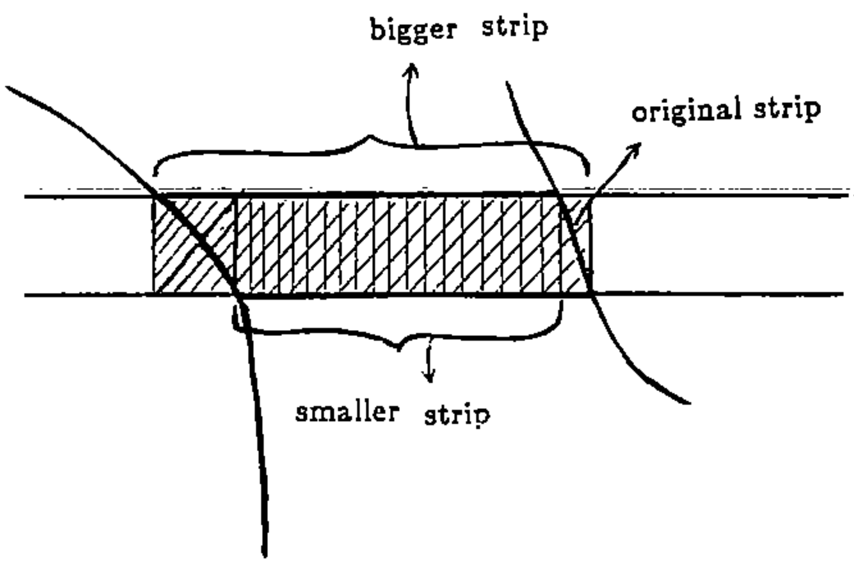

Figure 10: Evaluating the face area of a strip by taking the average of two strips, where one is bigger and the other is smaller than the original strip.

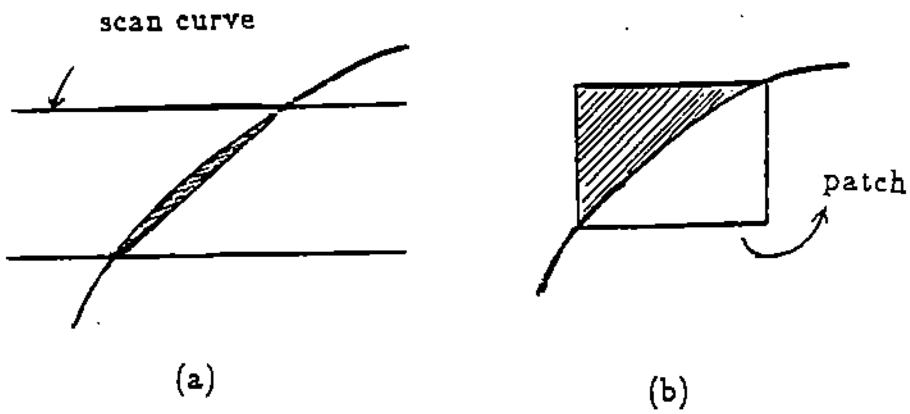

Figure 11: Error in area evaluation of (a) scan curve approach (b) patch decomposition approach

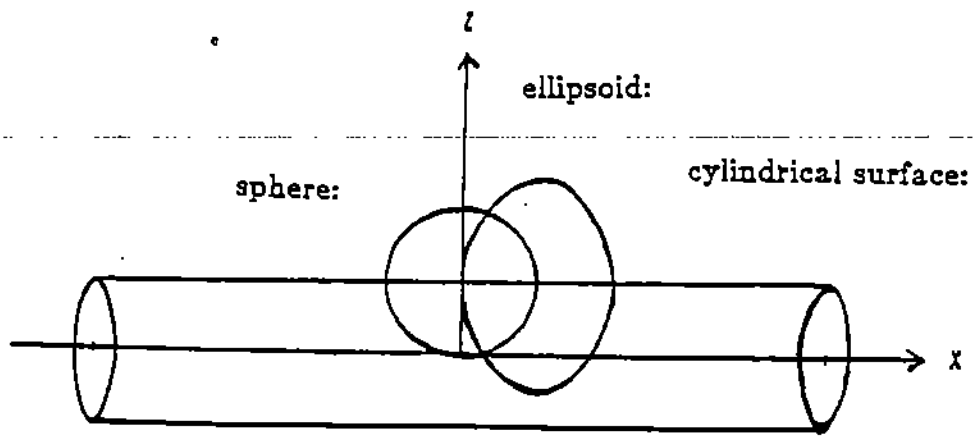

Figure 12: Testing solids consisting of three primitive solids

Cylinder: $Y^2 + Z^2 = 1$

$X = 5$

$X = -5$

Sphere: $X^2 + Y^2 + (Z-1)^2 = 1$

Ellipsoid: $\frac{(X-1)^2}{1^2} + \frac{Y^2}{(0.5)^2} + \frac{(Z-1)^2}{(1.5)^2} = 1$