THE EMERGENCE OF ALGEBRAIC CURVES
AND SURFACES IN GEOMETRIC DESIGN

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Abstract

This paper highlights algebraic curves and surfaces and illustrates its use in geometric design applications. It contrasts the implicit and parametric representations for algorithms in Geometric Modelling, Scattered Data fitting and Computer Graphics/Mesh Generation. In this framework the choice of which of the two representations to use is determined entirely by the desired optimality of the geometric algorithm.

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1   Introduction

Our approach to the design and analysis of geometric algorithms for operations on algebraic curves and surfaces is to take the view of abstract data types, that is, a data representation coupled together with the operations on them. In this framework, the choice of which representation of the algebraic curve or surface to use is determined by the desired optimality of the geometric algorithms for the operations. Algebraic curves and surfaces can be represented in an **implicit** form, and sometimes also in a **parametric** form. The implicit form of a real algebraic surface in $\mathbb{R}^3$ is

$$f(x, y, z) = 0 \tag{1}$$

where f is a polynomial with coefficients in $\mathbb{R}$. The parametric form, when it exists, for a real algebraic surface in $\mathbb{R}^3$ is

$$x = \frac{f_1(s, t)}{f_4(s, t)} \tag{2}$$

$$y = \frac{f_2(s, t)}{f_4(s, t)} \tag{3}$$

$$z = \frac{f_3(s, t)}{f_4(s, t)} \tag{4}$$

where the f_i are again polynomials with coefficients in $\mathbb{R}$. The above implicit form describes a two dimensional real algebraic variety (a surface) with a single polynomial equation in $\mathbb{R}^3$. The parametric form also describes a real two dimensional algebraic variety (a surface), however with a set of three independent polynomial equations in $\mathbb{R}^5$, with coordinate variables x, y, z, s, t. Alternatively, the parametric form of a real surface may also be interpreted as a rational mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$. We can thus compare the implicit and parametric representations of algebraic surfaces by considering the the parametric form either as a *mapping* or alternatively, an *algebraic variety*.

In this paper, we consider specific geometric operations in geometric modeling, scattered data fitting, computer graphics and finite element mesh generation, and compare the implicit and parametric forms for their superiority (or lack thereof) in optimizing algorithms for operations in each of these three categories. In these comparisons we factor out the choice of the polynomial basis, i.e. whether the polynomials in the implicit and parametric form are defined over the *power* [14], barycentric *Bernstein* or *B-spline* basis [44]. This, so as not to overly complicate the comparison and also because the numerical or geometric advantages of these forms occurs for both the implicit and parametric curve and surface representations.

Section 2 sets the terminology and introduces some mathematical definitions. Section 3 provides some simple facts about algebraic curves and surfaces. Section 4 compares the implicit and parametric representations for geometric modeling operations. Some of these operations (intersection, offset, blending, etc.) exploit the simpler algebraic variety of the implicit form while others (sorting, neighborhood classification, etc.) benefit from the rational mapping of the parametric form. Section 5 considers the tradeoff between implicit and parametric representations for scattered data fitting operations where primarily the algebraic variety interpretation for algebraic surfaces is the most natural. The rational mapping of the parametric form does

not seem to provide any simplification. Section 6 compares the implicit and parametric representations for graphics display and triangular mesh generation operations. Here the rational mapping gives a clear advantage to the parametric form, though the algorithms to solve this problem in this representation are still non-trivial. Several examples are given in the subsections to illustrate the above comparisons.

2 Mathematical Preliminaries

In this section we review some basic terminology from algebraic geometry that we shall be using in subsequent sections. These and additional facts can be found for example in [49, 53].

The set of real and complex solutions (or *zero* set $Z(C)$) of a collection C of polynomial equations

$$f_1(x_1, ..., x_d) = 0$$

$$...$$

$$f_m(x_1, ..., x_d) = 0 \tag{5}$$

with coefficients over the reals $\mathbb{R}$ or complexes $\mathbb{C}$, is referred to as an *algebraic set*. The algebraic set defined by a single equation $(m = 1)$ is also known as a hypersurface. A algebraic set that cannot be represented as the union of two other distinct algebraic sets, neither containing the other, is said to be *irreducible*. An irreducible algebraic set $Z(C)$ is also known as an algebraic variety V.

A hypersurface in R^d, some d dimensional space, is of *dimension* $d - 1$. The *dimension* of an algebraic variety V is k if its points can be put in $(1, 1)$ rational correspondence with the points of an irreducible hypersurface in $k + 1$ dimensional space. Let the *algebraic degree* of an algebraic variety V be the *maximum* degree of any defining polynomial. A degree 1 hypersurface is also called a *hyperplane* while a degree 1 algebraic variety of dimension k is also called a *k-flat*. The *geometric degree* of a variety V of dimension k in some R^d is the maximum number of intersections between V and a $(d - k)$-flat, counting both real and complex intersections and at infinity. Hence the geometric degree of an algebraic hypersurface is the maximum number of intersections between the hypersurface and a line, counting both real and complex intersections and at infinity.

The following theorem, perhaps the oldest in algebraic geometry, summarizes the resulting geometric degree of intersections of varieties of different degrees.

[Bezout] A variety of geometric degree p which *properly* intersects a a variety of geometric degree q does so in an algebraic set of geometric degree either at most $p * q$ or infinitely often.

The *normal* or *gradient* of a hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ is the vector $\nabla f = (f_{x_1}, f_{x_2}, ..., f_{x_n})$. A point $\mathbf{p} = (a_0, a_1, ... a_n)$ on a hypersurface is a *regular* point if the gradient at $\mathbf{p}$ is not null; otherwise the point is *singular*. A singular point $\mathbf{q}$ is of multiplicity e for a hypersurface $\mathcal{H}$ of degree d if any line through $\mathbf{q}$ meets $\mathcal{H}$ in at most $d - e$ additional points. Similarly a singular point $\mathbf{q}$ is of multiplicity e for a variety V in $\mathbb{R}^n$ of dimension k and degree d if any sub-space $\mathbb{R}^{n-k}$ through $\mathbf{q}$ meets V in at most $d - e$ additional points. It is important to note that even if two varieties intersect in a *proper* manner, their intersection in general may consist of sub-varieties of various multiplicites. The total degree of the intersection, however is bounded

by the above Bezout's theorem. Finally, one notes that a hypersurface $f(x_1, ..., x_n) = 0$ of degree d has $K = \binom{n+d}{n}$ coefficients and one less than that number of independent coefficients. Hypersurfaces $f(x_1, ..., x_n) = 0$ of degree d form K dimensional vector spaces over the field of coefficients of the polynomials.

Finally, two hypersurfaces $f(x_1, ..., x_n) = 0$ and $g(x_1, ..., x_n) = 0$ meet with C^k-continuity along a common subvariety V if and only if there exists functions $\alpha(x_1, ..., x_n)$ and $\beta(x_1, ..., x_n)$ such that all derivatives upto order k of $\alpha f - \beta g$ equals zero at all points along V, see for e.g., [28].

3 Algebraic Curves and Surfaces

We cast our real implicit and parametric curves and surfaces, in the terminology of section 2. A real implicit algebraic plane curves $f(x, y) = 0$ is a hypersurface of dimension 1 in $\mathbb{R}^2$, while a parametric plane curve $[f_3(s)x - f_1(s) = 0, f_3(s)y - f_2(s) = 0]$ is an algebraic variety of dimension 1 in $\mathbb{R}^3$, defined by the two independent algebraic equations in the three variables x, y, s. Similarly, a real implicit algebraic surface $f(x, y, z) = 0$ is a hypersurface of dimension 2 in $\mathbb{R}^3$, while a parametric surface $[f_4(s, t)x - f_1(s, t) = 0, f_4(s, t)y - f_2(s, t) = 0, f_4(s, t)z - f_3(s, t) = 0]$ is an algebraic variety of dimension 2 in $\mathbb{R}^5$, defined by three independent algebraic equations in the five variables x, y, z, s, t.

A plane parametric curve is a very special algebraic variety of dimension 1 in x, y, s space, since the curve lies in the 2-dimensional subspace defined by x, y and furthermore points on the curve can be put in $(1, 1)$ rational correspondence with points on the 1-dimensional sub-space defined by s. Parametric curves are thus a special subset of algebraic curves, and are often also called rational algebraic curves. Figure 1 depicts the relationship between the set of parametric curves and non-parametric curves at various degrees.

Example parametric (rational algebraic) curves are degree two algebraic curves (conics) and degree three algebraic curves (cubics) with a singular point. The non-singular cubic is not rational and are also known as elliptic cubics. In general, a necessary and sufficient condition for the rationality of an algebraic curve of arbitrary degree is given by the Cayley-Riemann criterion: a curve is rational iff $g = 0$, where g, the genus of the curve is a measure of the deficiency of the curve's singularities from its maximum allowable limit [51]. Algorithms for computing the genus of an algebraic curve and for symbolically deriving the parametric equations of genus 0 curves, are given in [1, 2, 3].

Similarly, a parametric surface is a very special algebraic variety of dimension 2 in x, y, z, s, t space, since the surface lies in the 3-dimensional subspace defined by x, y, z and furthermore points on the surface can be put in $(1, 1)$ rational correspondence with points on the 2-dimensional sub-space defined by s, t. Figure 2 depicts the relationship between parametric and non-parametric surfaces.

Example parametric (rational algebraic) surfaces are degree two algebraic surfaces (quadrics) and most degree three algebraic surfaces (cubic surfaces). The cylinders of nonsingular cubic curves and the cubic surface cone are of not rational. Other examples of rational algebraic surfaces are Steiner surfaces 11 which are degree four surfaces with a triple point, and Plücker surfaces which are degree four surfaces with a double curve. In general, a necessary and sufficient
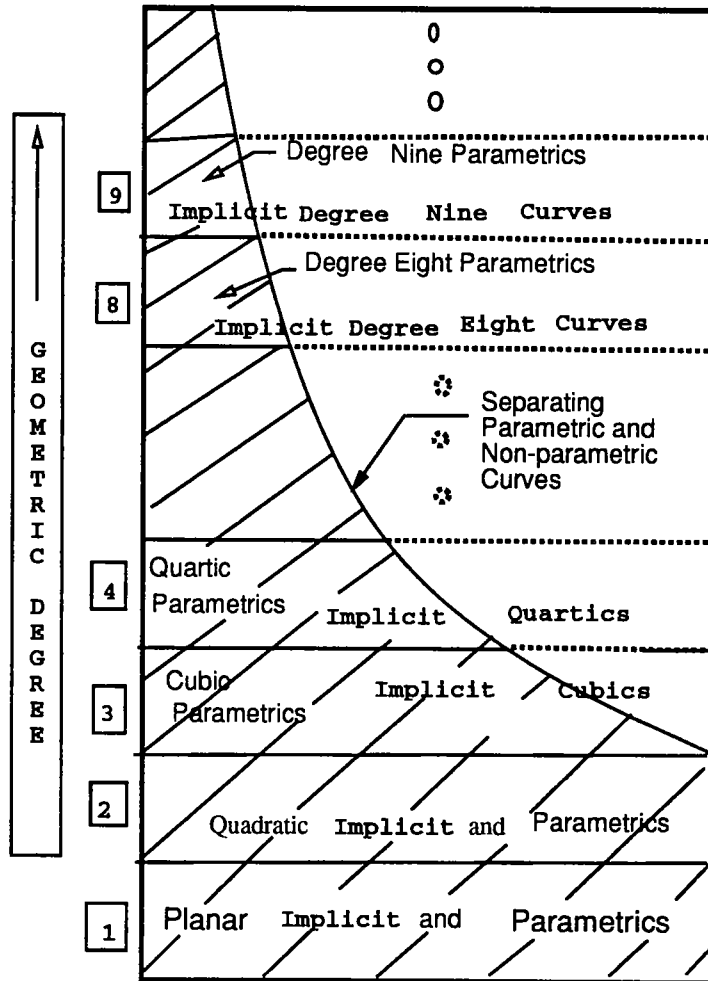
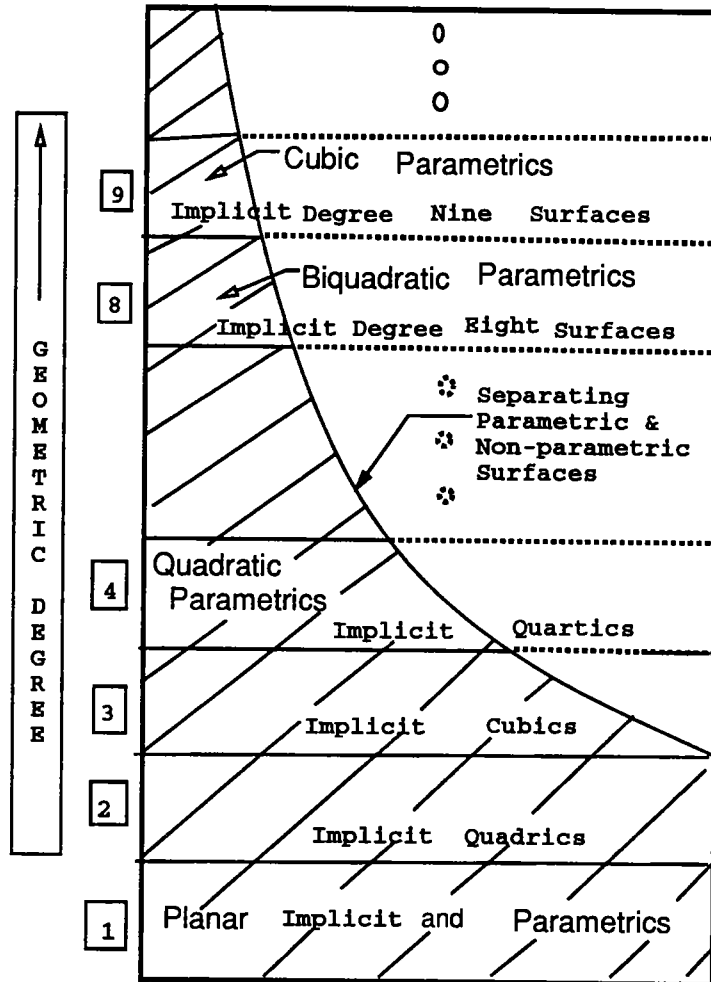Figure 1: A Classification of Low Degree Algebraic Curves

Figure 2: A Classification of Low Degree Algebraic Surfaces

condition for the rationality of an algebraic surface of arbitrary degree is given by Castelnuovo's criterion: $P_a = P_2 = 0$, where P_a is the arithmetic genus and P_2 is the second plurigenus [52]. Algorithms for symbolically deriving the parametric equations of degree two and three rational surfaces are given in [1, 2, 3, 4].

3.1 Degree & Singularities

For implicit algebraic plane curves and surfaces defined by polynomials of degree d, the maximum number of intersections between the curve and a line in the plane or the surface and a line in space, is equal to the maximum number of roots of a polynomial of degree d. Hence, here the geometric degree is the same as the algebraic degree which is equal to d. For parametric curves defined by polynomials of degree d, the maximum number of intersections between the curve and a line in the plane is again equal to the maximum number of roots of a polynomial of degree d. Hence here again the geometric degree is the same as the algebraic degree. For parametric surfaces defined by polynomials of degree d the geometric degree can be as large as $O(d^2)$, the square of the algebraic degree d. This can be seen as follows. Consider the intersection of a generic line in space $[a_1x + b_1y + c_1z - d_1 = 0, a_2x + b_2y + c_2z - d_2 = 0]$ with the parametric surface. The intersection yields two implicit algebraic curves of degree d which can potentially intersect in $O(d^2)$ points (via Bezout's theorem), corresponding to the intersection points of the line and the parametric surface.

A parametric curve of algebraic degree d is an algebraic curve of genus 0 and so may have $\frac{(d-1)(d-2)}{2} = O(d^2)$ singular (double) points. This number is the maximum number of singular points an algebraic curve of degree d may have. From Bezout's theorem, we realize that the intersection of two implicit surfaces of algebraic degree d can be a curve of geometric degree $O(d^2)$. Furthermore the same theorem implies that the intersection of two parametric surfaces of algebraic degree d (and geometric degree $O(d^2)$) can be a curve of geometric degree $O(d^4)$. Hence, while the potential singularities of the space curve defined by the intersection of two implicit surfaces defined by polynomials of degree d can be as many as $O(d^4)$, the potential singularities of the space curve defined by the intersection of two parametric surfaces defined by polynomials of degree d can be as many as $O(d^8)$.

4 Geometric Modeling Operations

Geometric modeling operations are dependent largely on the application for the modeling system. Our comparisons here are limited to operations such as Boolean set (intersection, union, difference), blending, rounding, joining and the properties these operations should have.

4.1 Boolean Set Operations

4.1.1 Closure

One desirable property which we would like to optimize by the choice of curve and surface representation is *closure* under most modeling operations. By closure we mean that the result of an operation (without any approximation) has the same representation as the input. The
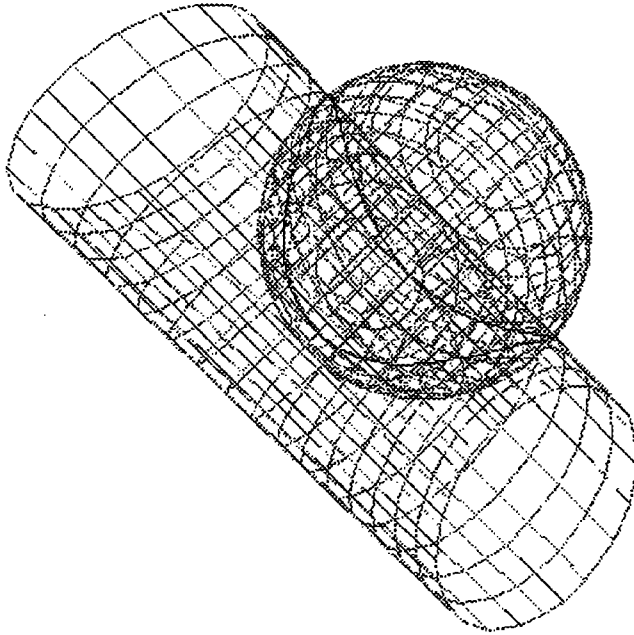
Figure 3: A non-parametric intersection curve of two parametric surfaces

closure property allows the cascading of similar operations with the output of one operation serving as the input of another. The set of rational curves and surfaces are a subset of algebraic curves and surfaces of the same degree. While all algebraic curves and surfaces have an implicit representation, only the class of rational algebraic curves and surfaces also have the alternate parametric representation. The *non-closure* property of the parametric representation stems from this restriction.

One of the generic sub-operations in intersecting two solid models is the intersection of two surfaces. Figure 3 depicts an example where the intersection of two parametric (degree two) surfaces produces a (degree four) curve which is non-parametric. This can be seen as follows:

Example 4.1 *The intersecting surfaces are a sphere $x^2 + y^2 + z^2 - 2 = 0$ and a circular cylinder $x^2 + (y - 1)^2 - 1 = 0$. Both being of degree two are parametric[49] and their parameterizations can be easily derived [1]. The projection of the intersection curve onto the $x - y$ plane is given by $C : z^4 + 4*x^2 - 4 = 0$. The plane curve C has no affine singularities but a simple double point*

at the origin. Hence, its genus and also the genus of the intersection curve is 2 and therefore non-rational.

This implies that the result of an intersection operation on parametric representations, in general, are curves and surfaces that are not rational. To compute parametric representations for these non-rational results requires approximation [16, 17]. As the solution to most modeling operations are expressible by polynomial equation, from the projection theorem of algebraic varieties it follows that the implicit representation is closed under modeling operations.

4.1.2 Space & Time Complexity

The implicit representation of an algebraic plane curve of degree d requires $\binom{d+2}{2} = O(d^2)$ real coefficients. On the other hand the parametric representation of a rational curve of algebraic degree d requires only $3(d+1) = O(d)$ coefficients. Similarly the implicit representation of an algebraic surface of degree d requires $\binom{d+3}{3} = O(d^3)$ real coefficients, while the parametric representation of a rational surface of algebraic degree d requires only $4\binom{d+2}{2} = O(d^2)$ coefficients.

The time complexity for Boolean set operations (amongst others) are governed by the number of polynomial equations and variables needed to express the result of the corresponding operation. The intersection of two implicit surfaces of algebraic degree d, viz., $f_1(x,y,z) = 0$ and $f_2(x,y,z) = 0$ where f_1 and f_2 are polynomials of degree d, is expressible as the simultaneous solution of the two polynomial equations $f_1 = 0 = f_2$ in the three variables x, y, z. On the other hand the intersection of two parametric surfaces of algebraic degree d, viz., $[x = \frac{f_1(s,t)}{f_4(s,t)}, y = \frac{f_2(s,t)}{f_4(s,t)}, z = \frac{f_3(s,t)}{f_4(s,t)}]$ and $[x = \frac{g_1(u,v)}{g_4(u,v)}, y = \frac{g_2(u,v)}{g_4(u,v)}, z = \frac{g_3(u,v)}{g_4(u,v)}]$ is expressible as the simultaneous solution of three polynomial equations

$$
\begin{aligned}
f_1(s,t)g_4(u,v) - g_1(u,v)f_4(s,t) &= 0 \\
f_2(s,t)g_4(u,v) - g_2(u,v)f_4(s,t) &= 0 \\
f_3(s,t)g_4(u,v) - g_3(u,v)f_4(s,t) &= 0
\end{aligned}
$$

in the four variables s, t, u, v. The simplicity of the intersection solution definitely makes the implicit representation as the more desirable form. Furthermore, it may noted again that while the geometric degree of the intersection curve of the two implicit surfaces of algebraic degree d is no larger than d^2, the geometric degree of the intersection curve of the two parametric surfaces of algebraic degree d may be as large as d^4.

4.2 Blending & Joining

The mechanics of blending and joining operations in geometric design are

1. geometric features of a surface to be designed are described in terms of a combination of points, curves, and possibly associated normal vectors, derived from the primary attachment surfaces

2. these properties are translated into a homogeneous linear system of equations with extra surface constraints
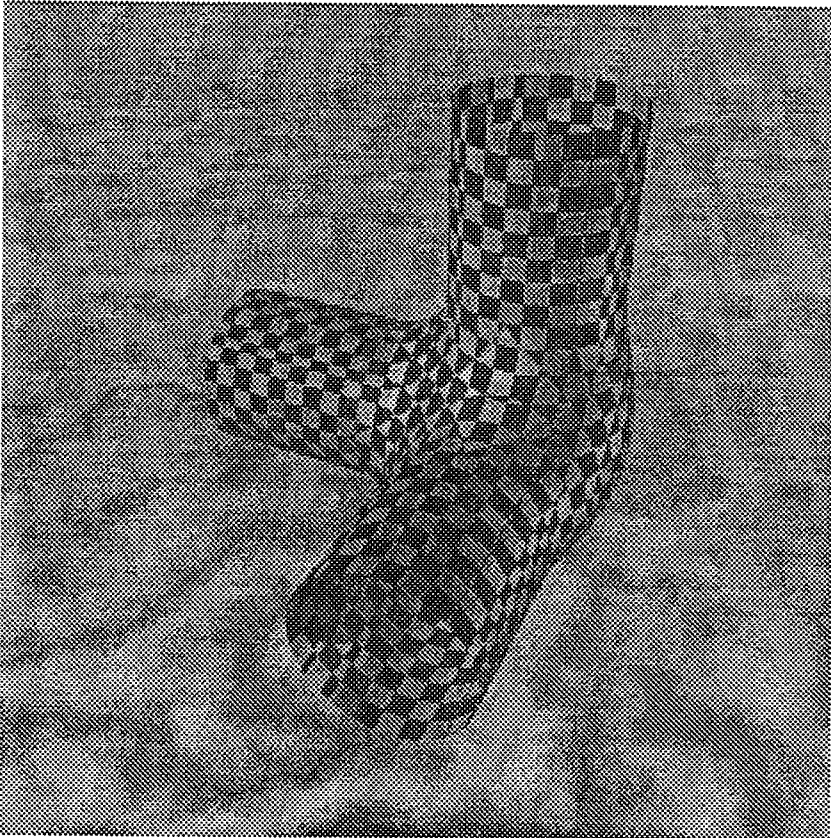
Figure 4: A smooth three way join using a degree four algebraic surface
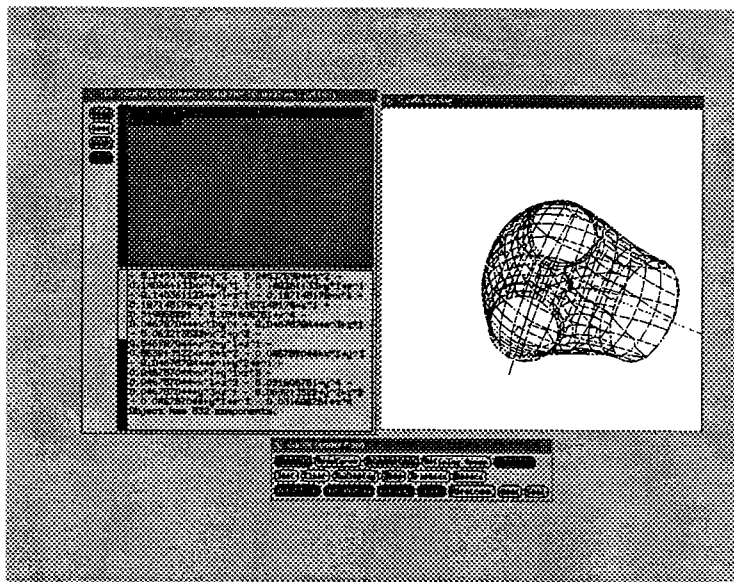
Figure 5: The joining surface as a C^1 interpolant

3. nontrivial solutions of the linear system are computed

A common technique for the translation in step 2. above is to use C^k interpolation and approximation. See Figures 4, 5. The basic problem here can be described as follows: Construct a single real algebraic surface S which C^k interpolates a collection of l points $\mathbf{p}_i$ in $\mathbb{R}^3$ with associated fixed "normal" unit vectors $\mathbf{m}_i$, and m given space curves C_j in $\mathbb{R}^3$, possibly with associated "normal" unit vectors $\mathbf{n}_j$ and additionally upto k^{th} order derivatives of $\mathbf{n}_j$ varying along the entire span of the curves.

4.2.1 Linearity

In comparing the implicit and parametric surface for this fitting problem in (x, y, z) space, one notes that the problem straightforwardly reduces to a linear problem for the implicit representation in (1) as the unknowns are the coefficients of the polynomial. The same problem is however non-linear for the parametric representation in (2) as the unknowns are the coefficients and the domain parameters s, t. To reduce the interpolation to a linear problem for the parametric surface requires some assignment of s, t values to each of the original data points in x, y, z space.

4.2.2 Degrees of Freedom

Note also that if one works with polynomials of degree d in both the implicit and parametric form, one has $O(d^3)$ degrees of freedom (independent coefficients) for the implicit representation as compared to only $O(d^2)$ degrees of freedom (independent coefficients) for the parametric.
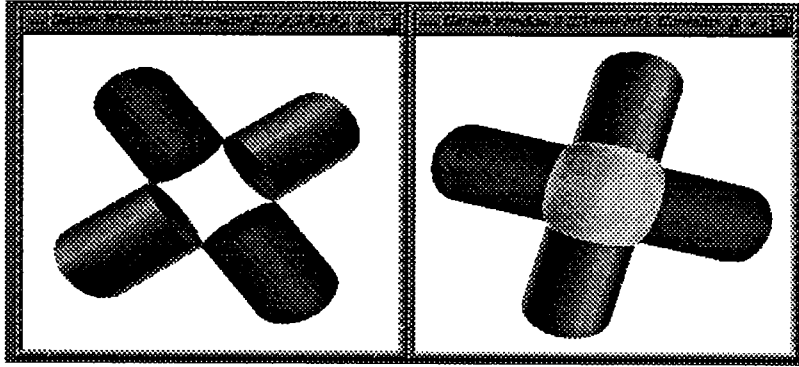
Figure 6: A smooth quartic surface blend of the four cylinders
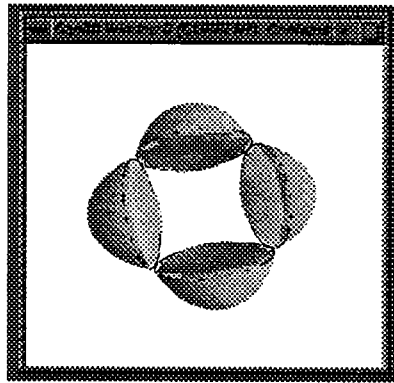

Figure 7: A degenerate blend of the four cylinders

A straightforward combinatorial argument shows that the larger number of degrees of freedom for the implicit representation, coupled with the smaller number of constraints, leads to lower degree interpolatory surfaces.

The total number of linear equations generated for a possible implicit surface of algebraic degree d to C^1 interpolate k points with fixed constant normal directions and also to contain, with C^1 continuity, l space curves of degree e with assigned normal directions, varying as a polynomial of degree $m \leq d$, is $3k + (2d - 1)el + 2l$. This number becomes $3k + (2d^2 - 1)el + 2l$ when the C^1 interpolating surface is a parametric surface of algebraic degree d [15].

For a given configuration of points, curves, and normal vectors, the above interpolation scheme allows one to both-upper and lower-bound the degree of the blending or joining surface.

1. **Lower Bound** Let $r(n)$ be the rank of a homogeneous system of linear equations, obtained from the given geometric configuration and surface degree n. The rank tells us the exact number of independent constraints on the coefficients of the desired algebraic surface of

degree d. Dependencies arise from spatial interrelationships of the given points and curves. From the rank, we can conclude that there exists no algebraic surface of a degree less than or equal to d_0 where d_0 is the largest d such that $F(d) < r(d)$ with $F(d) = \binom{d+3}{3} - 1$ for an implicit surface of algebraic degree d and with $F(d) = 4\binom{d+2}{2} - 4$ for a parametric surface of algebraic degree d.

2. **Upper Bound** Alternatively, the smallest d can be chosen such that $F(d) \geq r(d)$. The nontrivial solutions of the linear system represents a $(F(d) - r(d) + 1)$-parameter family (with $F(d) - r(d)$ degrees of freedom) of algebraic surfaces of degree d which interpolate the given geometric data. We select suitable surfaces from this family. See Figure 6.

The advantages of lower number of interpolatory contraints and higher degrees of freedom, both favour the implicit form. In general this translates to a lower geometric degree blending or joining solution. However, some of these interpolating surfaces of a interpolating family may not be suitable for the design application they were intended to benefit. These problems arise when the given points or curves are smoothly interpolated, but, lie on separate real components of the same nonsingular, irreducible algebraic surface. See Figure 7. Solutions to counter this problem in the implicit case have been given in [15, 9]. For parametric surfaces this problem is circumvented by restricting to polynomial parametric surfaces or to rational Bezier or rational B-splines[25]. For these classes of parametric surfaces, the domain parameters are confined to regions which have no real poles or base points and hence correspond to single sheeted surfaces.

5 Scattered Data Fitting

Consider the problem of constructing a C^k mesh of smooth surface patches or *splines* that interpolate or approximate scattered data in $\mathbb{R}^3$. Computations which we would like to optimize by our choice of curve and surface representation include:

- solution requiring a small number of surface patches

- reduction of the fitting problem to solving small linear systems

- low geometric degree of the solution surfaces

There are several possible variants of the problem depending on the nature of the interpolation problem on hand:- local versus non-local patch interpolation, splitting v.s. non-splitting of the surface patches per triangulation face, the convexity versus non-convexity of the given triangulation, etc. In each of these cases, the comparison between the implciit versus parametric representation does not yeild a clear winner. While the implicit representation yields lower geometric degree solutions (for reasons relating to degrees of freedom and the number of constraints, similar to subsection 4.2), the parametric surfaces shows a clear advantage when suitable surfaces need to be selected from an infinite family of interpolatory solutions. As discussed in subsection 4.2 straightforward conditions on the parameter domain can yield parametric surface solutions which are free of poles and base points. The selection of suitable implicit surfaces
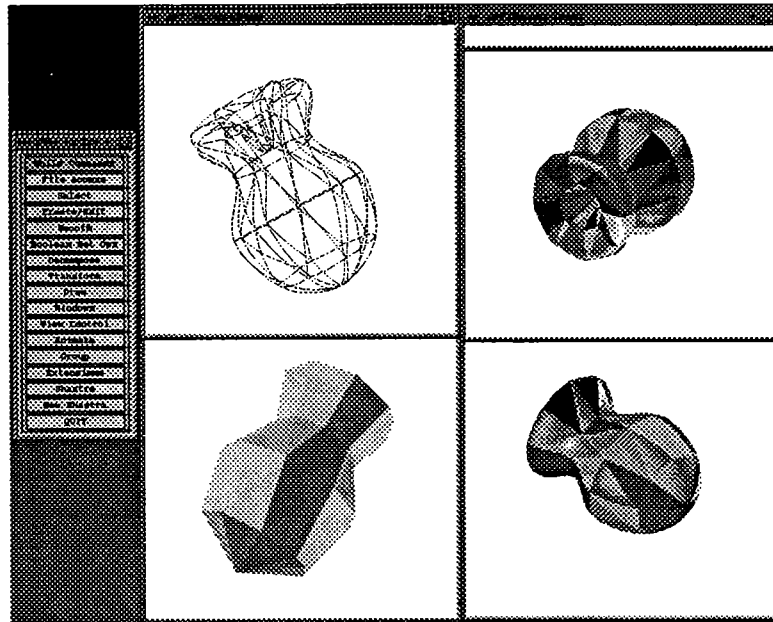
Figure 8: C^1 Implicit Splines over a Spatial Triangulation

The generation of a C^1 mesh of smooth surface patches or *splines* that interpolate or approximate *triangulated space data* is one of the central topics of geometric design. Alfeld [5], Chui [22], Dahmen and Michelli [24] and Hollig [33] summarize much of the history of scattered data fitting and multivariate splines. Prior work on splines have traditionally worked with a given planar triangulation using a polynomial function basis [5, 44, 48]. More recently surface fitting has been considered over closed triangulations in three dimensions using parametric surface patches [18, 21, 27, 29, 30, 32, 34, 37, 39, 40, 42, 45, 50]. Little work has been done on spline bases using implictly defined algebraic surface patches. Sederberg [47] showed how various smooth implicit algebriac surfaces in trivariate Bernstein basis can be manipulated as functions in Bezier control tetrahedra with finite weights. Patrikalakis and Kriezis [38] extended this by considering implicit algebriac surfaces in a tensor product B-spline basis. However the problem of selecting weights or specifying knot sequences for C^1 meshes of implicit algebraic surface patches which fit given spatial data, was left open. Dahmen [23] presented a scheme for constructing C^1 continuous, piecewise quadric surface patches over a data triangulation in space. In his construction each triangular face is split and replaced by six micro quadric triangular patches, similar to the splitting scheme of Powell-Sabin [43]. Dahmen's technique however works only if the original triangulation of the data set allows a transversal system of planes, and hence is quite restricted. Moore and Warren [36] extend the marching cubes scheme of [35] and compute a C^1 piecewise quadratic approximation (least-squares) to scattered data. They too use a Powell-Sabin like split, however over subcubes.

In paper [9] the authors consider an arbitrary spatial triangulation $\mathcal{T}$ consisting of vertices $\mathbf{p} = (x_i, y_i, z_i)$ in $\mathbb{R}^3$ (or more generally a simplicial polyhedron $\mathcal{P}$ when the triangulation is closed), with possibly "normal" vectors at the vertex points. An algorithm is given to construct a C^1 continuous mesh of low degree real algebraic surface patches S_i over $\mathcal{T}$ or $\mathcal{P}$. The algorithm first converts the given triangulation $\mathcal{T}$ or simplicial polyhedron $\mathcal{P}$ into a curvilinear wireframe (with at most cubic parametric curves) which C^1 interpolates all the vertices, followed by a fleshing of the wireframe with low degree algebraic surface patches. See Figure 8. The technique is completely general and uses a single implicit surface patch of degree at most 7, for each triangular face of $\mathcal{T}$ of $\mathcal{P}$, i.e. no local splitting of triangular faces. Furthermore, the C^1 interpolation scheme is local in that each triangular surface patch has independent degrees of freedom which may be used to provide local shape control. Extra free parameters may be adjusted and the shape of the patch controlled by using weighted least squares approximation from additional points and normals, generated locally for each triangular patch. Similar techniques exist for parametrics [21, 27, 29, 40, 45] however the geometric degree of the solution surfaces tend to be prohibitively high.

One should note that the above surface patches which C^1-interpolate the vertices of the spatial triangulation may be singular at the end vertices. A well known necessary condition [20] for the regularity of surfaces which C^1 interpolate a point is given by:

Theorem 5.1 *Let $C_1(u)$ and $C_2(v)$ be two parametric curves with parametric normal directions $N_1(u)$ and $N_2(v)$ such that $C_1(0) = C_2(0) = p$, and that $N_1(0)$ and $N_2(0)$ are proportional. Then, any surface S, which interpolates the curves with tangent plane continuity, is singular at p unless $\frac{(N_1'(0),C_2'(0))}{\|N_1(0)\|} = \frac{(C_1'(0),N_2'(0))}{\|N_2(0)\|}$.*

The above theorem implies that enforcing two curves to have the same normal vectors at intersection points, does not guarantee the regularity of an interpolating surface at those points. The equation in the theorem is a necessary condition for regularity, indicating that, if the bounding curves of the surface patches, and their normals do not satisfy the equation, any smoothly interpolating surface must be singular at p. In cases where there is a strict requirement for completely nonsingular C^1-interpolant surface patches, the above equation, also known as the *vertex enclosure constraint*, must be met by the curvilinear wireframes which replace the underlying given triangulation. This issue has been addressed in the literature of parametric surface fitting. Peters [40] showed that not every mesh of parametric curves with well-defined tangent planes at the mesh points can be interpolated by non-singular, regularly parametrized surfaces with one surface patch per mesh face.

One way to satisfy the vertex enclosure constraint is to generate curvlinear wireframes (meshes) which are C^2 at the vertex endpoints. For arbitrary triangulations such C^2 curvlinear meshes can be quickly generated using only cubic polynomial parametric curves [26]. Hence there is no increase in the surface degree of the C^1 mesh of interpolating patches and at most degree 7 surfaces still suffice for the implicit representation. For cases where the singularity at vertex end points of smooth patches is permissible, the vertex enclosure constraint is automatically resolved [9]. In [41], Peters also used singularly parametrized surfaces to enclose a curvilinear mesh where the mesh curves emanating from a point do not satisfy the vertex enclosure constraint.
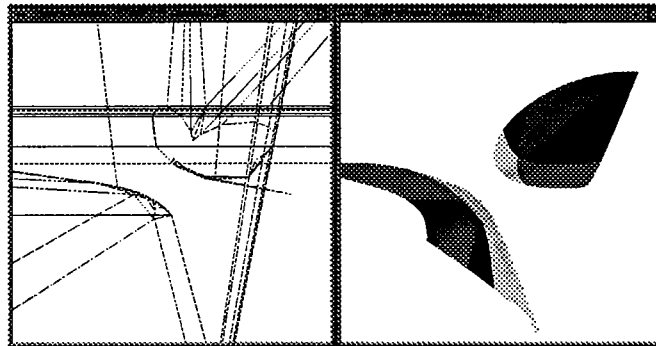
Figure 9: A Quadratic Parametric Surface with Domain Poles

6 Computer Graphics & Mesh Generation

We consider the problem of computing piecewise linear or polygonal approximations of real algebraic surfaces. Modern day computer graphics hardware accept such polygonal approximations and accurately render the complicated surfaces with sophisticated lighting and shading models. Similar, more structured, linear approximations of surfaces are required for finite element approaches to solving system of partial differential equations.

A well-known strength of the parametric representation (its mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$) is the ease by which real points can be generated on the parametric curve or surface. To compute real points on implicit algebraic surfaces requires the solution of polynomial equations. Furthermore, the problem of constructing a polygonal approximation, especially for finite element meshes, is complicated by the need for a correct topology of the mesh even in the presence of singularities and multiple sheets of the real algebraic surface. Direct schemes which work for arbitrary implicit algebraic surfaces are based on either the regular subdivision of the cube [19] or a finite subdivision of an enclosing tetrahedron [31]. However, such sampling methods fail in the presence of point and curve singularities of the algebraic surface, or yield ambiguous topologies in neighborhoods where multiple sheets of the surface come close together. Symbolic methods are necessary to disambiguate or calculate the correct topology for general algebraic curves and surfaces[3, 46]

While the issue of surface singularities are not as critical for rational parametric surfaces, the problems of constructing polygonal approximations with consistent topology is still highly non-trivial. Rational parametric surfaces have *pole curves* in their domain, where the denominators of the parameter functions vanish, domain *base points* for which all four numerator and denominator polynomials vanish simultaneously, and other features that cause naïve polygonal approximation algorithms to fail. These are ubiquitous problems occurring even among the natural quadrics. We illustrate the problems in more detail.

1. [Finite Parameter Range] To fully cover the parametric curve or surface, one must allow the parameters to somehow range over the entire parametric domain, which is infinite. For example, the unit sphere $f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$ has the standard rational para-
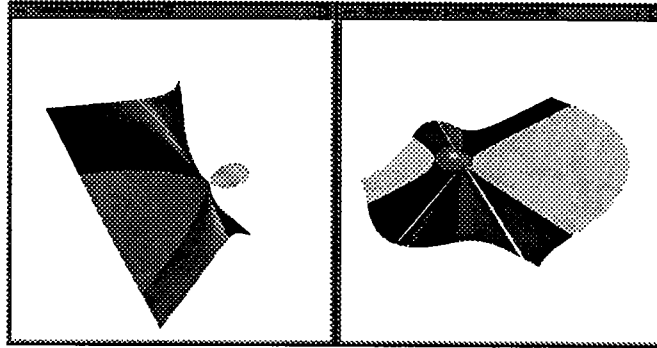
Figure 10: A Cubic Parametric Surface with Seam Curves Due to Base Points

metric representation $(x = \frac{2*s}{1+s^2+t^2}, y = \frac{2*t}{1+s^2+t^2}, z = \frac{1-s^2-t^2}{1+s^2+t^2})$ In this parameterization the point $(-1, 0, 0)$ origin can only be reached by the parameter values $s = t = \infty$.

2. [Complex Parameter Range] It is possible for real points of a curve or surface to be generated only by complex parameter values. For instance, the rational algebraic curve $f(x, y) = x^3 + x^2 + y^2 = 0$ has an isolated real point at the origin. A rational parametric representation of this curve is $(x(s), y(s)) = (-(s^2 + 1), -s(s^2 + 1))$. In this parameterization the origin can only be reached by the complex parameter value $s = \sqrt{-1} = i$.

3. [Poles] Even when restricting the surface to a bounded real part of the parametric domain, the rational functions describing the surface may have poles over that domain. A hyperboloid of two sheets, with implicit equation $z^2 + y*z + x*z - y^2 - x*y - x^2 - 1 = 0$, has the parametric representation

$$x(s, t) = \frac{4s}{5t^2 + 6st + 5s^2 - 1}$$

$$y(s, t) = \frac{4t}{5t^2 + 6st + 5s^2 - 1}$$

$$z(s, t) = \frac{5t^2 + 6st - 2t + 5s^2 - 2s + 1}{5t^2 + 6st + 5s^2 - 1}$$

then problems arise because of the pole curve described by $5t^2 + 6st + 5s^2 - 1$ in the parameter domain. See Figure 9.

4. [Base Points] The rational parameter functions describing curves and surfaces are generally assumed to be reduced to lowest common denominators, i.e., the numerator and denominator of each rational function are relatively prime. Thus for a curve, there is no parameter value that can cause both numerator and denominator of a rational parameter function to vanish. For surfaces, the situation is different. A surface is defined by three bivariate rational functions

$$x(s, t) = \frac{F_1(s, t)}{F_4(s, t)}$$

$$y(s,t) \quad = \quad \frac{F_2(s,t)}{F_4(s,t)}$$

$$z(s,t) \quad = \quad \frac{F_3(s,t)}{F_4(s,t)}$$

Even if F_1, F_2, F_3, F_4 are relatively prime polynomials, it is still possible that there are a finite number of points (a,b) such that $F_1(a,b) = F_2(a,b) = F_3(a,b) = F_4(a,b) = 0$. Each such point is called a *base point* of the parametric surface. There may also be base points at infinity in the parameter domain, and the base points can be complex as well as real-valued. Information about base points can be found in books on algebraic geometry such as [49, 52]. Base points are problematic since there is no one surface point for the corresponding domain point. To each base point there actually corresponds a curve on the surface [49], and since there is no parameter value for surface points on such a curve, the entire curve will be missing from the parametric surface. Such a curve is called a *seam curve*. See the right side of Figure 10 which corresponds the cubic parametric surface $x = \frac{t^3-t+s^3-s^2+1}{t^3+s^3+1}, y = \frac{2t^3-t^2-s^2t+2s^3+2}{t^3+s^3+1}, z = \frac{-st-s^3}{t^3+s^3+1}$. Thus for a truly accurate display of a parametric surface, one should also display the seam curves, alongside the parametric surface. See the left side of Figure 10 whee the seam curves are bridged.

In [13] we give solutions to the above problems for the C^0 meshing of rational parametric curves, surfaces and hypersurfaces of any dimension. The technique is based on homogeneous linear (projective) reparameterizations and yields a complete and accurate C^0 planar mesh of free-form, discontinuous rational parametric domains. For the Cartan surface $(x = s, y = \frac{s^2}{t^2}, z = t)$, a single reparameterization $(x = st, y = s^2, z = t)$ removes the pole $t = 0$ of the original parameterization. For the Steiner surface $(x = \frac{2st}{1+s^2+t^2}, y = \frac{2s}{1+s^2+t^2}, z = \frac{2t}{1+s^2+t^2})$, and the cubic elbow surface $(x = \frac{4t^2+(s^2+6s+4)t-4s-8}{2t^2-4t+s^2+4s+8} y = \frac{4t^2+(-s^2-6s-20)t+2s^2+8s+16}{2t^2-4t+s^2+4s+8} z = \frac{(2s+6)t^2+(-4s-12)t-s^2-4s}{2t^2-4t+s^2+4s+8})$, four different projective reparameterizations yield a complete covering of the rational parametric surface. See figures 11 and 12. In [13] for surfaces, four reparameterizations always suffice. In general 2^d projective reparameterizations suffice for a d dimensional parametric hypersurface, see [12]. The algorithms which computes these reparameterizations as well as generates the C^0 planar meshes have been implemented in C in our GANITH toolkit[11]. The pictures have been generated using this toolkit.

7   The SHASTRA Geometric Toolkits

We have built a geometric software environment called SHASTRA[1], allowing manipulations of algebraic curves and surfaces, and using both the implicit and parametric representations[7]. Our selection of geometric algorithms and data structures is based on the paradigm of abstract data types, i.e. the representation implicit or parametric chosen is the one which optimizes the operation or application on hand. The algorithms of the previous sections have all been implemented in this environment. SHASTRA is a good supplement to geometric algorithm research as it provides us with a testbed for new approaches to solving complicated geometric

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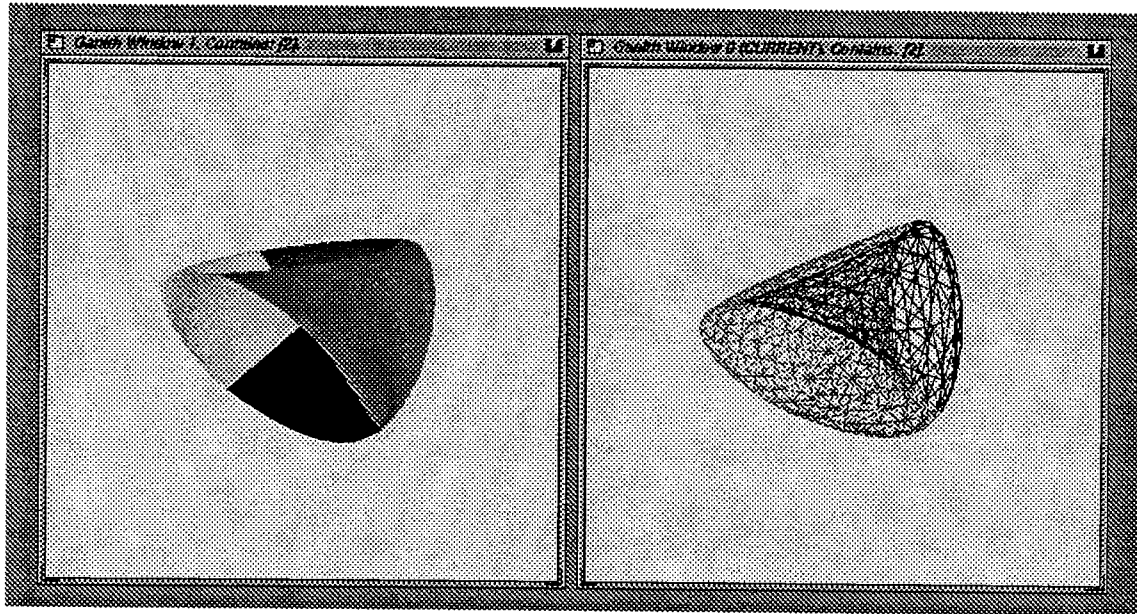[1]SHASTRA is the Sanskrit word for Science

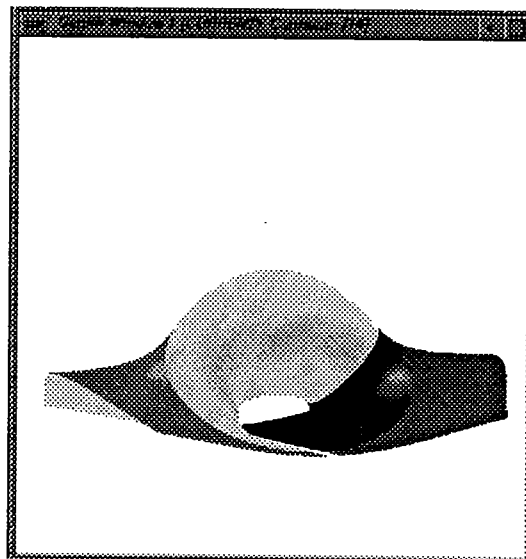Figure 11: Complete Display of the Steiner Parametric Surface

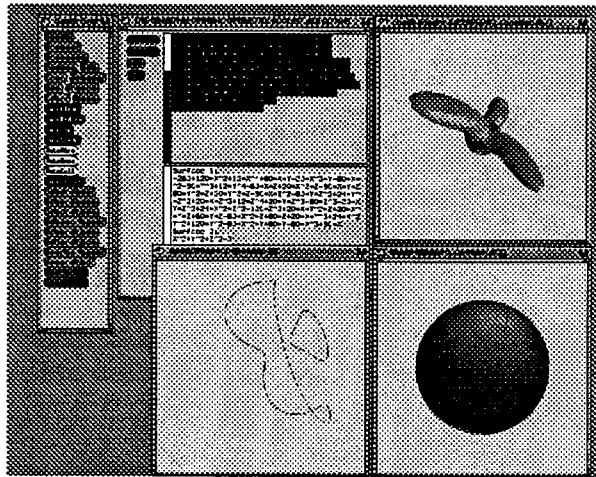Figure 12: Complete Display of a Cubic Parametric Surface

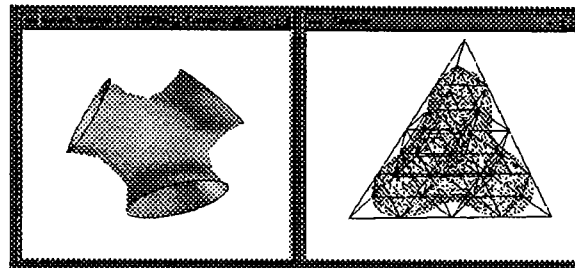Figure 13: Surface - Surface Intersections in Ganith

Figure 14: Implicit Surface Patches in Bezier Basis

problems. In this section we give a quick overview of SHASTRA and its individual geometric toolkits.

SHASTRA is a highly extensible, distributed and collaborative geometric software environment consisting of a growing set of individually powerful and interoperable (client-server) toolkits which support collaborative design sessions. In the SHASTRA environment multiple users (say, a collaborative engineering design team) interactively create, share, manipulate, simulate and visualize complex geometric designs over a heterogeneous network of workstations and supercomputers.

1. The GANITH algebraic surface modeling toolkit[11] provides symbolic and numeric computations on algebraic varieties. Example applications of this are curve and surface piecewise linear approximations for display, curve-curve intersections, surface-surface intersections, global and local parameterizations, implicitizations, inversions, curve and surface interpolation, approximation, etc. See figures 13 and 14.
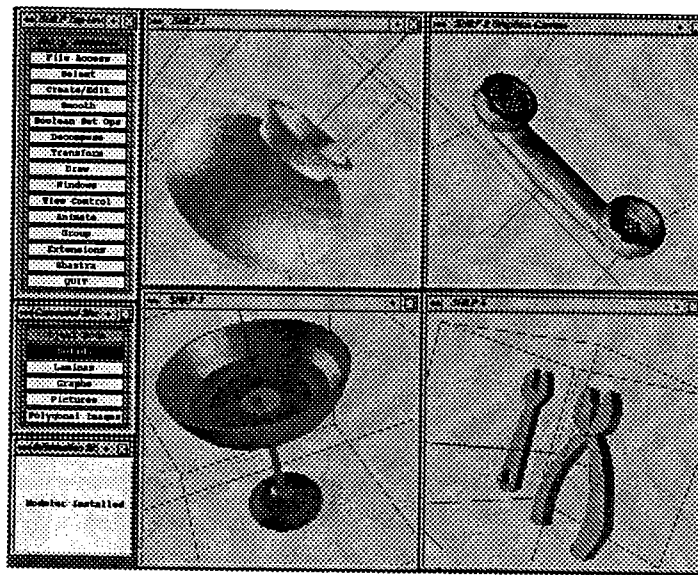
Figure 15: Interactively Designed Models with Piecewise Algebraic Surfaces in SHILP

2. The SHILP solid modeling and display toolkit[6] manipulates curved solid objects with piecewise algebraic surfaces. It can be used for the interactive design (creation, editing, etc.) and display of solid models with algebraic surface boundaries. See figure 15.

3. The VAIDAK medical imaging and model reconstruction toolkit[8] manipulates medical image volume data. It can be used to construct accurate surface and solid models of skeletal and soft tissue structures from CT (Computed tomography), MRI (Magnetic Resonance Imaging) or LSI(Laser Surface Imaging) data. See figure 16.

4. The BHAUTIK physical analysis toolkit[10] provides a graphical interface and functionality to set up and perform scientific and engineering simulations on geometric models. Its capabilities include finite element mesh generation, a graphical editor for setting up a physical problem domain and boundary conditions, a database of material properties and a growing database of physical (PDE) models for heat transfer and structural (stress/strain). See figure 17.

GANITH provides the surface modeling infrastructure for SHILP and VAIDAK. Further, SHILP provides all the solid model manipulation and display functionality to skeletal structures reconstructed from CT/MRI image data in VAIDAK. GANITH, SHILP and VAIDAK provide BHAUTIK with a varied source of geometric domains. Collectively these toolkits provide a vast infrastructure of numeric and symbolic algorithms manipulating complex geometric objects.

In the SHASTRA environment the above toolkits run as independent processes on separate workstations having separate user interfaces (using X-11 and Motif). The application toolkits make use of a custom designed network library to communicate data structures conveniently
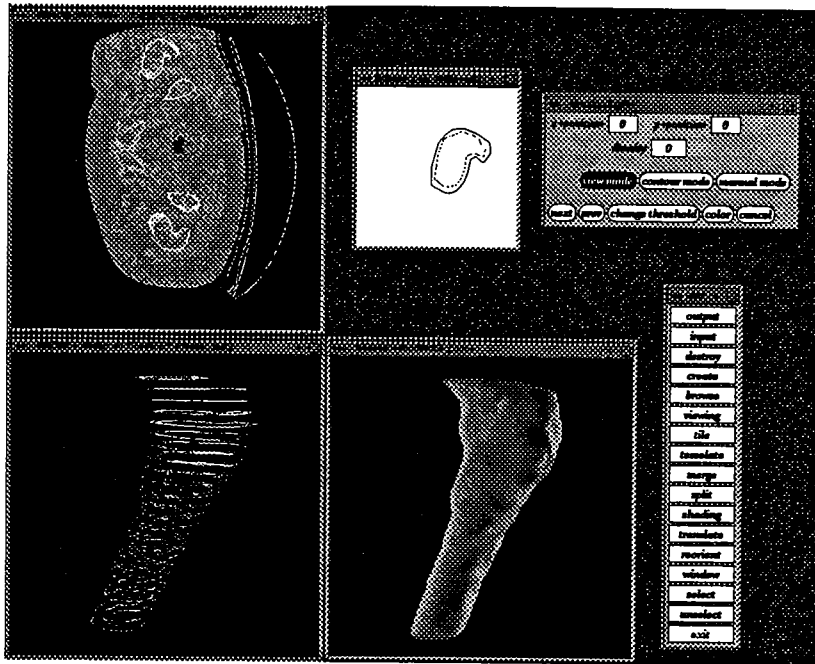
Figure 16: Femur Model Reconstruction from CT data in VAIDAK

with each other and manage multiple connections across a network. The network library is designed around the highly extensible **client-server** paradigm and utilizes TCP/IP. Each application runs as a server for the functionality it offers, and as a client capable of requesting functionality from other sibling systems. Applications maintain multiple concurrent connections to other applications on multiple hosts. This is effected using the multiplexing facility accorded by the "select" system call. The application dynamically opens connections to different systems and registers handlers with the multiplexing layer or closes such connections. In server mode, the application sets up shop at a well known port, and awaits requests for connections from other systems. In client mode, the application attempts to connect to a well known port for a specific service. Both of these actions can be performed identically in all systems. The data communication aspect is also standardizable. For example, in the SHASTRA environment the applications need to be able to exchange parts of three dimensional solid models (curve segments, surface patches, solids, etc.) to perform various operations. This motivated the development of a modular data communication library where data objects can be exchanged between systems.

One current industrial application of the SHASTRA environment is the interactive design and physical prototyping of artificial implants. Figure 17 shows a model of a 3D reconstructed model of a human femur and an artificial hip implant built using the interactive tools in SHASTRA. The implant design task is accomplished via SHILP in conjunction with remote calls to GANITH for surface fitting operations. VAIDAK is used to reconstruct accurate models of
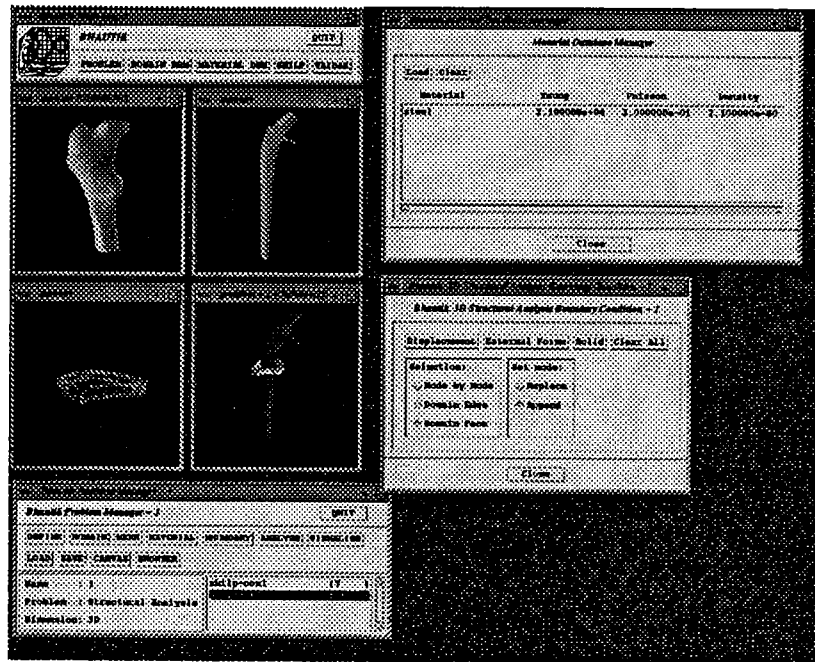
Figure 17: Finite Element Stress Transfer Modeling from Prosthesis to Femur in BHAUTIK

the appropriate part of the human anatomy, and BHAUTIK is used to conduct stress-strain analyses of the bone model for varying loads on the designed implant. The collaborative layer in SHASTRA allows multi-user creation, manipulation and visualization of geometric models.

8 Conclusion

We have presented several tradeoffs between implicit and parametric representation for a range of problems stemming from geometric modelling, scattered data fitting, computer graphic and mesh generation. The parametric representations possesses good properties which include: easier to order, easily generate points on, simpler patches, compact storage, irreducibility, etc. The implicit representations on the other hand are easier for halfspace queries, easier for representing complete surfaces, and generally yields lower degree surface interpolatory solutions. Hence in geometric design on needs to follow the abstract data type paradigm as discussed in this paper and implemented in the software system SHASTRA. The geometric data structures for curves and surfaces in the GANITH toolkit of SHASTRA unifies the implicit and the parametric forms by representing and manipulating them as algebraic varieties. In SHASTRA the user thus has complete freedom in select the more appropriate representation (implicit, parametric or both) for the desired operation on hand.

References

[1] Abhyankar, S., and Bajaj, C. Automatic Rational Parameterization of Curves and Surfaces I: Conics and Conicoids. *Computer Aided Design*, 19(1):11–14, 1987.

[2] Abhyankar, S., and Bajaj, C. Automatic Rational Parameterization of Curves and Surfaces II: Cubics and Cubicoids. *Computer Aided Design*, 19(9):499–502, 1987.

[3] Abhyankar, S., and Bajaj, C. Automatic Rational Parameterization of Curves and Surfaces III: Algebraic Plane Curves. *Computer Aided Geometric Design*, 5(1):309–321, 1988.

[4] Abhyankar, S., and Bajaj, C. Automatic Rational Parameterization of Curves and Surfaces IV: Algebraic Space Curves. *ACM Transactions on Graphics*, 8(4):324 – 333, 1989.

[5] Alfeld, P.,. Scattered Data Interpolation in Three or More Variables. In T. Lyche and L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design*, pages 1–34. Academic Press, 1989.

[6] Anupam, A., Bajaj, C., and Royappa, A.,. *The SHASTRA Distributed and Collaborative Geometric Design Environment*. Computer Science Technical Report, CAPO-91-38, Purdue University, 1991.

[7] Anupam, V., Bajaj, C., Dey, T., and Ihm, I.,. *The SHILP Solid Modeling and Display Toolkit*. Computer Science Technical Report, CAPO-91-29, Purdue University, 1991.

[8] Bailey, B., Bajaj, C., and Fields, M.,. *The VAIDAK Medical Imaging and Model Reconstruction Toolkit*. Computer science technical report, capo-91-31, Purdue University, 1991.

[9] Bajaj, C. Geometric modeling with algebraic surfaces. In D. Handscomb, editor, *The Mathematics of Surfaces III*, pages 3–48. Oxford Univ. Press, 1988.

[10] Bajaj, C., and Ihm, I. Algebraic surface design with Hermite interpolation. *ACM Transactions on Graphics*, 19(1):61–91, January 1992.

[11] Bajaj, C., and Ihm, I.,. C^1 Smoothing of Polyhedra with Implicit Algebraic Splines. *Computer Graphics*, 26:79–88, 1992.

[12] Bajaj, C., and Royappa, A.,. *The GANITH Algebraic Geometry Toolkit*. In Proceedings of the First International Symposium on the Design and Implementation of Symbolic Computation Systems, pages 268–269. Springer-Verlag, *Lecture Notes in Computer Science No. 429*, 1990.

[13] Bajaj, C., and Royappa, A.,. *Finite Representation of Parametric Curves and Surfaces*. Computer science technical report, Purdue University, 1992.

[14] Bajaj, C., and Royappa, A.,. The Robust Display of Arbitrary Rational Parametric Surfaces. In *Curves and Surfaces in Computer Vision and Graphics III*, pages x–y, Boston, MA, 1992.

[15] Bajaj, C., and Xu, G. *Piecewise Rational Approximation of Real Algebraic Curves.* Computer Science Technical Report, CAPO-91-19, Purdue University, 1991.

[16] Bajaj, C., and Xu, G. *NURBS Approximation of Surface-Surface Intersection Curves.* Computer Science Technical Report, CAPO-92-17, Purdue University, 1992.

[17] Bajaj, C., Okamura, K., and Schikore, D.,. *The BHAUTIK Physical Analysis Toolkit.* Computer Science Technical Report, Purdue University, 1992.

[18] Beeker, E. Smoothing of Shapes Designed with Free Form Surfaces. *Computer Aided Design*, 18(4):224–232, 1986.

[19] J. Bloomenthal. Polygonization of implicit surfaces. *Computer Aided Geometric Design*, 5:341–355, 1988.

[20] Chiyokura, H., and Kimura, F. Design of Solids with Free-form Surfaces. Computer Graphics, 17(3):289–298, 1983.

[21] Chui, C. *Multivariate Splines.* Regional Conference Series in Applied Mathematics, 1988.

[22] Dahmen, W. Smooth piecewise quadratic surfaces. In T. Lyche and L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design*, pages 181–193. Academic Press, Boston, 1989.

[23] Dahmen, W. and Michelli, C. Recent Progress in Multivariate Splines. In L. Schumaker C. Chui and J. Word, editors, *Approximation Theory IV*, pages 27–121. Academic Press, 1983.

[24] C. DeBoor. *A Pracitcal Guide to Splines.* Springer-Verlag, New York, 1978.

[25] DeBoor, C., and Hollig, K., and Sabin, M. High Accuracy Geometric Hermite Interpolation. *Computer Aided Geometric Design*, 4(00):269–278, 1987.

[26] Do Carmo, M.,. *Differential Geometry of Curves and Surfaces.* Englewood Cliffs, New Jersey, 1976.

[27] Farin, G. Triangular Bernstein-Bézier patches. *Computer Aided Geometric Design*, 3:83–127, 1986.

[28] Garrity, T., and Warren, J. Geometric continuity. *Computer Aided Geometric Design*, 8:51–65, 1991.

[29] Gregory, J., and Charrot, P. A C^1 Triangular Interpolation Patch for Computer Aided Geometric Design. *Computer Graphics and Image Processing*, 13:80–87, 1980.

[30] Hagen, H., and Pottmann, H. Curvature Continuous Triangular Interpolants. *Mathematical Methods in Computer Aided Geometric Design*, pages 373–384, 1989.

[31] M. Hall and J. Warren. Adaptive polygonalization of implicitly defined surfaces. *IEEE Computer Graphics and Applications*, pages 33–42, November 1990.

[32] Herron, G. Smooth Closed Surfaces with Discrete Triangular Interpolants. *Computer Aided Geometric Design*, 2(4):297–306, 1985.

[33] Hollig, K. Multivariate Splines. *SIAM J. on Numerical Analysis*, 19:1013–1031, 1982.

[34] Liu, D., and Hoschek, J. GC^1 Continuity Conditions Between Adjacent Rectangular and Triangular Bezier Surface Patches. *Computer Aided Design*, 21:194–200, 1989.

[35] Lorensen, W., and Cline, H. Marching Cubes: A High Resolution 3D Surface Construction Algorithm. *Computer Graphics*, 21:163–169, 1987.

[36] Moore, D., and Warren, J. Approximation of dense scattered data using algebraic surfaces. In *Proc. of the 24th Hawaii Intl. Conference on System Sciences*, pages 681–690, Kauai, Hawaii, 1991.

[37] Nielson, G. A Transfinite Visually Continuous Triangular Interpolant. In G. Farin, editor, *Geometric Modeling Applications and New Trends*. SIAM, 1986.

[38] Patrikalakis, N., and Kriezis, G. Representation of piecewise continuous algebraic surfaces in terms of B-splines. *The Visual Computer*, 5(6):360–374, Dec. 1989.

[39] Peters, J. Local Cubic and BiCubic C^1 Surface Interpolation with Linearly Varying Boundary Normal. *Computer Aided Geometric Design*, 7:499–516, 1990.

[40] Peters, J. Smooth interpolation of a mesh of curves. *Constructive Approximation*, 7:221–246, 1991.

[41] Peters, J. Parametrizing singularly to enclose data points by a smooth parametric surface. In *Proc. of Graphics Interface*, Calgary, Alberta, June 1991. Graphics Interface '91.

[42] Piper, B. Visually Smooth Interpolation with Triangular Bezier Patches. In G. Farin, editor, *Geometric Modeling: Algorithms and New Trends*. SIAM, 1987.

[43] Powell, M., and Sabin, M. Piecewise Quadratic Approximations on Triangles. *ACM Trans. on Math. Software*, 3:316–325, 1977.

[44] Ramshaw, L. Beziers and B-splines as Multiaffine Maps. In *Theoretical Foundations of Computer Graphics and CAD*. Springer Verlag, 1988.

[45] Sarraga, R. G^1 interpolation of generally unrestricted cubic Bézier curves. *Computer Aided Geometric Design*, 4:23–39, 1987.

[46] Sederberg, T.,. Piecewise Algebraic Surface Patches. *Computer Aided Geometric Design*, 2:53–59, 1985.

[47] Sederberg, T., and J. Snively, J.,. Parameterization of Cubic Algebraic Surfaces. In Oxford University Press R. Martin, editor, *The Mathematics of Surfaces II*, 1987.

[48] Seidel, H-P. A New Multiaffine Approach to B-splines. *Computer Aided Geometric Design*, 6:23–32, 1989.

[49] Semple, J., and Roth, L. *Introduction to Algebraic Geometry.* Oxford University Press, Oxford, U.K., 1949.

[50] Shirman, L., and Sequin, C. Local Surface Interpolation with Bezier Patches. *Computer Aided Geometric Design*, 4:279–295, 1987.

[51] Walker, R.,. *Algebraic Curves.* Springer Verlag, 1950.

[52] Zariski, O.,. *Algebraic Surfaces.* Ergebnisse der Mathematik und ihre Grenzgebiete 4, 1935.

[53] Zariski, O., and Samuel, P.,. *Commutative Algebra (Vol. I, II).* Springer Verlag, 1958.