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Spline Approximations of Real Algebraic Surfaces

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We use a combination of both symbolic and numerical techniques to construct several degree bounded G^0 and G^1 continuous, piecewise spline approximations of real implicit algebraic surfaces for both computer graphics and geometric modeling. These approximations are based upon an adaptive triangulation (a G^0 planar approximation) of the real components of the algebraic surface, and include both singular points and singular curves on the surface. A curvilinear wireframe is also constructed using minimum bending energy, parametric curves with additionally normals varying along them. The spline approximations over the triangulation or curvilinear wireframe could be one of several forms: either low degree, implicit algebraic splines (triangular A-patches) or multivariate functional B-splines (B-patches) or standardized rational Bernstein–Bézier patches (RBB), or triangular rational B-Splines. The adaptive triangulation is additionally useful for a rapid display and animation of the implicit surface.

1. Introduction

Real algebraic surfaces are often used to cope with the problem of modeling complicated shapes Bajaj *et al.* (1995), Blinn (1982); Pratt (1987); Wyvill *et al.* (1986). Implicitly defined algebraic surfaces have both advantages, and disadvantages over functional and parametric surfaces (Bajaj, 1993). The class of implicit algebraic surfaces is closed under several geometric operations (intersections, union, offset, etc.), often desired in a solid modeling system. On the other hand, free-form geometric modeling (display and shape control) is much easier with parametric curve and surface spline representations (and evidenced by available software systems). This largely motivates the need for constructing parametric spline approximations of real algebraic surfaces.

In computer graphics, most rendering algorithms for implicit surfaces [besides ray tracing (Hanrahan, 1983; Kalra and Barr, 1989; Sederberg and Zundel, 1989)] rely on piecewise linear approximations (triangles, polygons) based on space subdivision or polyhedron continuation. Peterson (1984) and Bloomenthal (1988) use octrees based on regular spatial partitioning to achieve a polygonal approximation. Hall and Warren (1990) use a tetrahedral subdivision. Allgower and Gnutzmann (1987, 1991) use simplicial continuation or pivoting algorithms to generate a triangular or quadrilateral polygonal approximation.

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In this paper, we use neither space subdivision nor polyhedron continuation. Instead, we use a novel triangular expansion scheme on the real algebraic surface starting from a seed point and conforming to point and curve singularities on the surface within an arbitrary bounding box. We then construct a curvilinear wire frame (minimum bending energy parametric curves with normals for G^1 continuity) and finally fit low degree parametric or implicit surface patches to cover the wire frame smoothly to achieve a spline approximation. Compared with prior approaches the surface expansion scheme fully uses the differential properties of the surface, is second order adaptive and conforms to the surface singularities. This expansion based spline approximation scheme is also a generalization of our curve marching schemes for spline approximation of real algebraic curves with singularities (Bajaj and Xu, 1991, 1994b).

Once a topologically correct triangulation of the implicit surface is achieved, there exist interpolation and approximation spline fitting schemes which work, however under various restrictions on the triangulation. Mann *et al.* (1992) surveyed several parametric patch interpolation schemes for triangulated data. Nasri (1991) considered parametric surface interpolation on irregular curvilinear networks with prespecified normal conditions at vertices but without singularities. A similar problem is discussed in Peters (1990). Herron (1985a) used parametric surfaces interpolating function values and tangential derivatives at the vertices of a triangle. His method is generalized in Herron (1985b) to cover the G^1 continuity case for smooth (non-singular) closed surfaces. Smooth interpolation of a curvilinear wireframe with implicit algebraic patches is given by Bajaj and Ihm (1992). Kolb *et al.* (1995) fit parametric Bezier patches to minimum norm curve networks and Bajaj *et al.* (1995) fit iso-contours of trivariate cubic implicit Bernstein–Bézier surfaces (A-patches), to closed surface triangulations.

In this paper, we additionally present two other parametric curve and surface fitting solutions to the closed surface interpolation problem for curvilinear networks with singular vertices. One uses a functional patch by subdividing each triangular patch into three smaller patches. The other solution requires rational parametric patches to cover a constructed curvilinear space wire frame. Compared with the method given in Herron (1985b), our second approach leads to a lower rational degree of approximation and is singularity conforming. Using well known transformation techniques such as Bajaj and Xu (1994a), one can furthermore convert the smooth rational parametric patches to standardised rational Bernstein–Bézier (RBB) form or even triangular rational B-splines.

2. Sketch of Approximation Algorithm

1. **Singularity and Seed Point Computation**. In this step we compute real singular points and real singular curves of the real surface in a given bounding box, as well as a real seed point per real surface connected component within the box region (see Section 3 for details). For singular curves, we construct a piecewise linear approximation of each real component of the curve.
2. **Triangulation**. Next, we construct a piecewise linear (triangular) approximation of the surface with the topology dictated by the surface singularities. The approximation of the smooth part is based on a power series expansion and from which a surface triangulation is produced (see Section 4 for details). Expansion vertices and edges of the triangulation which approach singular points or linear approximations of the singular curves, are "stitched" to the singular points and edges by additional edges and by splitting existing triangles.

3. **Wire Construction**. In this step we construct a G^1 curvilinear wire frame by computing normals at each vertex of the triangulation and then building a parametric space curve with minimum bending energy, and a normal function along the curve such that the curve G^1 interpolates the edge vertices and the curve normal function has the given normal at the vertices and orthogonal to the tangent of the curve (see Section 5 for details). The normal at a smooth point of the original surface is provided by the gradient of the surface. At singular points (and curves), the surface normals are not uniquely defined, and hence the incident normals of the wire frame are deemed to conform automatically (i.e. the incident surface shall be just G^0 continuous at those points).
4. **Patch fitting**. For each triangular face consisting of three parametric curve wires together with normal functions, this step constructs a G^0 or a G^1 parametric surface patch that interpolates the wires and has the normal function along the boundary (see Section 6).

3. Computation of Singularities and Seed Points

The set of solutions (or *zero* set $Z(S)$) of a collection S of polynomial equations

$$\begin{aligned} S_1 : f_1(x_1, \ldots, x_n) &= 0 \\ \vdots \qquad &= \vdots \\ S_m : f_m(x_1, \ldots, x_n) &= 0 \end{aligned} \tag{3.1}$$

is referred to as an *algebraic set.* Algebraic curves and surfaces are algebraic sets of dimension 1 and 2 respectively. Problems dealing with zero sets $Z(S)$, such as the intersection of curves and surfaces, or the decision whether a surface contains a set of curves, are often first versed in an ideal-theoretic form and then solved using Gröbner basis manipulations. Bajaj (1990) presented an alternative technique based on constructing bi-rational mappings between algebraic varieties and hypersurfaces.

Given m independent equations in n variables (3.1), let S be the algebraic variety of dimension $n - m$ defined by these equations. Then the bi-rational map construction of Bajaj (1990) produces a new "triangulated" polynomial system of equations

$$\begin{aligned} \tilde{f}(x_1, \ldots, x_{n-m+1}) &= 0 \\ x_{n-m+2} &= \frac{h_{2m-4}(x_1, \ldots, x_{n-m+1})}{h_{2m-3}(x_1, \ldots, x_{n-m+1})} \\ \vdots \quad &= \quad \vdots \\ x_{n-1} &= \frac{h_2(x_1, \ldots, x_{n-2})}{h_3(x_1, \ldots, x_{n-2})} \\ x_n &= \frac{h_0(x_1, \ldots, x_{n-1})}{h_1(x_1, \ldots, x_{n-1})}\,. \end{aligned} \tag{3.2}$$

This bi-rational map construction is based on the multi-polynomial resultant (Macaulay, 1916) and multi-polynomial remainder sequences.

Cases of intersection computation of interest in this paper are the computation of singularities on algebraic surfaces. These are special cases of the above system of equations and bi-rational map construction. Algebraic surfaces can possess both point and curve singularities. Curve singularities reduce to the special case of for $n = 3$ and $m = 2$ and $x_1 = x$, $x_2 = y$, $x_3 = z$. In particular the system of equations to compute curve singularities on the surface is given by

$$\begin{aligned} f(x_1, x_2, x_3) &= 0 \\ \alpha_1 f_{x_1}(x_1, x_2, x_3) + \alpha_2 f_{x_2}(x_1, x_2, x_3) + \alpha_3 f_{x_3}(x_1, x_2, x_3) &= 0 \end{aligned}$$

for algebraically independent α_1, α_2 and α_3. In practice one computes the common intersection curves of $f = 0$ and $f_{x_1}(x_1, x_2, x_3) = 0$ which are also completely contained simultaneously by the surfaces $f_{x_2}(x_1, x_2, x_3) = 0$ and $f_{x_3}(x_1, x_2, x_3) = 0$. To test if a curve is completely contained by a surface one needs to test only if $(mn + 1)$ points of the curve lie on the surface, where m and n are the degrees of the curve and surface respectively. Points (x_1, x_2, x_3) on the singular space curve lying on the algebraic surface are then obtained from the special case "triangulated" system (3.2) by first computing points on the plane curve $\tilde{f}(x_1, x_2) = 0$ and then substituting these into $x_3 = \frac{h_0(x_1,x_2)}{h_1(x_1,x_2)}$. $f_{x_3} = 0$.

Point singularities on the algebraic surface reduce to the special case of for $n = 3$ and $m = 3$ and $x_1 = x$, $x_2 = y$, $x_3 = z$, and given by

$$\begin{aligned} f(x_1, x_2, x_3) &= 0 \\ \alpha_1 f_{x_1}(x_1, x_2, x_3) + \alpha_2 f_{x_2}(x_1, x_2, x_3) + \alpha_3 f_{x_3}(x_1, x_2, x_3) &= 0 \\ \beta_1 f_{x_1}(x_1, x_2, x_3) + \beta_2 f_{x_2}(x_1, x_2, x_3) + \beta_3 f_{x_3}(x_1, x_2, x_3) &= 0 \end{aligned}$$

for algebraically independent α_i and β_i. Again, in practice one computes the common intersection points of $f = 0$, $f_{x_1} = 0$ and $f_{x_2} = 0$ and keeps only those points which also satisfy $f_{x_3} = 0$. Singular points (x_1, x_2, x_3) on the algebraic surface are then obtained from the special case "triangulated" system (3.2) by first computing zeros of the univariate polynomial $\tilde{f}(x_1) = 0$ and then substituting these into $x_2 = \frac{h_0(x_1)}{h_1(x_1)}$ and $x_3 = \frac{h_2(x_1)}{h_3(x_1)}$.

Seed points covering each real component of the surface are computed from the intersections of the surface with a bounding box as well as x_1 extreme points (for bounded components lying completely within the bounding box). Details of the polynomial system solution are similar to the computation of point singularities and given in Canny (1988). Bounds on the numerical approximation (number of bits of precision) required for the seed points and singular points on the surface are such that the G^0 triangulations can be correctly determined, are similar to ones given for curves in Bajaj and Xu (1991, 1994b) and are based on the gap theorem of Canny (1988).

In this paper, we assume the singular curve of the surface is smooth, that is the curve can be redefined by the intersection of two smooth surfaces and the two surfaces are not tangent at their intersection. A piecewise linear approximation is generated with specified density for each singular curve component (see Bajaj and Xu, 1994b, for getting the point list). The line segments of the piecewise linear curve approximation will become the edges of the surface triangulation to be described in the next section.

4. Adaptive Triangulation

4.1. Edge expansion approach

We begin with a few notational definitions.

Expansible edge. During the process of expansion of the triangular polygon, an edge is called expansible if we can go further outward from this edge to get a new triangle. That is

(a) this edge is on the boundary of the present constructed polygon,
(b) this edge is inside the given boundary box.

P-plane, P-expression, Expansion point.
Let $p_0 = (x_0, y_0, z_0)$ be a point on the surface $f(x,y,z) = 0$. Then the orthogonal transform

$$T: \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} c_2 & s_2 & 0 \\ -c_1 s_2 & c_1 c_2 & s_1 \\ s_1 s_2 & -s_1 c_2 & c_1 \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix}$$

with

$$\begin{aligned} c_1 &= f_z(p_0)/\|\nabla f(p_0)\|, & c_2 &= -f_y(p_0)/\sqrt{f_x(p_0)^2 + f_y(p_0)^2}, \\ s_1 &= \sqrt{f_x(p_0)^2 + f_y(p_0)^2}/\|\nabla f(p_0)\|, & s_2 &= f_x(p_0)/\sqrt{f_x(p_0)^2 + f_y(p_0)^2} \end{aligned}$$

establishes a one-to-one map between (x,y,z) space and (X,Y,Z) space. It is easy to see that the XY-plane is the tangent plane of the surface $f(x,y,z) = 0$ at the point p_0. We call this plane as the projection plane of $f = 0$ at p_0, and denote it the P-plane. The projection of any point p onto the P-plane is denoted by $P(p)$, and consists of the first two components of $T(p)$. On the P-plane, $f(x,y,z) = 0$ can be expressed locally as a power series $Z = \phi_{p_0}(X,Y)$. We call its truncation up to degree k a P-expression, denoted by $\phi_{p_0,k}(X,Y)$. The point p_0 is referred to as an expansion point.

(p,k,ϵ)-circle, (p,k,ϵ)-sphere, (p,k,ϵ)-radius
If the maximal $r = r(p,k,\epsilon)$ for which

$$\|\phi_{p,k}(X,Y) - \phi_p(X,Y)\| < \epsilon \qquad \text{for } X^2 + Y^2 \le r^2$$

then we say that the circle $X^2 + Y^2 = r^2$ is a (p,k,ϵ)-circle and $X^2 + Y^2 + Z^2 = r^2$ is a (p,k,ϵ)-sphere of f at p and r is the (p,k,ϵ)-radius.
It is easy to see that $r(p,k,\epsilon)$ converges to the convergence radius of $\phi_p(X,Y)$ at p as $k \to \infty$.

Algorithm 1

Let S be the collection of singular points and point lists of the singular curves of the given real algebraic surface.

1. Initial Step. For a given smooth seed point p_0 on one component of the surface $f(x,y,z) = 0$ and in the given bounding box, we first compute the P-expression $Z = \phi_{p_0,k}(X,Y)$. On the P-plane, compute the (p_0,k,ϵ)-radius. Take three uniformly distributed points on the (p_0,k,ϵ)-circle, say q_0, q_1, q_2, and refine the points $(q_i, \phi_{p_0,k}(q_i))$ by Newton's method such that the resulting points V_i are on the surface. If Newton's method fails to converge, reduce the radius of the circle and repeat. If the point V_i is outside the bounding box, then adjust it to the boundary curve (intersection of the surface and the box). The triangle $[V_0, V_1, V_2]$ is the first in the triangulation. Next compute the outside angle at each vertex of the triangle. In this initial case, each edge is expansible except the one that is on the boundary curve.

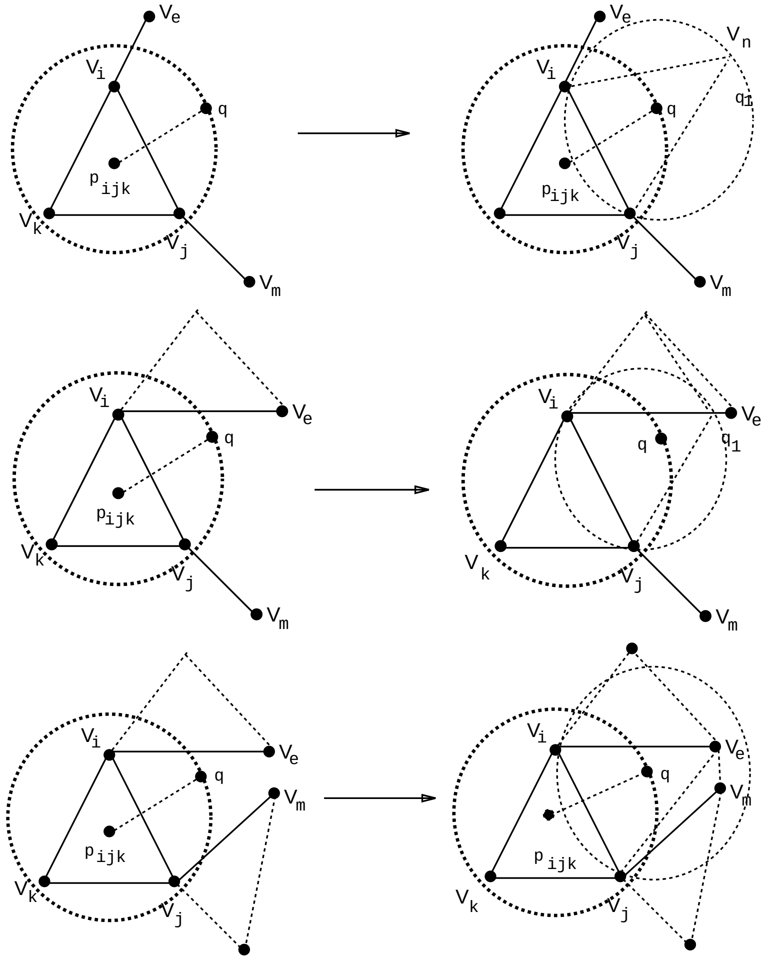


Figure 1. Expansion steps for an adaptive triangulation on real algebraic surfaces.

2. General Step. Suppose we have constructed several space triangles that form one or more than one connected mesh of triangles. For each mesh, we keep boundary information such as edges with related expansion points, vertices with angles, etc. Assume now that at least one of the edges is expansible. Then the general step is to construct one more triangle that joins the original one and enlarges the mesh. Refer also to Figure 1.

   (a) Find a vertex on the present boundary such that the angle at this point is minimal and the related two boundary edges are expansible. Start from one of the two expandible edges that has longer length, say $[V_1, V_2]$ which is also the edge of triangle $[V_0, V_1, V_2]$ with expansion point p_0 and P-expression $Z = \phi_{p_0,k}(X, Y)$. Choose one point q on the P-plane away from the present triangle and within the (p_0, k, ϵ)-circle such that q is on the middle-perpendicular line of $[P(V_1), P(V_2)]$ and as far as possible from $P(p_0)$.

(b) Refine the point $(q, \phi_{p_0,k}(q))$ by Newton's method to get a new expansion point p_1. As before, if Newton's method fails to converge, a nearer point q to the circle center is used.
(c) Compute the new P-expression $Z = \phi_{p_1,k}(X, Y)$ and new (p_1, k, ϵ)-circle $X^2 + Y^2 = r_1^2$.
(d) In the new P-plane at p_1, choose a point q_1 on the intersection of the middle-perpendicular line of $[P(V_1), P(V_2)]$ and the new (p_1, k, ϵ)-circle, then form a new triangle according to the following cases:

- If the line segment $[(P(V_1) + P(V_2))/2, q_1]$ intersects a previous edge's projection on the P-plane and the point $T^{-1}(q_1, \phi_{p_1}(q_1))$ lies on a previous surface patch, we take the intersection point to be q_1 and a new triangle $[V_1, V_2, T^{-1}(q_1, \phi_{p_1}(q_1))]$ is formed. Alternatively, if the vertices are near(within ϵ) to a singular point in S, then a new edge is added by connecting the vertices to the nearest singular point. In practice, this ϵ could be chosen interactively.
- Let $[V_2, V_4]$ be the other edge connecting to the present vertex. If the angle $\angle P(V_4)P(V_2)P(V_1) \leq \frac{\pi}{2}$, the new triangle is formed by the three points V_4, V_2, V_1.Otherwise, for the case when the angle $\angle P(V_4)P(V_2)P(V_1) > \frac{\pi}{2}$, the new triangle is $(V_1, V_2, P^{-1}(q_1, \phi_{p_1}(q_1))$.

3. Final Step. Repeat the general step iteratively, until every edge is non-expansible. This finishes the generation of the triangle approximation for one real component of the implicit surface.

4.2. VERTEX EXPANSION APPROACH

In this subsection, we describe an alternate (but related) scheme for constructing the triangular approximation using vertex expansions. We find this approach is superior to the edge expansion approach for some surfaces with point singularities. See also the Examples section.

The approach starts from an initial point (vertex) on the surface and then expands outward by using degree k power series expansions as a tool and an ϵ as a controller. We refer to it as the (k, ϵ)-triangulation. For easy description of the algorithm, we again introduce some terminology.

Expansible vertex. During the process of expansion of the triangular polygon, a vertex is called expansible if it is a smooth point of the surface $f = 0$, and it is in the interior of the given boundary box and it is on the boundary of the present constructed polygon.

Binary partition process.
For a given point p on the surface, consider its (p, k, ϵ)-circle and two points q_1 and q_2 on the circle, and let $p_i = (q_i, \phi_p(q_i))$. The binary partition process is to produce a series of triangles by the following process.
(a) If angle $\angle p_1 p p_2 \geq \pi$, then choose a point q_3 in the middle of q_1 and q_2 and on the circle. If the point p_3 is outside the given bounding box, then set it on the boundary. Repeat this step till the angle is less than π.
(b) If $\angle p_1 p p_2 \leq \pi$, then if p_2 is in the (p_1, k, ϵ)-sphere and p_1 is in the (p_2, k, ϵ)-sphere, then $[p_1, p_2]$ is a new edge and a new triangle $[p_1, p_2, p]$ is formed. Otherwise,

a new point q_3 is taken on the (p, k, ϵ)-circle and in the middle of q_1 and q_2. If p_3 is outside the given bounding box, then set it on the boundary. This step is repeated until the (p, k, ϵ)-sphere of every vertex p contains its neighboring vertices. The connection of all the points whose projection on the edges with p form triangles.

Algorithm 2

1. Initial Step. For a given smooth point p_0 on one component of the surface $f(x, y, z) = 0$, first compute the P-expression $Z = \phi_{p_0,k}(X, Y)$. On the P-plane, then find the (p_0, k, ϵ)-circle with center $P(p_0) = (0, 0)$. Take three equally distributed points q_i, $i = 1, \ldots, 3$ on the circle and let $p_i = (q_i, \phi_{p_0,k}(q_i))$. For each pair of points p_i and p_{i+1}, a binary partition process is conducted. In this initial case, all the boundary vertices are expandible except those that are not inside the given box. Compute the angle at each boundary vertex of the triangle.
2. General Step. Suppose we have constructed several space triangles that form one or more than one connected mesh. For each mesh, we keep the boundary information such as incident edges, vertices with angles, etc. Assume now that at least one of the boundary vertices is expandible. Then the general step is to construct more triangles that join the original ones and enlarge the mesh. We try to keep the mesh as convex as possible, so we always expand around the vertex that has the sharpest angle.
   Find an expandible vertex, say p_0, on the present boundary such that the angle at p_0 is minimal. Let the two related boundary edges be $[p_1, p_0]$ and $[p_0, p_2]$, Compute the P-expression $Z = \phi_{p_0,k}(X, Y)$ and with (p_0, k, ϵ)-radius $r(p_0, k, \epsilon)$. Let $[q_1', q_0']$ and $[q_0', q_2']$ be the projection of $[p_1, p_0]$ and $[p_0, p_2]$ on the P-plane and q_1 and q_2 be the intersection of rays $[q_0', q_1'\rangle$ and $[q_0', q_2'\rangle$ with the (p_0, k, ϵ)-circle. Next, perform the binary partition process for q_1 and q_2. If the line segment $[q_i, q_0]$ intersects a previous edge's projection on the P-plane and the point $T^{-1}(q_i, \phi(q_i))$ lies on a previous surface patch, then the new vertex becomes non-expandible. Here a local re-triangulation is needed.
3. Final Step. Repeat the general step iteratively, until every vertex is non-expandible. This finishes the generation of the triangulation for one connected component of the real surface.

5. Constructions of Curvilinear Wire Frames

For each edge of the triangulation, we shall construct a parametric space curve and a normal function (for G^1 smoothness) such that the curve interpolates the vertices of the edge and the normal function along the curve interpolates the given vertex normals and is orthogonal to the tangent of the curve. The normals at vertices are defined by the original surface normals. However, at the singular points of the surface, the normals are not uniquely defined. Hence the parametric space curve and the normal function will not have normal conditions there and will be G^0 continuous. In the following, the construction of the wire frame on an edge is considered in different cases according to having two normals, one normal and no normal:

Problem 1. Given an edge $[p_0, p_1]$ and two vertex normals n_0, n_1 (i) find a parametric space curve $C(t) = [X(t), Y(t), Z(t)]^{\mathrm{T}}$ such that

$$C(0) = p_0, \qquad C(1) = p_1 \tag{5.1}$$

$$n_0^{\mathrm{T}} C'(0) = 0, \qquad n_1^{\mathrm{T}} C'(1) = 0 \tag{5.2}$$

and (ii) find a normal function $n(t)$ on $C(t)$ such that

$$n(0) = n_0, \qquad n(1) = n_1 \tag{5.3}$$

$$n^{\mathrm{T}}(t) C'(t) \equiv 0, \qquad t \in [0,1]\,. \tag{5.4}$$

Problem 1 is the general case that occurs on smooth (non-singular) regions of the surface.

Problem 2. Given an edge $[p_0, p_1]$ and one vertex normal n_0 or n_1, (i) find a parametric space curve $C(t) = [X(t), Y(t), Z(t)]^{\mathrm{T}}$ such that (5.1) holds and

$$n_0^{\mathrm{T}} C'(0) = 0, \qquad \text{or} \qquad n_1^{\mathrm{T}} C'(1) = 0 \tag{5.5}$$

and (ii) find a normal function $n(t)$ on $C(t)$ such that

$$n(0) = n_0, \qquad \text{or} \qquad n(1) = n_1 \tag{5.6}$$

and (5.4) holds.

Problem 2 arises when an edge has one smooth vertex and one singular vertex.

Problem 3. Given an edge $[p_0, p_1]$, (i) find a parametric space curve $C(t) = [X(t), Y(t), Z(t)]^{\mathrm{T}}$ such that (5.1) holds and (ii) find a normal function $n(t)$ on $C(t)$ such that (5.4) holds.

Problem 3 occurs when an edge has two singular end-points.

5.1. DEGREE TWO PARAMETRIC SPACE CURVE WITH MINIMUM BENDING ENERGY

Solution of Problem 1(i). Let $C(t) = At^2 + Bt + C$, with A, B, $C \in \mathbb{R}^3$. Then by (5.1), we have $C = p_0$, $A = p_1 - p_0 - B$. From (5.2), it follows that

$$[n_0, n_1]^{\mathrm{T}} B = \begin{bmatrix} 0 \\ 2n_1^{\mathrm{T}}(p_1 - p_0) \end{bmatrix} \tag{5.7}$$

A. If n_0, n_1 are linearly dependent, we must have

$$n_1^{\mathrm{T}}(p_1 - p_0) = 0, \tag{5.8}$$

otherwise, equation (5.7) has no solution. If (5.8) is true, we take $A = 0$, $B = p_1 - p_0$, and thus equations (5.1)–(5.2) are satisfied.

B. If n_0, n_1 are linearly independent, then equation (5.7) has many solutions. Let $n_2 = n_0 \times n_1 / \|n_0 \times n_1\|$, where $\times$ denotes cross product and $\|\cdot\|$ denotes the Euclidean norm. Then B can be expressed as $B = \alpha n_2 + [n_0, n_1]\beta$, $\beta \in \mathbb{R}^2$. From equation (5.7), we have

$$[n_0, n_1]^T [n_0, n_1] \beta = \begin{bmatrix} 0 \\ 2n_1^{\mathrm{T}}(p_1 - p_0) \end{bmatrix}. \tag{5.9}$$

That is, β is determined uniquely by (5.9) and α is arbitrary. For simplicity, denote

$$B = n_3 + \alpha n_2 \qquad \text{with } n_3 = [n_0, n_1]\beta$$

$$A = p_1 - p_0 - B = n_4 - \alpha n_2\,.$$

Now we take α, such that the bending strain energy $\int_0^1 \|C'(t)\|^2\,dt = \min$ of the curve $C(t)$ is minimized. Since $C'(t) = 2At + B$,

$$\begin{aligned}\int_0^1 \|C'(t)\|^2\,dt &= \frac{4}{3}A^{\mathrm{T}}A + 2A^{\mathrm{T}}B + B^{\mathrm{T}}B \\ &= \frac{1}{3}n_2^{\mathrm{T}}n_2\alpha^2 - \frac{2}{3}n_4^{\mathrm{T}}n_2\alpha + \frac{4}{3}n_4^{\mathrm{T}}n_4 + 2n_4^{\mathrm{T}}n_3 + n_3^{\mathrm{T}}n_3\,.\end{aligned}$$

From $\frac{d}{d\alpha}\int_0^1 \|C'(t)\|^2\,dt = 0$, we get the α that minimizes the bending energy: $\alpha = n_4^{\mathrm{T}}n_2 = (p_1 - p_0)^{\mathrm{T}}n_2$. Therefore

$$B = [n_0, n_1]\beta + n_2(p_1 - p_0)^{\mathrm{T}}n_2\,.$$

Lemma 5.1. If n_0, n_1 are linearly independent, the parametric space curve interpolation problem 1(i) has a unique minimum bending energy solution.

Solution of Problem 2(i). Suppose we are given a single normal n_0 at p_0, we shall construct $C(t) = At^2 + Bt + C$ such that the curve is in the plane $\mathrm{span}(n_0, p_1 - p_0)$ spanned by n_0 and $p_1 - p_0$.

If $n_0^T(p_1 - p_0) \neq 0$, then as before, $C = p_0$, $A = p_1 - p_0 - B$ and $n_0^{\mathrm{T}}B = 0$. Furthermore, B is in the plane $\mathrm{span}(n_0, p_1 - p_0)$. These requirements lead to

$$B = \alpha n_2, \qquad \text{with } n_2 = n_0(p_1 - p_0)^{\mathrm{T}}n_0 - (p_1 - p_0)n_0^{\mathrm{T}}n_0$$

where α is parameter that is determined by minimizing the bending energy of the curve. This results in $\alpha = (p_1 - p_0)^{\mathrm{T}}n_2/n_2^{\mathrm{T}}n_2$. If $n_0^{\mathrm{T}}(p_1 - p_0) = 0$, we take $A = 0$, $B = p_1 - p_0$, $C = p_0$.

Similarly, if we are given a normal n_1 at p_1, then if $n_1^{\mathrm{T}}(p_1 - p_0) \neq 0$, we have

$$B = 2(p_1 - p_0) + \alpha n_2, \qquad C = p_0, \qquad A = p_1 - p_0 - B$$

with

$$\alpha = (p_0 - p_1)^{\mathrm{T}}n_2/n_2^{\mathrm{T}}n_2, \qquad n_2 = n_1(p_0 - p_1)^{\mathrm{T}}n_1 - (p_0 - p_1)n_1^{\mathrm{T}}n_1\,.$$

Again, if $n_1^{\mathrm{T}}(p_1 - p_0) = 0$, we take $A = 0$, $B = p_1 - p_0$, $C = p_0$.

Therefore, problem 2(i) always has a degree two solution.

Solution of Problem 3(i). Now we simply take $C(t)$ to be a linear curve. That is $A = 0$, $B = p_1 - p_0$, $C = p_0$, Therefore, problem 3(i) always has a linear solution.

5.2. NORMAL FUNCTION ON DEGREE TWO PARAMETRIC SPACE CURVE

Solution of Problem 1(ii). Let

$$n(t) = (Dt + E)/(1 + wt),$$

be the normal function, where D, $E \in \mathbb{R}^3$, $w \in \mathbb{R}$. Then by (5.3) we have

$$E = n_0, \qquad D = n_1 - n_0 + wn_1\,.$$

Since the numerator of $n^{\mathrm{T}}(t)C'(t)$ is a polynomial of degree 2 in t and $n^{\mathrm{T}}(t)C'(t) = 0$ when $t = 0$ and $t = 1$, (5.4) holds if there is another point in [0,1] such that (5.4) holds. Take $t = \frac{1}{2}$ then by $n^{\mathrm{T}}(\frac{1}{2})C'(\frac{1}{2}) = 0$, we have $1 + w = -\frac{n_0^{\mathrm{T}}C'(1)}{n_1^{\mathrm{T}}C'(0)}$. Since $C'(0) = B$,

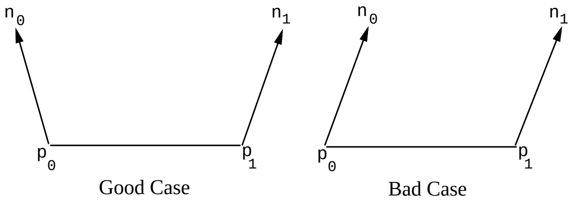


Figure 2. The normal pattern.

$C'(1) = 2A + B = 2(p_1 - p_0) - B$, it follows from (5.7) that

$$1 + w = -\frac{n_0^{\mathrm{T}}(p_1 - p_0)}{n_1^{\mathrm{T}}(p_1 - p_0)}\,.$$

The good w should make $1 + w > 0$, i.e., $n(t)$ has no pole in [0,1]. This requires that $n_0^{\mathrm{T}}(p_1 - p_0)$ and $n_1^{\mathrm{T}}(p_1 - p_0)$ have opposite signs (see Figure 2).

Lemma 5.2. If $n_0^{\mathrm{T}}(p_1 - p_0)/n_1^{\mathrm{T}}(p_1 - p_0) < 0$, there exists a unique linear rational normal function $n(t)$ on $C(t)$ such that (5.3) and (5.4) are satisfied.

Solution of Problem 2(ii). Now we are given only one normal, say, n_0 at p_0. We specify a normal n_1 at p_1 by taking n_1 to be the normal of $C(t)$ at p_1 such that n_1 lies in the plane defined by $\mathrm{span}(n_0, p_1 - p_0)$ and points to the same side of the edge $[p_0, p_1]$ as n_0. This normal is uniquely defined and the condition in Lemma 2 is satisfied if $n_0^{\mathrm{T}}(p_1 - p_0) \neq 0$. Hence the above results can be used. If $n_0^{\mathrm{T}}(p_1 - p_0) = 0$, the constant normal function $n(t) = n_0$ satisfies the required condition.

Solution of Problem 3(ii). This case occurs when the edge is on a singular curve. Now we cannot expect that the constructed surface is G^1 smooth. Since an edge on singular curve of the original surface will be shared by several triangles, the normal function will be defined once for each triangle. For a specified triangle that contains the edge, we take the normals at the vertices of the edge to be the other edge curve's normal defined at the corresponding vertices. The normal function along the singular edge is defined to be the linear function that has the two normals at its end points. However, the space curve on this edge is defined uniquely and the surface constructed here is continuous (G^0), but not G^1 smooth.

5.3. Cubic parametric space curve with minimum bending energy

Since a conic space curve does not always exist for problem 1, we may use cubic space curves instead. Let $C(t) = At^3 + Bt^2 + Ct + D$. We determine $A, B, C, D \in \mathbb{R}^3$ such that

$$C(0) = p_0, \qquad C(1) = p_1, \qquad C(1/2) = p_2, \tag{5.10}$$

$$n_0^{\mathrm{T}} C'(0) = 0, \qquad n_1^{\mathrm{T}} C'(1) = 0\,. \tag{5.11}$$

From (5.10)

$$D = p_0, \qquad A = p_1 - p_0 - B - C, \qquad B = 8(p_2 - p_0) - (p_1 - p_0) - 3C$$

and hence we need to determine C. It follows from (5.11) that

$$[n_0, n_1]^{\mathrm{T}} C = \begin{bmatrix} 0 \\ 4n_1^{\mathrm{T}}(2p_2 - p_1 - p_0) \end{bmatrix}. \tag{5.12}$$

A. If n_0, n_1 are linearly dependent, we must choose p_2 such that

$$n_1^{\mathrm{T}} p_2 = \frac{1}{2} n_1^{\mathrm{T}} (p_1 + p_0)\,. \tag{5.13}$$

Let n_3, n_4 satisfy $n_0^{\mathrm{T}} n_3 = n_0^{\mathrm{T}} n_4 = n_3^{\mathrm{T}} n_4 = 0$ and $\|n_3\| = \|n_4\| = 1$. The solution of (5.12) can be expressed as $C = \alpha n_3 + \beta n_4$. It is not difficult to calculate that when $\alpha = (4p_2 - p_1 - 3p_0)^{\mathrm{T}} n_3$, $\beta = (4p_2 - p_1 - 3p_0)^{\mathrm{T}} n_4$, the energy $\int_0^1 \|C'(t)\|^2\,dt$ of the curve $C(t)$ is minimum. We can take

$$n_4 = (2p_2 - p_1 - p_0)/\|2p_2 - p_1 - p_0\|, \qquad n_3 = n_1 \times n_4\,.$$

Then

$$\alpha = (p_1 - p_0)^{\mathrm{T}} n_3, \qquad \beta = 2\|2p_2 - p_1 - p_0\| + (p_1 - p_0)^{\mathrm{T}} n_4\,.$$

Theorem 5.3. If n_0, n_1 are linearly dependent, then if p_2 satisfies (5.13) and

$$(p_2 - p_0)^{\mathrm{T}} n_1 \neq 0, \qquad \det[p_2 - p_1, p_2 - p_0, n_1] \neq 0$$

then the matrix $[A, B, C]$ is non-singular.

Proof. Since $n_1^{\mathrm{T}} n_3 = n_0^{\mathrm{T}} n_4 = n_3^{\mathrm{T}} n_4 = 0$, we have $\alpha = (p_1 - p_0)^{\mathrm{T}} n_3 \neq 0$. Otherwise we are lead to $(p_2 - p_0)^{\mathrm{T}} n_3 = 0$ and then $[p_2 - p_1, p_2 - p_0, n_1]^{\mathrm{T}} n_3 = 0$. This contradicts the non-singularity of $[p_2 - p_1, p_2 - p_0, n_1]$ and $n_3 \neq 0$. Hence $[A, B, C] \cong [p_2 - p_1, p_2 - p_0, C] \cong [n_4, p_2 - p_0, n_3]$. Since $n_4, p_2 - p_0$ and n_1 are linearly independent by the assumption of the theorem, n_3 can be expressed as $n_3 = a n_4 + b(p_2 - p_0) + c n_1$. By multiplying n_3 on this equality we know that $b \neq 0$ and by multiplying n_1 on the same equality we get $c \neq 0$. Therefore the matrix $[n_4, p_2 - p_0, n_3]$ is non-singular. $\square$

B. If n_0, n_1 are linearly independent, then equation (5.12) has many solutions. Let $n_2 = n_0 \times n_1$, $\|n_2\| = 1$. Then C can be expressed as $C = \alpha n_2 + [n_0, n_1]\beta$, $\beta \in \mathbb{R}^2$ where β is determined uniquely by $[n_0, n_1]^{\mathrm{T}} [n_0, n_1]\beta = \begin{bmatrix} 0 \\ 4n_1^{\mathrm{T}}(2p_2 - p_1 - p_0) \end{bmatrix}$, and α, which makes the energy of the curve $C(t)$ to be minimum, is $\alpha = (4p_2 - p_1 - 3p_0)^{\mathrm{T}} n_2$.

5.4. NORMAL FUNCTION ON CUBIC PARAMETRIC SPACE CURVE

Let the normal function $n(t)$ be in the form $n(t) = Et^2 + Ft + G$ that satisfies

$$n(0) = n_0, \qquad n(1) = n_1, \qquad n^{\mathrm{T}}(t) C'(t) \equiv 0, \qquad t \in [0, 1]\,. \tag{5.14}$$

Since $n^{\mathrm{T}}(t) C'(t)$ is a polynomial of degree 4 and it vanishes at $t = 0$ and $t = 1$, we need to choose three points, say $t = 1/4, 1/2, 3/4$, such that (5.14) holds. Since $G = n_0$, $E = n_1 - n_0 - F$, we have, for unknown vector F, the following equations

$$\begin{cases} C'(1/4)^{\mathrm{T}}(1/16(n_1 - n_0 - F) + 1/4F + n_0) = 0 \\ C'(1/2)^{\mathrm{T}}(1/4(n_1 - n_0 - F) + 1/2F + n_0) = 0 \\ C'(3/4)^{\mathrm{T}}(9/16(n_1 - n_0 - F) + 3/4F + n_0) = 0\,. \end{cases}$$

The coefficient matrix of this equation is equivalent to the matrix $[A, B, C]$. Hence the equation has a unique solution iff the matrix $[A, B, C]$ is invertible. If the matrix is singular, one can solve the equation by least-squares approximation.

6. Interpolation with Parametric Surface Patches

Suppose we are given a triangular wire frame $C_i(t)$, $i = 0,1,2$ with vertices V_i, $i = 1,2,3$ and furthermore normal functions $n_i(t)$, $i = 0,1,2$, such that $p_1 = C_0(0) = C_2(1)$, $p_2 = C_0(1) = C_1(0)$, $p_3 = C_1(1) = C_2(0)$, and $C_i^{\mathrm{T}}(t)n_i(t) = 0$. We wish to construct a parametric patch $X(u,v) = [x(u,v)\, y(u,v)\, z(u,v)]$. that covers the given wire frame (for G^0 continuity) and further has the given normal (for G^1 continuity) on the wire frame, where u, v, w are barycentric coordinate systems with $w = 1 - u - v$.

6.1. G^0 INTERPOLATION

Covering conic wire frames

Since the degree of the space curve is 2, we choose one more point on each edge of the triangle in addition to the vertices. Let $p_{i+4} = C_i(\frac{1}{2})$, $i = 0,1,2$ yielding totally six points (p_i, $i = 1,\ldots,6$). Now find a polynomial P_2 of degree 2 such that

$$\begin{aligned} P_2(V_i) &= p_i, \qquad i = 1,2,3 \\ P_2\left(\frac{V_1+V_2}{2}\right) &= p_4 \\ P_2\left(\frac{V_2+V_3}{2}\right) &= p_5 \\ P_2\left(\frac{V_3+V_1}{2}\right) &= p_6\,. \end{aligned} \tag{6.1}$$

The coefficient matrix of (6.1)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} & 0 & 0 \end{bmatrix}$$

is non-singular, hence equation (6.1) has a unique solution.

Covering cubic wire frames

Since the degree of the space curve now is 3, we need to choose two more points on each edge of the triangle in addition to the vertices. Let $p_{i+4} = C_i(\frac{1}{3})$, $p_{i+7} = C_i(\frac{2}{3})$, $i = 0,1,2$. Then we have nine points (p_i, $i = 1,\ldots,9$). Now find a polynomial P_3 of degree 3 such that

$$\begin{gathered} P_3(V_i) = p_i, \qquad i = 1,2,3 \\ P_3\left(\frac{V_1+2V_2}{3}\right) = p_4, \qquad P_3\left(\frac{2V_1+V_2}{3}\right) = p_7 \\ P_3\left(\frac{V_2+2V_3}{3}\right) = p_5, \qquad P_3\left(\frac{2V_2+V_3}{3}\right) = p_8 \\ P_3\left(\frac{V_3+2V_1}{3}\right) = p_6, \qquad P_3\left(\frac{2V_3+V_1}{3}\right) = p_9\,. \end{gathered}$$

Since P_3 has 10 coefficients, one more equation $P_3(V) = p$ is needed, where $V = (u,v,w)$

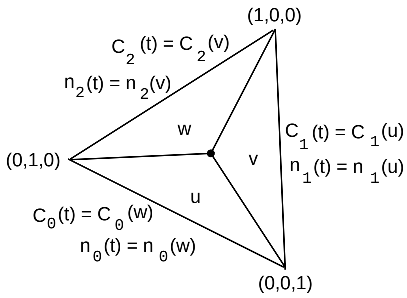


Figure 3. The split triangular wireframe.

is any given point inside the triangle and $p \in \mathbb{R}^3$ that can be used to control the shape of the patch.

It is easy to check that the resulting system of linear equations has a non-singular coefficient matrix for any V inside the triangle.

6.2. G^1 interpolation

We now wish to construct a G^1 surface patch interpolating the given normal functions on the curvilinear wire frame.

Covering conic wire frame

The patch is defined in the following form

$$P(u,v,w) = P_2(u,v,w) + uvwP_1(u,v,w)$$

where P_2 is a BB-form (Bernstein–Bézier) polynomial of degree 2 that covers the conic wire frame. P_1 is a rational function in the form of

$$P_1(u,v,w) = \frac{uvP_w + uwP_v + vwP_u}{uv + uw + vw} \tag{6.2}$$

and P_u, P_v and P_w are constants (polynomial of degree 0) to be determined such that G^1 continuity is guaranteed. Since variable t in $C_i(t)$ and $n_i(t)$ can be changed into $(1-t)$, we may assume, without loss of generality, that t is increased from 0 to 1 when point goes from $(0,1,0)$ to $(0,0,1)$, from $(0,0,1)$ to $(1,0,0)$ and from $(1,0,0)$ to $(0,1,0)$ (see Figure 3). Hence the variable of C_0 and n_0 is w, C_1 and n_1 is u, C_2 and n_2 is v. Now we determine P_1 such that tangent plane determined by the span of $(\frac{\partial P}{\partial u}, \frac{\partial P}{\partial v})$ are orthogonal to normal functions $n_i(t)$ on $C_i(t)$

A. when $u = 0$

$$\frac{\partial P}{\partial u} = \frac{\partial P_2}{\partial u} - \frac{\partial P_2}{\partial w} + vwP_1 \tag{6.3}$$

$$\frac{\partial P}{\partial v} = \frac{\partial P_2}{\partial v} - \frac{\partial P_2}{\partial w} = \frac{d}{dw}(P_2(0,1-w,w) = -C_0'(w) \tag{6.4}$$

B. when $v = 0$

$$\frac{\partial P}{\partial u} = \frac{\partial P_2}{\partial u} - \frac{\partial P_2}{\partial w} = \frac{d}{du}P_2(u,0,1-u) = C_1'(u) \tag{6.5}$$

$$\frac{\partial P}{\partial v} = \frac{\partial P_2}{\partial v} - \frac{\partial P_2}{\partial w} + uwP_1 \tag{6.6}$$

C. when $w = 0$

$$\frac{\partial P}{\partial u} = \frac{\partial P_2}{\partial u} - \frac{\partial P_2}{\partial w} - uvP_1 \tag{6.7}$$

$$\frac{\partial P}{\partial v} = \frac{\partial P_2}{\partial v} - \frac{\partial P_2}{\partial w} - uvP_1 \tag{6.8}$$

and further

$$\frac{\partial P}{\partial v} - \frac{\partial P}{\partial u} = \frac{\partial P_2}{\partial v} - \frac{\partial P_2}{\partial u} = \frac{d}{dv}(P_2(1-v,v,0)) = C_2'(v)\,. \tag{6.9}$$

By the definition of $n_i(t)$, we have $C_i'(t)^{\mathrm{T}} n_i(t) = 0$, so we need to have by (6.3), (6.6) and (6.7)

$$\left(\frac{\partial P_2}{\partial u} - \frac{\partial P_2}{\partial w} + (1-w)wP_u\right)^{\mathrm{T}} n_0(w) = 0 \qquad u = 0 \tag{6.10}$$

$$\left(\frac{\partial P_2}{\partial v} - \frac{\partial P_2}{\partial w} + u(1-u)P_v\right)^{\mathrm{T}} n_1(u) = 0 \qquad v = 0 \tag{6.11}$$

$$\left(\frac{\partial P_2}{\partial u} - \frac{\partial P_2}{\partial w} - (1-v)vP_w\right)^{\mathrm{T}} n_2(v) = 0 \qquad w = 0 \tag{6.12}$$

Consider first the left-hand side of equation (6.10). Since

(i) for $u = w = 0$, $v = 1$, it follows from (6.4) and (6.9) that

$$\begin{aligned}\frac{\partial P_2}{\partial u} - \frac{\partial P_2}{\partial w} &= \left(\frac{\partial P_2}{\partial v} - \frac{\partial P_2}{\partial w}\right) - \left(\frac{\partial P_2}{\partial v} - \frac{\partial P_2}{\partial u}\right)\\ &= -C_0'(0) - C_2'(1)\end{aligned}$$

(ii) for $u = v = 0$, $w = 1$ it follows from (6.5) that $\frac{\partial P_2}{\partial u} - \frac{\partial P_2}{\partial w} = C_1'(0)$, (6.10) holds for $w = 0$ and $w = 1$. Similarly, we can show that (6.11) and (6.12) hold for end points of the unit interval.

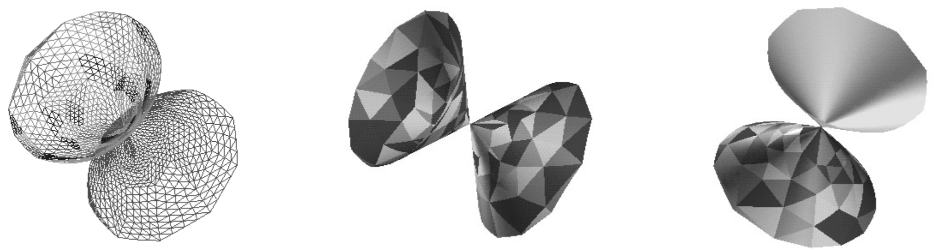
Therefore, (6.10)–(6.12) is equivalent to

$$\begin{cases} P_u(t)^{\mathrm{T}} n_0(t) = a \\ P_v(t)^{\mathrm{T}} n_1(t) = b \\ P_w(t)^{\mathrm{T}} n_2(t) = c \end{cases} \tag{6.13}$$

where a, b, and c are constants. The left-hand side of (6.13) are polynomials of degree 1. At point $u = v = w = \frac{1}{3}$, we specify a vector p of P_1 that controls the shape of the patch. We have the following system of equations.

$$\begin{bmatrix} n_0^{\mathrm{T}}(0) & & \\ n_0^{\mathrm{T}}(1) & & \\ & n_1^{\mathrm{T}}(0) & \\ & n_1^{\mathrm{T}}(1) & \\ & & n_2^{\mathrm{T}}(0) \\ & & n_2^{\mathrm{T}}(1) \\ I & I & I \end{bmatrix} \begin{bmatrix} P_u \\ P_v \\ P_w \end{bmatrix} = \begin{bmatrix} a \\ a \\ b \\ b \\ c \\ c \\ 3p \end{bmatrix}.$$

In order to study the singularity of the coefficient matrix, we assume $n_0^{\mathrm{T}}(1) = n_1^{\mathrm{T}}(0)$,



(a) (b) (c)

Figure 4. Spline approximation of a circular cone: (a) shows the surface triangulation; (b) shows the different triangular rational Bezier G^1 patches; (c) partly shaded display.

$n_1^{\mathrm{T}}(1) = n_2^{\mathrm{T}}(0)$, $n_1^{\mathrm{T}}(1) = n_0^{\mathrm{T}}(0)$. By simple elimination we know that the coefficient matrix is non-singular if the matrix $[n_0, n_1, n_2]$ is non-singular. Therefore we have

Lemma 6.1. If $[n_0, n_1, n_2]$ is non-singular, the G^1 interpolation problem with one free (control) point has a unique solution.

Note. One way to choose the control value p at the middle point is to take $p = 0$.

The condition that the matrix $[n_0, n_1, n_2]$ is non-singular in Lemma 4.1 can be relaxed since the vectors n_0, n_1 and n_2 are pairwise independent. In this case, equation (6.13) can be solved separately with one degree of freedom left that can be used to control the shape at point $u = v = w = \frac{1}{3}$ in the least-squares sense.

Covering cubic wire frame

The patch is now defined in the following form

$$P(u, v, w) = P_3(u, v, w) + uvwP_1(u, v, w)$$

where P_3 is a BB-form polynomial of degree 3 that covers the cubic wire frame. P_1 is a rational function in the same form as (6.2).

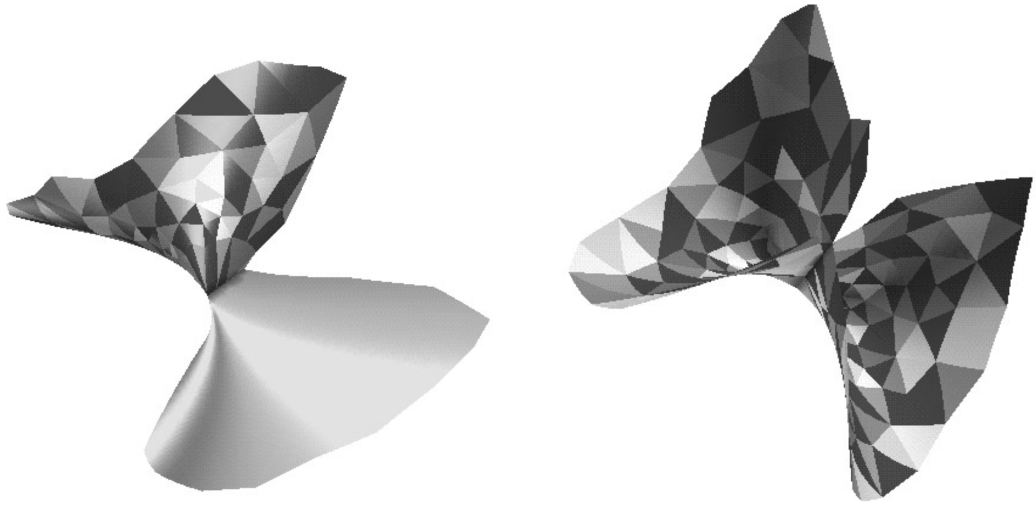
Parallel to the case of smoothly covering a conic wire frame, we are lead to a system (see (6.13)) of equations

$$\begin{cases} P_u^{\mathrm{T}} n_0(t) = a(t) \\ P_v^{\mathrm{T}} n_1(t) = b(t) \\ P_w^{\mathrm{T}} n_2(t) = c(t) \end{cases}$$

where the normal functions $n_i(t)$ and $a(t)$, $b(t)$, and $c(t)$ are polynomials of degree 2. In fact, this system can be solved separately for P_u, P_v and P_w. Each equation has a unique solution iff the coefficient vectors of the corresponding normal function are linearly independent. In practice, we solve these equations by least-squares approximation.

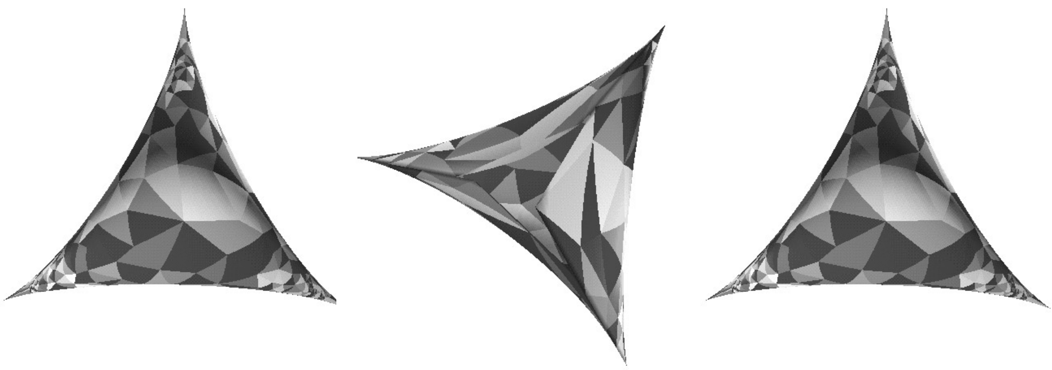
6.3. EXAMPLES

We present some non-trivial examples of piecewise rational approximations of implicit algebraic surfaces, implemented in GANITH, (Bajaj and Royappa, 1990), an X11 based



(a) (b)

Figure 5. Spline approximation of the Cartan umbrella surface: (a) shows the top lobe with triangular rational Bezier G^1 patches and the bottom lobe is Gouraud shaded; (b) both lobes are shown with triangular rational Bezier G^1 patches.



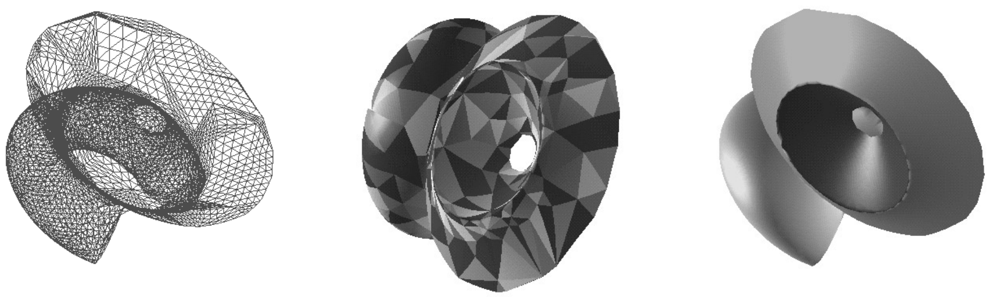
(a) (b) (c)

Figure 6. Spline approximation of a portion of the Steiner surface: (a), (b) and (c) show different views of the triangular rational Bezier G^1 patches.

interactive algebraic geometry toolkit, using Common Lisp for the symbolic computation (resultants) and C for all other numeric and graphical computations.

The examples shown in Figures 6.3, 6.3 and 6.3, are handled by the vertex expansion approach.

1. $f = x^2 + y^2 - z^2$.
   The conical surface $f = 0$ in Figure 6.3 has a singular point $(0,0,0)$. The rational parametric spline approximation (shown by approximately 190 multi-colored



(a) (b) (c)

Figure 7. Spline approximation of a toroidal surface: (a) shows the surface triangulation; (b) shows the different triangular rational Bezier G^1 patches; (c) shaded display.

patches) and the shaded display are within a bounding box $[-3, 3, -3, 3, -2, 2]$. The triangulation algorithm is started with a seed point $(1, 0, 1)$.

2. $f = x^2 - y * z^2$.

   The Cartan umbrella surface $f = 0$ in Figure 6.3 has a singular point $(0, 0, 0)$ and a singular line $(x = 0, z = 0)$. The rational parametric spline approximation (shown by 280 multi-colored patches) and the shaded display are within a bounding box $[-3, 3, -3, 3, -2, 2]$. The triangulation algorithm is started with a seed point $(1, 0, 1)$.

3. $f = x^2 * y^2 + y^2 * z^2 + x^2 * z^2 - 4 * x * y * z$.

   The Steiner surface $f = 0$ (a portion shown in Figure 6.3) has singular curves (lines) on the x-axis, y-axis, and z-axis and a triple point at the origin. The rational parametric spline approximation (shown by 370 multi-colored patches) and the shaded display are shown within a bounding box $[-3, 3, -3, 3, -2, 2]$. The triangulation algorithm is started with seed points $(1, 1, 1)$, $(-1, -1, 1)$, $(-1, 1, -1)$ and $(1, -1, -1)$. Here, one needs to reduce the triangulation error considerably and stitch distance close to the origin to allow the edges to stitch through the singular origin.

4. $f = (y^2 + z^2 - x^2 + 0.5 * x^3 - 4)^2 - 16 * x^2 + 8 * x^3$.

   The toroidal-like surface of revolution $f = 0$ in Figure 6.3 has a singular curve $y^2 + z^2 = 2$. The rational parametric spline approximation (shown by 540 multi-colored patches) and the shaded display are within a bounding box $[-1.2, 4, -4, 4, -4, 4]$.

Acknowledgements

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