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A-splines: local interpolation and approximation using G^k-continuous piecewise real algebraic curves

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Abstract

We provide sufficient conditions for the Bernstein–Bézier (BB) form of an implicitly defined bivariate polynomial over a triangle, such that the zero contour of the polynomial defines a smooth and single sheeted real algebraic curve segment. We call a piecewise G^k-continuous chain of such real algebraic curve segments in BB-form as an A-spline (short for algebraic spline). We prove that the degree n A-splines can achieve in general G^{2n-3} continuity by local fitting and still have degrees of freedom to achieve local data approximation. As examples, we show how to construct locally convex cubic A-splines to interpolate and/or approximate the vertices of an arbitrary planar polygon with up to G^4 continuity, to fit discrete points and derivatives data, and approximate high degree parametric and implicitly defined curves. Additionally, we provide computable error bounds. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Designing fonts with piecewise smooth curves or fitting curves to scattered data for image reconstruction are just two of the diverse applications of spline curve constructions.

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In this paper, we generalize past curve fitting schemes with conics (Bookstein, 1979; Farin, 1989; Pavlidis, 1983; Pottmann, 1991; Pratt, 1985; Sampson, 1982) and parametric spline fitting (Curry and Schoenberg, 1966; Nürnberger et al., 1984; Schoenberg and Whitney, 1953), achieving fits with fewer number of pieces or with higher order of smoothness/continuity. We exhibit efficient techniques to deal with higher degree implicitly defined algebraic curves, $f(x, y) = 0$, with $f(x, y)$ a bivariate real polynomial. The spline techniques of this paper simplify and extend prior approaches and applications of algebraic curves to problems in geometric modeling (Bajaj, 1997; Bajaj and Ihm, 1992; Bajaj and Xu, 1994, 1996; Bajaj et al., 1999; Paluszny and Patterson, 1992, 1993; Sederberg et al., 1985; Sederberg et al., 1988; Sederberg, 1984). The main advantages of the implicitly defined algebraic spline curve over the functional and parametric curves are: (1) the class of algebraic curves is closed under several geometric operations (intersections, union, offset, etc.), often required in a solid modeling system. For example, the offset of a parametric curve may not be parametric but is always algebraic and has an implicit representation. (2) Implicit algebraic curve segments (of degree n) have more degrees of freedom $(= (n + 2)(n + 1)/2 - 1 = n(n + 3)/2)$ compared with rational function $(= 2n + 1)$ and rational parametric $(= 3n - 1)$ curves of the same degree. Thus, implicit algebraic curve segments appear to be more flexible to approximate a complicated curve with fewer number of pieces or to achieve higher order of smoothness. For example, by local interpolation, implicit algebraic curves have the potential to achieve G^k continuity with $k \leqslant n(n + 3)/4 - 1$, while functional or parametric curve can achieve $G^{k'}$ continuity with $k' = n - 1$. Note that the degrees of freedom for the rational parametric curves are consistent with the well known theorem that rational parametric curves are exactly the irreducible implicit algebraic curves of genus 0 (Walker, 1978). An irreducible implicit curve of genus 0 and degree n possesses the maximum number of singularities an irreducible curve can have, viz., $(n - 1)(n - 2)/2$ and this is exactly the difference between the degrees of freedom of arbitrary implicit algebraic curves and rational parametric algebraic curves of the same degree n. The primary drawback, however, for the widespread use of the implicit algebraic curve is that the curve can have singularities (see Walker, 1978) and possess several disconnected real components. For example, fitting or interpolating a cluster of points within a triangle by an algebraic curve, the resulting curve could have either singular points or several disconnected components (sheets). Hence it is natural in some applications to require the curves to be smooth (no singularity) and not disconnected (single sheeted) within the triangle considered. In this paper we show how to isolate a nonsingular and single sheeted segment of an implicit algebraic curve and furthermore how to stitch these segments together to form a spline with continuity as high as G^{2n-3} using degree n curve pieces.

In Section 3 we provide sufficient conditions for the Bernstein–Bézier (BB) form of an implicitly defined bivariate polynomial over a triangle, such that the zero contour of the polynomial defines a smooth and single sheeted real algebraic curve segment. We call a piecewise G^k-continuous chain of such real algebraic curve segments in BB-form as an A-spline (short for algebraic spline). In Section 4 we prove that the degree n A-splines can achieve in general G^{2n-3} continuity by local fitting. As examples, we show how to construct locally convex cubic A-splines to interpolate and/or approximate the vertices of an arbitrary planar polygon with up to G^4 continuity, to fit discrete points and derivatives data, and approximate high degree parametric and implicitly defined curves. Additionally, we provide computable error bounds in Section 5.

1.1. Related prior work

Considerable work on polynomial spline interpolation and approximation has been done in the last decades(see (deBoor, 1978) for a bibliography). In general, spline interpolation has been viewed as a global fitting problem to arbitrary scattered data (Bookstein, 1979; Curry and Schoenberg, 1966; Nürnberger et al., 1984; Pavlidis, 1983; Pratt, 1985; Sampson, 1982; Schoenberg and Whitney, 1953). Here we consider local interpolation to an ordered set of points, defining an arbitrary polygon. Local interpolation by polynomials and rational functions is rather an old and simple technique that trace back to Hermite and Cauchy (1821). However, local interpolation by the zero sets of algebraic polynomials (implicit algebraic curves, surfaces etc.) is relatively new (Bajaj, 1997; Bajaj and Ihm, 1992; Floater, 1996; Paluszny and Patterson, 1992, 1993; Patterson, 1988; Sederberg, 1984). We lay emphasis in this paper on using connected and nonsingular real segments of implicit algebraic curves. Towards the same goal, Sederberg, in (Sederberg, 1984), has specified the coefficients of the BB form of an implicitly defined bivariate polynomial on a triangle in such a way that if the coefficients on the lines that parallel to one side, say L, of the triangle all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the algebraic curve segment at most once. Our conditions in Theorem 3.1 is more general, with Sederberg's condition forming a special case. In (Sederberg et al., 1988), Sederberg, Zhao and Zundel gave another similar set of conditions which guarantees the single sheeted property of their TPAC by requiring that $b_{i0} \geqslant 0$, that $b_{0i}, b_{m-1,i} \leqslant 0$ and that the directional derivative of PAC (*piecewise algebraic curves*) with respect to any direction $s = \alpha u$ be nonzero within the triangle domain, here b_{ij} denotes the Bézier coefficient. This condition is also much more restrictive than ours.

Related papers which construct families of G^1 and G^2 continuous cubic algebraic splines are given by Paluszny and Patterson (1992, 1993, 1994, 1998). They use the following reduced form of the cubic

$$F(s,t,u) = as^2u + bsu^2 - cst^2 - dt^2u + estu,$$

with $a > 0, b > 0, c > 0, d > 0$, and (s,t,u) in BB-coordinates over a triangle and guarantee that the segment of the curve inside the triangle is convex. These results are special cases of the present paper as we consider the general implicit cubic. Our cubic A-splines can always achieve G^3-continuity, and even G^4-continuity for some special cases.

2. Notation and preliminaries

Let $f(x,y)$ be a bivariate polynomial of degree n with real coefficients, and p_1, p_2, v_1 be three affine independent points in the xy-plane. Then the transform

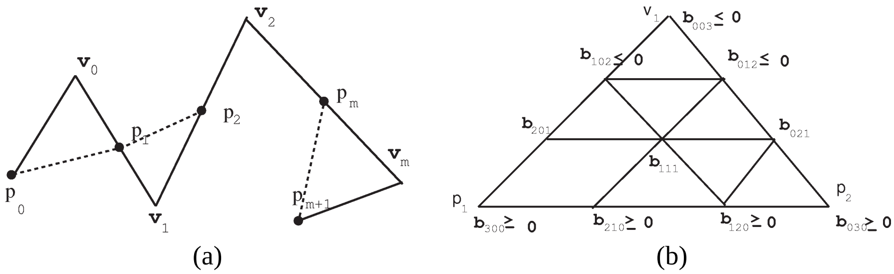


Fig. 1. (a) A G^1 polygon; (b) Bézier coefficients of cubic.

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & v_1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{2.1}$$

maps $f(x, y)$ into its barycentric coordinate form $F(\alpha_1, \alpha_2, \alpha_3) = f(x, y)$ on the triangle $[p_1 p_2 v_1]$, where $0 \leqslant \alpha_i \leqslant 1$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$. In the barycentric coordinate system, $F(\alpha_1, \alpha_2, \alpha_3)$ can be expressed in BB form (see (Farin, 1990)).

$$F(\alpha_1, \alpha_2, \alpha_3) = \sum_{i+j+k=n} b_{ijk} B^n_{ijk}(\alpha_1, \alpha_2, \alpha_3), \tag{2.2}$$

where

$$B^n_{ijk}(\alpha_1, \alpha_2, \alpha_3) = \frac{n!}{i!j!k!} \alpha_1^i \alpha_2^j \alpha_3^k.$$

Let p_1, v_1 and p_2 be three affine independent points in the xy-plane (see Fig. 1(a)). Then we consider the two line segments $[p_1 v_1]$ and $[v_1 p_2]$ as a segment of a polygon, denoted by $\widehat{p_1 v_1 p_2}$. We shall consider v_1 as a controller and p_1 and p_2 as interpolation points. An arbitrary polygon chain(or polygon for brevity) consists of a sequence of consecutive polygon segments denoted by $\{p_i \widehat{v_i p}_{i+1}\}_{i=0}^m$. A polygon $\{p_i \widehat{v_i p}_{i+1}\}_{i=0}^m$ is said to be of type G^1(see Fig. 1(a)) if

$$(v_i - p_{i+1}) = \alpha_i (v_{i+1} - p_{i+1}), \ \alpha_i < 0, \quad \text{for } i = 0, \ldots, m.$$

If $p_0 = p_{m+1}$, then the polygon is closed. Note that a G^1 polygon can be trivially constructed from an arbitrary polygon by inserting one vertex per edge of the polygon.

3. A sufficient condition of A-splines

Let $F(\alpha_1, \alpha_2, \alpha_3)$ be defined as (2.2) on the triangle $[p_1 p_2 v_1]$. Since there is constant multiplier to the equation $F(\alpha_1, \alpha_2, \alpha_3) = 0$, we may assume $b_{00n} = -1$ if $b_{00n} \neq 0$.

Theorem 3.1. *For the given polynomial $F(\alpha_1, \alpha_2, \alpha_3)$ defined as* (2.2)*, if there exists an integer $K (0 < K < n)$ such that* (*see Fig.* 1(b) *for $n = 3$ and $K = 1$*)

$$b_{ijk} \geqslant 0 \quad \textit{for } j = 0, 1, \ldots, n-k; \ k = 0, 1, \ldots, K-1, \tag{3.1}$$

$$b_{ijk} \leqslant 0 \quad \textit{for } j = 0, 1, \ldots, n-k;\ k = K+1, \ldots, n, \tag{3.2}$$

and

$$\sum_{j=0}^{n} b_{n-j,j0} > 0, \quad \sum_{j=0}^{n-k} b_{n-j-k,jk} < 0$$

for at least one k $(K < k \leqslant n)$, *then for any* β *that* $0 < \beta < 1$, *the straight line*

$$(\alpha_1, \alpha_2, \alpha_3)(t) = (1-t)(\beta, 1-\beta, 0) + t(0, 0, 1) \tag{3.3}$$

that passes through v_1 *and* $\beta p_1 + (1-\beta)p_2$, *intersects the curve* $F(\alpha_1, \alpha_2, \alpha_3) = 0$ *one and only one time* (*counting multiplicity*) *in the interior of the triangle* $[p_1 p_2 v_1]$.

The proofs of this and other theorems are given in the Appendix A.

This theorem guarantees that there is one and only one curve segment of $F(\alpha_1, \alpha_2, \alpha_3) = 0$ within the triangle under the given condition. The term “algebraic spline” or A-spline that we use in this paper is a chain of such curve segments with fixed continuity at the join points. We should mention that the curve $F(\alpha_1, \alpha_2, \alpha_3) = 0$ passes through v_1 if $b_{00n} = 0$. However, we do not use this part of the curve. In our application in Section 4, we take b_{00n} to be -1.

Note 3.1. Formulas (3.3) and (A.1) could be used to evaluate the curve $F(\alpha_1, \alpha_2, \alpha_3) = 0$. That is, for a given sequence of points of $\beta \in (0, 1)$, solve the equation $B_\beta(t) = 0$ for $t \in (0, 1)$. Then substituting (β, t) into (3.3), we then obtain a sequence of points, in terms of barycentric coordinates, on the curve. For $n \leqslant 4$, a closed form for the solution of $B_\beta(t) = 0$ exists. For $n \geqslant 5$, numerical methods have to be employed to solve the equation. However, since the equation has a single root in $(0, 1)$, Newton iterations combined with bisection suffice.

The next theorem goes further about the smoothness of the curve $F(\alpha_1, \alpha_2, \alpha_3) = 0$ and the properties on the boundary of the triangle.

Theorem 3.2. *Let* $F(\alpha_1, \alpha_2, \alpha_3)$ *be defined as Theorem* 3.1, *then*

(i) *The curve* $F(\alpha_1, \alpha_2, \alpha_3) = 0$ *is smooth in the interior of the triangle* $[p_1 p_2 v_1]$.

(ii) *If we further assume* $b_{n-k,0k} = 0$ *for* $k = 0, \ldots, K$, $b_{n-(K+1),0,K+1} < 0$ *and* $b_{n-1,10} > 0$, *then the curve in the triangle passes through* p_1, *tangent with the line* $\alpha_2 = 0$ *with multiplicity* $K+1$ *at* p_1 *and no other intersection with* $\alpha_2 = 0$ *for* $\alpha_1 > 0, \alpha_3 > 0$. *Similarly, if* $b_{0,n-k,k} = 0$ *for* $k = 0, \ldots, K$, $b_{0,n-(K+1),K+1} < 0$, *and* $b_{1,n-1,0} > 0$, *then the curve passes through* p_2, *tangent with the line* $\alpha_1 = 0$ *with multiplicity* $K+1$ *at* p_2 *and no other intersection with* $\alpha_1 = 0$ *for* $\alpha_2 > 0, \alpha_3 > 0$.

(iii) *If* $b_{n00} = b_{n-1,01} = b_{n-1,10} = 0$, *then* p_1 *is a singular point of the curve. Similarly, if* $b_{0n0} = b_{1,n-1,0} = b_{0,n-1,1} = 0$, *then* p_2 *is a singular point of the curve.*

Since it is obvious that the quadratic A-spline is convex, we consider now the convexity of the cubic spline. At present, the convexity characterization of the general case for $n \geqslant 4$

degree A-spline is left as an open problem. Even for the cubic case, the convexity is not always guaranteed. If the curve segment is tangent with the sides of the triangle at p_1 and p_2, i.e., a G^1 A-spline as in Theorem 3.2(ii), then it is convex. This is of course a special case, but it is the case we most often use in this paper (see Section 4).

Theorem 3.3. *The cubic A-spline defined in Theorem* 3.2(ii) *has no inflection point inside its reference triangle.*

4. G^k A-splines

In this section, we connect the A-spline segments to form a piecewise G^k continuous spline curve. For simplicity we assume that we are given a polygon in the plane, that is we have an ordered point set $\{p_i\}_{i=0}^{m+1}$, and additionally a vertex set $\{v_i\}_{i=0}^{m}$ (see Fig. 1(a)), such that the three points p_i, v_i and p_{i+1} are affine independent (noncollinear). The G^k continuity of an A-spline is achieved by the following steps:

(1) Form a G^1 control polygon $\{\widehat{p_i v_i p_{i+1}}\}$.
(2) Compute the first k terms of the local power series expansion of the A-spline at the join points p_i from the given data at these points.
(3) Determine the coefficients of F such that $F = 0$ has the same first k terms of local power series at the join points.

Several schemes exist which produce a desired polygon chain from scattered data (Edelsbrunner et al., 1983; Fairfield, 1979; Walker, 1978). To produce a G^1 polygon from a polygonal chain is trivial and amounts to inserting a single additional vertex per polygon edge. In Section 4.1, we first define the local power series and then compute the coefficients of F. Then in Section 4.2 we compute the power series for three fitting problems: (a) fit to a parametric curve; (b) fit to discrete data; (c) fit to a higher degree implicit curve.

4.1. Coefficients of F from local power series expansion

We consider first a two segment A-spline curve

$$F_l(\alpha_1, \alpha_2, \alpha_3) = \sum_{i+j+k=n} b_{ijk}^{(l)} B_{ijk}^n(\alpha_1, \alpha_2, \alpha_3) = \sum_{i+j+k=n} \tilde{b}_{ijk}^{(l)} \alpha_1^i \alpha_2^j \alpha_3^k = 0$$

on triangles $[p_1^{(l)} p_2^{(l)} v_1^{(l)}]$ for $l = 1, 2$ with $p_1^{(1)} = p_2^{(2)}$ as join point (see Fig. 2), where $\tilde{b}_{ijk}^{(l)} = (n!/(i!j!k!)) b_{ijk}^{(l)}$. We want to join these curve segments with the desired smoothness at $p_1^{(1)}$.

In the triangle $[p_1^{(l)} p_2^{(l)} v_1^{(l)}]$, we require our A-spline passing through $p_1^{(1)}$ and tangent with the line $[p_1^{(1)} v_1^{(1)}]$ at $p_1^{(1)}$. Hence we assume $b_{n-1,1,0}^{(1)} > 0$, $b_{1,n-1,0}^{(2)} > 0$. This implies that the curves $F_l(\alpha_1, \alpha_2, \alpha_3) = 0$ are regular at $p_1^{(1)}$. Therefore, the curve $F_1(1 - \alpha_2 - \alpha_3, \alpha_2, \alpha_3) = 0$ can be represented as a power series at $p_1^{(1)}$

$$\alpha_2 = \sum_{i=0}^{\infty} a_i^{(1)} \alpha_3^i = \sum_{i=0}^{\infty} a_i^{(1)} \big(p_1^{(1)}\big) \alpha_3^i, \quad \alpha_1 = 1 - \alpha_2 - \alpha_3, \tag{4.1}$$

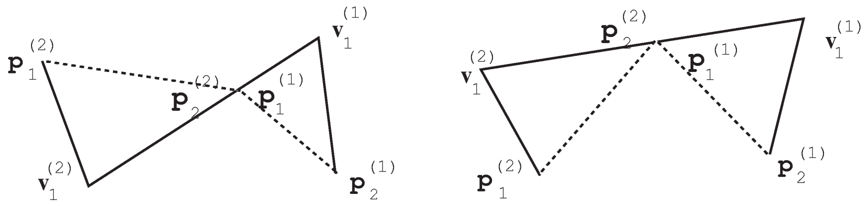


Fig. 2. The two different cases of G^1 join polygon segments.

with $a_0^{(1)} = a_0^{(1)}(p_1^{(1)}) = 0$, where we relate the coefficients $a_i^{(1)}$ to $p_1^{(1)}$ to emphasize that the expansion is performed at $p_1^{(1)}$. Similarly, $F_2(\alpha_1, 1-\alpha_1-\alpha_3, \alpha_3) = 0$ can be represented as

$$\alpha_1 = \sum_{i=0}^{\infty} a_i^{(2)} \alpha_3^i = \sum_{i=0}^{\infty} a_i^{(2)}\bigl(p_2^{(2)}\bigr)\alpha_3^i, \quad \alpha_2 = 1-\alpha_1-\alpha_3, \tag{4.2}$$

at $p_2^{(2)}$ with $a_0^{(2)} = 0$. It follows from Theorem 3.2 that the curve $F_1 = 0$ is tangent with $[p_1^{(1)} v_1^{(1)}]$ $n-2$ times at $p_1^{(1)}$ if and only if

$$b_{n-k,0,k}^{(1)} = 0, \qquad \text{for } k = 0, 1, \dots, n-2, \tag{4.3}$$

or if and only if $a_k^{(1)} = 0$, for $k = 0, 1, \dots, n-2$. The same is true for the curve $F_2 = 0$ at $p_2^{(2)}$, that is, the curve is tangent with $[p_2^{(2)} v_1^{(2)}]$ at $p_2^{(2)}$ $n-2$ times if and only if

$$b_{0,n-k,k}^{(2)} = 0, \qquad \text{for } k = 0, \dots, n-2. \tag{4.4}$$

Now we assume (4.3) and (4.4) hold. Hence (4.1) and (4.2) become

$$\alpha_2 = \sum_{i=n-1}^{\infty} a_i^{(1)} \alpha_3^i, \quad \alpha_1 = 1-\alpha_2-\alpha_3, \tag{4.5}$$

$$\alpha_1 = \sum_{i=n-1}^{\infty} a_i^{(2)} \alpha_3^i, \quad \alpha_2 = 1-\alpha_1-\alpha_3, \tag{4.6}$$

respectively. Substitute (4.5) into $F_1(1-\alpha_2-\alpha_3, \alpha_2, \alpha_3) = 0$, we get

$$F_1\left(1-\alpha_2(\alpha_3)-\alpha_3, \alpha_2(\alpha_3), \alpha_3\right) = \sum_{i=n-1}^{\infty} g_i^{(1)} \alpha_3^i = 0.$$

From $g_i^{(1)} = 0$ for $i = n-1, \dots, 2n-3$, we derive

$$\tilde{b}_{n-1,10}^{(1)} = -\frac{\tilde{b}_{10,n-1}^{(1)}}{a_{n-1}^{(1)}}, \tag{4.7}$$

$$\tilde{b}_{n-2,11}^{(1)} = -\frac{-\tilde{b}_{10n-1}^{(1)} + \tilde{b}_{00n}^{(1)} + \tilde{b}_{n-1,10}^{(1)}[a_n^{(1)} - (n-1)a_{n-1}^{(1)}]}{a_{n-1}^{(1)}}, \tag{4.8}$$

$$\tilde{b}^{(1)}_{n-i-2,1,i+1} = -\frac{\sum_{j=0}^{i} \tilde{b}^{(1)}_{n-1-j,1,j} \sum_{l-0}^{n-1-j} (-1)^l C^l_{n-1-j} a^{(1)}_{n+i-l-j}}{a^{(1)}_{n-1}}, \tag{4.9}$$

for $i = 1, 2, \ldots, n-3$, where $C_n^k = n!/(k!(n-k)!)$, $a_j^{(1)} = 0$ if $j < n-1$. That is, the coefficients determined by (4.7)–(4.9) will lead to the curve $F_1(\alpha_1, \alpha_2, \alpha_3) = 0$ matching the power series (4.5) up to the first $2n-3$ terms. It is noted that, each of the formulas (4.7)–(4.9) determines one of the coefficients b's, and introduces one of the coefficients a's. Among all the coefficients b's, there is one degree of freedom.

Since $b^{(1)}_{n-1,10} > 0$, $b^{(1)}_{10,n-1} < 0$, (4.7) implies that $a^{(1)}_{n-1} > 0$. The correct sign of $b^{(1)}_{n-1-k,1k}$ can be obtained by giving $a^{(1)}_{n+k-1}$ properly.

For the curve $F_2 = 0$ at $p_2^{(2)}$, we similarly have,

$$F_2(\alpha_1(\alpha_3), 1 - \alpha_1(\alpha_3) - \alpha_3, \alpha_3) = \sum_{i=n-1}^{\infty} g_i^{(2)} \alpha_3^i = 0.$$

From which we have

$$\tilde{b}^{(2)}_{1,n-1,0} = -\frac{\tilde{b}^{(2)}_{01,n-1}}{a^{(2)}_{n-1}}, \tag{4.10}$$

$$\tilde{b}^{(2)}_{1n-2,1} = -\frac{-\tilde{b}^{(2)}_{01,n-1} + \tilde{b}^{(2)}_{00n} + \tilde{b}^{(2)}_{1,n-1,0}[a_n^{(2)} - (n-1)a^{(2)}_{n-1}]}{a^{(2)}_{n-1}}, \tag{4.11}$$

$$\tilde{b}^{(2)}_{1,n-i-2,i+1} = -\frac{\sum_{j=0}^{i} \tilde{b}^{(2)}_{1,n-1-j,j} \sum_{l=0}^{n-1-j} (-1)^l C^l_{n-1-j} a^{(2)}_{n+i-l-j}}{a^{(2)}_{n-1}}, \tag{4.12}$$

for $i = 1, 2, \ldots, n-3$. If we further assume

$$a^{(1)}_{n-1} = b^{(1)}_{1,0,n-1} = 0, \tag{4.13}$$

then similar to the discussion above, we have

$$\tilde{b}^{(1)}_{n-1,1,0} = -\frac{\tilde{b}^{(1)}_{00n}}{a_n^{(1)}}, \tag{4.14}$$

$$\tilde{b}^{(1)}_{n-i-1,1,i} = -\frac{1}{a_n^{(1)}} \left[\sum_{j=0}^{i-1} \tilde{b}^{(1)}_{n-1-j,1,j} \sum_{l=0}^{n-1-j} (-1)^l C^l_{n-1-j} a^{(1)}_{n+i-l-j} \right], \tag{4.15}$$

for $i = 1, 2, \ldots, n-2$. Similarly, for curve $F_2 = 0$ at $p_2^{(2)}$, if we assume

$$a^{(2)}_{n-1} = b^{(2)}_{0,1,n-1} = 0, \tag{4.16}$$

then

$$\tilde{b}^{(2)}_{1,n-10} = -\frac{\tilde{b}^{(2)}_{00n}}{a_n^{(2)}}, \tag{4.17}$$

$$\tilde{b}^{(2)}_{1,n-i-1,i} = -\frac{1}{a_n^{(2)}}\left[\sum_{j=0}^{i-1}\tilde{b}^{(2)}_{1,n-1-j,j}\sum_{l=0}^{n-1-j}(-1)^l C^l_{n-1-j}a^{(2)}_{n+i-l-j}\right], \tag{4.18}$$

for $i = 1, 2, \ldots, n-2$.

Formulas (4.13)–(4.18) match the power series up to the first $2n-2$ terms. If we only fit the first $2n-3$ terms, $b^{(l)}_{1,1,n-2}$ could be free. For the G^3 fitting with cubics in Section 4, we choose it to be zero.

Now we explain why we consider both of the cases of $a^{(l)}_{n-1} > 0$ and $a^{(l)}_{n-1} = 0$. As before, let

$$p_1^{(1)}\widehat{v^{(1)}}_1 p_2^{(1)} \quad \text{and} \quad p_1^{(2)}\widehat{v^{(2)}}_1 p_2^{(2)}$$

be two segments of a polygon. If they G^1 join at $p_1^{(1)}$, then there are two join configurations (see Fig. 2): *nonconvex* join and *convex* join. In the nonconvex join, $p_2^{(1)}$ and $p_1^{(2)}$ lie on different sides of the line $[v_1^{(1)}v_1^{(2)}]$, while in the convex join, $p_2^{(1)}$ and $p_1^{(2)}$ lie on the same side of the line $[v_1^{(1)}v_1^{(2)}]$. Since our A-splines are always contained within the triangles considered, if $p_1^{(1)}$ is of a nonconvex join, then the curve will be tangent with the line $[v_1^{(1)}v_1^{(2)}]$ an odd number of times, otherwise, it will be tangent with the line $[v_1^{(1)}v_1^{(2)}]$ an even number of times. Therefore,

(i) If $p_1^{(1)}$ is of a nonconvex join, n is an even number then $a^{(l)}_{n-1} > 0$ for $l = 1, 2$; if n is an odd number, then $a^{(l)}_{n-1} = 0$.

(ii) If $p_1^{(1)}$ is of a convex join, n is an even number, then $a^{(l)}_{n-1} = 0$ for $l = 1, 2$; if n is an odd number, then $a^{(l)}_{n-1} > 0$.

Theorem 4.1. *The degree n A-spline can achieve G^{2n-3} continuity by fitting locally the given parametric or implicit curve at the join points.*

Note 4.1. If $n > 3$, the coefficients b_{ijk} are free for $i > 1$ and $j > 1$. These degrees $(= (n-2)(n-3)/2)$ of freedom can be used to interpolate/approximate points in the triangle, fairing the constructed curve, or to achieve even higher order continuity at the joins.

Note 4.2. If n is an odd/even number and all the points are nonconvex/ convex join, then each $b_{1,1,n-2}$ is free. This degree of freedom can be used to interpolate points in the triangle or to achieve G^{2n-2} continuity (see Section 4.3).

4.2. Local power series computation

A. Fitting to a parametric curve

Suppose we are given a parametric curve $X(t) = [\phi(t), \psi(t)]^T$ in the neighborhood of $p_1^{(1)}$ and assume $X(0) = p_1^{(1)}$. Now we compute $a_i^{(1)}$ defined in (4.1) ($a_i^{(2)}$ are similarly computed). It follows from (2.1) that the curves (4.5) and (4.6) in Cartesian xy-coordinates can be expressed as

$$\begin{aligned} Y_1(\alpha_3) &= \left[x_1(\alpha_3), y_1(\alpha_3)\right]^T \\ &= p_1^{(1)} + \left[v_1^{(1)} - p_1^{(1)}\right]\alpha_3 + \left[p_2^{(1)} - p_1^{(1)}\right] \sum_{i=n-1}^{\infty} a_i^{(1)}\left(p_1^{(1)}\right)\alpha_3^i, \end{aligned} \tag{4.19}$$

$$\begin{aligned} Y_2(\alpha_3) &= \left[x_2(\alpha_3), y_2(\alpha_3)\right]^T \\ &= p_2^{(2)} + \left[v_1^{(2)} - p_2^{(2)}\right]\alpha_3 + \left[p_1^{(2)} - p_2^{(2)}\right] \sum_{i=n-1}^{\infty} a_i^{(2)}\left(p_2^{(2)}\right)\alpha_3^i. \end{aligned} \tag{4.20}$$

Now we need to determine the so-called β-matrix (see (Seidel, 1993))

$$C^{(l)} = \begin{bmatrix} \beta_1 & & & & \\ \beta_2 & \beta_1^2 & & & \\ \beta_3 & 3\beta_1\beta_2 & \beta_1^3 & & \\ \beta_4 & 3\beta_2^2 + 4\beta_1 & \beta_3 & 6\beta_1^2\beta_2 & \beta_1^4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} = \left[\beta_{ij}^{(l)}(p_1^{(1)})\right]$$

and $a_i^{(l)}(p_1^{(1)})$ for $l = 1, 2$ and $i = n - 1, \ldots$, so that

$$\begin{bmatrix} Y_l'(0) \\ Y_l''(0) \\ \vdots \\ Y_l^k(0) \end{bmatrix} = C^{(l)} \begin{bmatrix} X'(0) \\ X''(0) \\ \vdots \\ X^k(0) \end{bmatrix}, \quad l = 1, 2. \tag{4.21}$$

Eq. (4.21) is the condition of G^k continuity between two parametric curves. From (4.19)–(4.20) and (4.21), we have

$$v_1^{(l)} - p_1^{(1)} = \beta_{11}^{(l)}\left(p_1^{(1)}\right)X'(0).$$

Hence

$$\beta_{11}^{(l)}\left(p_1^{(1)}\right) = (-1)^{l-1}\frac{\|v_1^{(l)} - p_1^{(1)}\|}{\|X'(0)\|}, \quad l = 1, 2.$$

Let

$$n_x = \left(v_1^{(1)} - p_1^{(1)}\right)/\left\|v_1^{(1)} - p_1^{(1)}\right\|$$

and n_y be two unit vectors such that $n_x^T n_y = 0$ and $\det[n_x, n_y] = 1$. Let

$$X^{(i)}(0) = \gamma_i\left(p_1^{(1)}\right)n_y + \delta_i\left(p_1^{(1)}\right)n_x, \quad i = 1, 2, \ldots. \tag{4.22}$$

Then, $\gamma_1(p_1^{(1)}) = 0$, $\delta_1(p_1^{(1)}) = \|X'(0)\|$ and

$$\gamma_i\left(p_1^{(1)}\right) = \det\left[n_x, X^{(i)}(0)\right], \quad \delta_i\left(p_1^{(1)}\right) = \det\left[X^{(i)}(0), n_y\right]$$

and

$$Y_l^{(k)}(0)=\sum_{i=1}^{k}\beta_{ki}^{(l)}\big(p_1^{(1)}\big)X^{(i)}(0)$$
$$=\left[\sum_{i=1}^{k}\beta_{ki}^{(l)}\big(p_1^{(1)}\big)\gamma_i\big(p_1^{(1)}\big)\right]n_y+\left[\sum_{i=1}^{k}\beta_{ki}^{(l)}\big(p_1^{(1)}\big)\delta_i\big(p_1^{(1)}\big)\right]n_x,$$

where $\beta_{ki}^{(l)}(p_1^{(1)})$ are known for $i=2,\dots,k$ and $l=1,2$. Let

$$p_2^{(1)}-p_1^{(1)}=s^{(1)}n_x+t^{(1)}n_y,\qquad p_1^{(2)}-p_2^{(2)}=s^{(2)}n_x+t^{(2)}n_y.$$

It follows from (4.19) and (4.20) that

$$Y_l^{(k)}(0)=k!a_k^{(l)}\big(p_1^{(1)}\big)\big[s^{(l)}n_x+t^{(l)}n_y\big],\quad k\geqslant 2,\ l=1,2.$$

We have

$$\beta_{k1}^{(l)}\big(p_1^{(1)}\big)=\frac{\sum_{i=2}^{k}\beta_{ki}^{(l)}(p_1^{(1)})[s^{(l)}\gamma_i(p_1^{(1)})-t^{(l)}\delta_i(p_1^{(1)})]}{t^{(l)}\|X'(0)\|},$$

$$a_k^{(l)}\big(p_1^{(1)}\big)=\frac{1}{k!t^{(l)}}\sum_{i=2}^{k}\beta_{ki}^{(l)}\big(p_1^{(1)}\big)\gamma_i\big(p_1^{(1)}\big).$$

B. Fitting to discrete data

Suppose we are given a set of points $\{p_i\}$. Let

$$\Delta^0 p_i=p_i,$$
$$\Delta^{j+1}p_i=\frac{\sigma(\Delta^j p_{i+1}-\Delta^j p_i)}{||p_{i+1}-p_i||}+\frac{(1-\sigma)(\Delta^j p_i-\Delta^j p_{i-1})}{||p_i-p_{i-1}||},$$

where $\sigma=||p_{i-1}-p_i||/(||p_{i+1}p-p_i||+||p_i-p_{i-1}||)$. Then $\Delta^j p_i$ can be an approximation of $X^j(t)$ at p_i with $X(t)$ as an imaginary space curve.

The computation of $a_i^{(j)}$ from $X^j(t)$ is the same as before.

C. Fitting to an implicit curve

Let $g(x,y)=0$ be a given implicit curve to be approximated. First compute the singular points and inflection points. These points will divide the curve into smooth and convex segments. For each segment, form a point list by a tracing (see (Bajaj and Xu, 1997)) scheme, such that the normals at two adjacent points have angle $<\pi/2$. Then a G^1 polygon for one segment is formed by the tangent lines at the point list.

For each triangle, say $[p_1p_2v_1]$, the curve $g(x,y)=0$ passes through p_1,p_2 and is tangent with the line $[p_iv_1]$ for $i=1,2$. Let $G(\alpha_1,\alpha_2,\alpha_3)$ be the barycentric form of $g(x,y)$ over $[p_1p_2v_1]$. Let

$$G_1(\alpha_2,\alpha_3)=G(1-\alpha_2-\alpha_3,\alpha_2,\alpha_3),$$

then at p_1, $G_1(\alpha_2,\alpha_3)=0$ can be expressed as a power series

$$\alpha_2=\sum_{i=0}^{\infty}a_i\alpha_3^i$$

by the following algorithm for $f(x,y)=0$. Let

$$f^0(x,y)=f(x,y)=y-a_2x^2+a_0^0(x)+a_1^0(x)y+\cdots+a_n^0(x)y^n$$

with $\mathrm{ord}(a_0^0)>2$. As a function of x, $y=y(x)$ has order $\geqslant 2$. Let $y_1=y-a_2x^2$. Then the order of $y_1=y_1(x)$ is $\geqslant 3$. Let

$$f^1(x,y_1)=f^0(x,y_1+a_2x^2)=y_1-a_3x^3+a_0^1(x)+a_1^1(x)y_1+\cdots+a_n^1(x)y_1^n,$$

then $\mathrm{ord}(a_0^1)>3$. Repeating this procedure, we get $a_2x^2, a_3x^3, \ldots$. Then $\sum_{i=2}^{\infty}a_ix^i$ is the power series expansion.

This algorithm is simple and easy to implement. If we want to compute $a_2x^2, a_3x^3, \ldots$ up to a_kx^k, then the terms in $a_i^j(x)$ with degree $> k-(j+2)^i$ can be deleted during the computation, since these terms have no contribution to $\sum_{i=2}^{k}a_ix^i$. Hence the algorithm is also space effective.

4.3. Cubic A-splines example

As an example, we describe cubic A-splines in detail. We omit the detail discussion of quadratic A-splines, since it is easier and the conclusions arrived are similar to the ones in the literature (Farin, 1990) and (Pratt, 1985).

Suppose we are given parametric data at the join points, that is X^k, $k=1,2,\ldots$. We shall determine the coefficients b_{ijk} so that G^{2n-3} continuity is achieved. Furthermore, for the error estimation (see Section 5), we require

$$b_{ijk+e_3}<\frac{b_{ijk+e_1}}{2}+\frac{b_{ijk+e_2}}{2}. \tag{4.23}$$

G^3 continuity. Consider a two segments cubic A-spline as in Section 4.1. Now suppose p_1 is the join point and assume that the coefficients $b_{111}^{(l)}=0$ for both segments. There are two cases that need to be considered:

Case 1. *p_1 is of nonconvex join.*

In this case, we have

$$a_2^{(1)}(p_1)=a_2^{(2)}(p_1)=0,\quad \tilde b_{210}^{(1)}=\frac{1}{a_3^{(1)}(p_1)},\quad a_3^{(1)}(p_1)>0,$$
$$\tilde b_{120}^{(2)}=\frac{1}{a_3^{(2)}(p_1)},\quad a_3^{(2)}(p_1)>0.$$

Since

$$\begin{aligned}a_3^{(l)}&=\frac{1}{6t^{(l)}(p_1)}\bigl[\beta_{32}^{(l)}(p_1)\gamma_2(p_1)+\beta_{33}^{(l)}(p_1)\gamma_3(p_1)\bigr]\\&=\frac{1}{6t^{(l)}(p_1)}\bigl[\beta_{33}^{(l)}(p_1)\gamma_3(p_1)\bigr]=\frac{1}{6t^{(l)}(p_1)}\bigl[\bigl(\beta_{11}^{(l)}(p_1)\bigr)^3\gamma_3(p_1)\bigr]\end{aligned}$$

and $(-1)^{l-1}\beta_{11}^{(1)}(p_1)>0$, we have $(-1)^{l-1}t^{(l)}(p_1)\gamma_3(p_1)>0$. The geometric meaning is $X'''(0)$ and p_2-p_1 are on the same side of the line $\langle p_1v_1\rangle$.

Case 2. *p_1 is of convex join.*

In this case, we have

$$\tilde{b}_{210}^{(1)} = -\frac{\tilde{b}_{102}^{(1)}}{a_2^{(1)}(p_1)}, \qquad \tilde{b}_{120}^{(2)} = -\frac{\tilde{b}_{012}^{(2)}}{a_2^{(2)}(p_1)},$$

$$\tilde{b}_{111}^{(1)} = \frac{\tilde{b}_{102}^{(1)}[a_3^{(1)}(p_1) - a_2^{(1)}(p_1)] + a_2^{(1)}(p_1)}{[a_2^{(1)}(p_1)]^2} = 0, \tag{4.24}$$

$$\tilde{b}_{111}^{(2)} = \frac{\tilde{b}_{012}^{(2)}[a_3^{(2)}(p_1) - a_2^{(2)}(p_1)] + a_2^{(2)}(p_1)}{[a_2^{(2)}(p_1)]^2} = 0. \tag{4.25}$$

Since $\tilde{b}_{102}^{(1)} < 0$, $\tilde{b}_{012}^{(2)} < 0$, we require $a_2^{(l)}(p_1) > 0$, $l = 1, 2$. Since

$$a_2^{(l)} = \beta_{22}^{(l)}(p_1)\gamma_2(p_1)/2t^{(l)}(p_1),$$

we need to have $t^{(l)}(p_1)\gamma_2(p_1) > 0$. Hence $X''(0)$ points to the inside of the polygon. It follows from (4.24) and (4.25) that

$$\tilde{b}_{102}^{(1)} = -\frac{a_2^{(1)}(p_1)}{a_3^{(1)}(p_1) - a_2^{(1)}(p_1)}, \qquad \tilde{b}_{012}^{(2)} = -\frac{a_2^{(2)}(p_1)}{a_3^{(2)}(p_1) - a_2^{(2)}(p_1)}.$$

In order to satisfy (4.23), we require

$$3a_3^{(l)}(p_1) - 4a_2^{(l)}(p_1) \geqslant 0, \quad l = 1, 2.$$

These two inequalities, which have three unknowns $\gamma_2(p_1), \delta_2(p_1), \gamma_3(p_1)$, will have infinitely many solutions. Therefore, we have proved the following theorem.

Theorem 4.2. *At each nonconvex join point, if X', X'' and X''' are given such that $\gamma_1 = \gamma_2 = 0$, $(-1)^{l-1}t^{(l)}\gamma_3 > 0$ and at each convex join point, if X', X'' and X''' are given such that $\gamma_1 = 0$, $t^{(l)}\gamma_2 > 0$ and $3a_3^{(l)} - 4a_2^{(l)} \geqslant 0$, $l = 1, 2$, then G^3 continuous cubic A-splines exist that fit the given data X', X'' and X'''* (*with possibly different magnitudes*).

G^4 continuity. In order to achieve G^4 continuity, we assume each join point is a nonconvex join. Consider the curve

$$F = \sum_{i+j+k=3} b_{ijk} B_{ijk}^n = 0$$

on the triangle $[p_1 p_2 v_1]$. All the coefficients, except b_{111} that is free, are determined as in the G^3 continuity case. Now we use the free b_{111} to achieve G^4 continuity. It follows from (4.14)–(4.18) that

$$\tilde{b}_{111}^{(1)} = -\frac{a_4^{(1)}(p_1) - 2a_3^{(1)}(p_1)}{[a_3^{(1)}(p_1)]^2} = -\frac{a_4^{(2)}(p_2) - 2a_3^{(2)}(p_2)}{[a_3^{(2)}(p_2)]^2}. \tag{4.26}$$

Since

$$a_4^{(l)}(p_l) = \frac{\beta_{43}^{(l)}(p_l)\gamma_3(p_l) + \beta_{44}^{(l)}(p_l)\gamma_4(p_l)}{24t^{(l)}(p_l)}, \quad l = 1, 2,$$

$a_4^{(l)}(p_l)$ depend linearly on $\gamma_4(p_l)$. Hence (4.26) can be written as

$$c_1\gamma_4(p_1) + c_2\gamma_4(p_2) = c_3. \tag{4.27}$$

This system of linear equations always have solutions and has one degree of freedom. Therefore, we have

Theorem 4.3. *If each join point is a nonconvex join, and the data* $X^{(i)}$ $(i = 1, \ldots, 4)$ *at each join point are given such that*

$$\gamma_1 = \gamma_2 = 0, \qquad (-1)^{l-1}t^{(l)}\gamma_3 > 0, \quad l = 1, 2,$$

and (4.27) *holds, then the* G^4 *continuous cubic A-splines exist that fits the given points and derivative data.*

5. Computable error bounds

First we define the notion of approximation error. We consider a A-spline segment defined in a given triangle $\Delta = [p_1 p_2 v_1]$ which approximate either a discrete points set or a parametric polynomial curve or an implicit curve within the same triangle. Our purpose here is to provide a computable error bound when the approximant is obtained within the triangle. In all the cases, without loss of generality, we assume that we are given a points set A and an A-spline S: $F(\alpha_1, \alpha_2, \alpha_3) = 0$. We define the error between A and S to be

$$E(A, S) = \sup_{x\in A} \inf_{y\in S} \|x - y\|.$$

It should be noted that $E(A, S)$ and $E(S, A)$ are not equal in general. Let $d = (-1/2, -1/2, 1)$ be a direction in the triangle Δ, that is parallel to the line $[v_1, (1/2)(p_1 + p_2)]$. Then $E(A, S)$ can be bounded by (see (Sederberg et al., 1988))

$$\frac{\sup_{p\in A} |F(p)|}{\inf_{q\in\Delta} |D_d F(q)|},$$

where D_d stands for the directional derivative in the direction d. Hence the problem is how to compute $\sup_{p\in A} |F(p)|$ and $\inf_{q\in\Delta} |D_d F(q)|$. If A is a discrete points set then $F(p)$ can be computed by Casteljau algorithm. If A is polynomial curve $X(t)$ (assume it is in Bézier form), then the composition $g(t) = F(X(t))$ can also be computed. Hence $F(X(t))$ is bounded by the Bézier coefficients of $g(t)$. If A is an implicit curve, that is A: $G(\alpha_1, \alpha_2, \alpha_3) = 0$, then first increase the degree of F or G so that they have same degree. Since $F = 0$ or $G = 0$ can have a constant multiplier, we normalize G by a factor α with

$$\alpha = \sum b_{ijk}c_{ijk} \Big/ \sum c_{ijk}^2,$$

where b_{ijk} and c_{ijk} are the coefficients of F and G. Then

$$\sup_{p\in A} \bigl|F(p)\bigr| \leqslant \sup_{p\in\Delta} \bigl|F(p) - G(p)\bigr| \leqslant \max_{ijk} |b_{ijk} - c_{ijk}|.$$

Now we compute $\inf_{q\in\Delta} |D_d F(q)|$. Let

$$F(\alpha_1,\alpha_2,\alpha_3) = \sum_{i+j+k=n} b_{ijk} B^n_{ijk}(\alpha_1,\alpha_2,\alpha_3).$$

Then

$$D_d F(\alpha_1,\alpha_2,\alpha_3) = \sum_{i+j+k=n-1} b'_{ijk} B^{n-1}_{ijk}(\alpha_1,\alpha_2,\alpha_3)$$

with

$$b'_{ijk} = n\bigl(b_{ijk+e_3} - \tfrac{1}{2} b_{ijk+e_1} - \tfrac{1}{2} b_{ijk+e_2}\bigr).$$

If $b'_{ijk} < 0$ for $i+j+k = n-1$, then $D_d F(\alpha_1,\alpha_2,\alpha_3) < 0$ in the triangle Δ. Hence

$$\inf_{q\in\Delta} |D_d F(q)| \geqslant \min_{ijk} |b'_{ijk}|.$$

For quadratic A-splines, condition (4.23) holds. Hence $b'_{ijk} < 0$ and

$$b'_{100} = -b_{110}, \qquad b'_{010} = -b_{110}, \qquad b'_{001} = -2.$$

Therefore, $\inf_{q\in\Delta} |D_d F(q)| \geqslant \min\{2,\ b_{110}\}$. For cubic G^3 A-splines, our construction guarantees that

$$b'_{200} < 0, \qquad b'_{110} < 0, \qquad b'_{020} < 0, \qquad b'_{101} \leqslant 0, \qquad b'_{011} \leqslant 0, \qquad b'_{002} < 0.$$

It is then not difficult to show that

$$\inf_{q\in\Delta} |D_d F(q)| \geqslant -\frac{b'_{200} b'_{020} b'_{002}}{b'_{200} b'_{020} + b'_{020} b'_{002} + b'_{200} b'_{002}}.$$

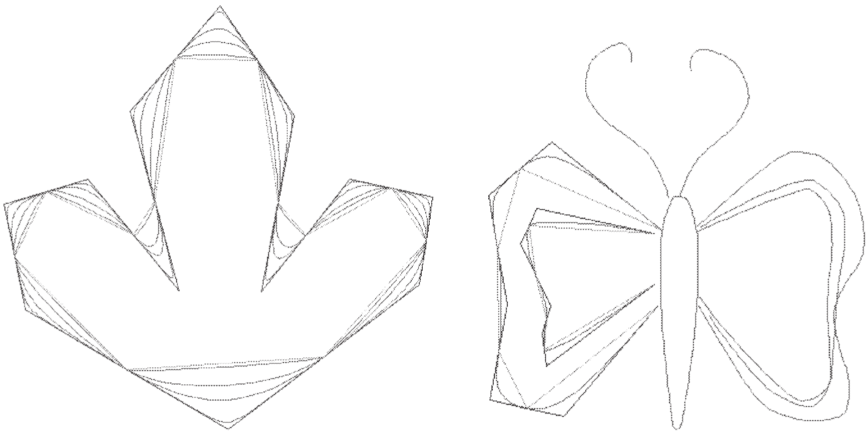
6. Conclusion and examples

We have presented sufficient conditions of the BB form of bivariate algebraic polynomials such that the zero contour of the polynomials define a single sheeted real curve segment in the given triangle. We have shown that the degree n ($n \geqslant 2$) A-splines can achieve in general G^{2n-3} continuity by local fitting and still have degrees of freedom to achieve locally data approximated.

As an example, the cubic A-splines are carefully analyzed, and resulting smoothness conditions are derived for local interpolation and approximation. Cubic A-splines can G^3 approximate a polygon with one free parameter $b_{111} \leqslant 0$. This parameter can be used to control the shape of the curve, but its influence on the curve is of limited when the derivatives at the end points are fixed. Therefore, we take b_{111} to be zero. However, with a change of derivatives at the end points, a desirable shape of the curve can be obtained (see Fig. 3(a)). Furthermore, if the polygon is of a nonconvex join, then a G^4 smooth curve can also be constructed.

Fig. 3(a) shows the G^3 cubic A-spline curve family for a given rather regular closed polygon. Fig. 3(b) shows G^4 quartic A-spline curves for several open polygons. Fig. 4 is to use G^3 cubic A-splines to fit a cluster of points with different control errors in breaking the points. As an application of A-splines, we show an example, in Fig. 5, of iso-contour reconstructions of a human head from volume MRI (Magnetic Resonance Imaging) data, using G^3 cubic A-splines. Fig. 6 shows G^3 cubic A-splines approximations of degree six and degree four algebraic plane curves:

$$\text{(a) } \left(x^2+y^2\right)^3-4x^2y^2=0, \quad \text{(b) } x^3-3x-(1/9)\left(y^4-12y^2+18\right).$$



(a) (b)

Fig. 3. (a) Family of G^3 cubic A-splines defined for an input closed polygon (dark line) with C^3 data at vertices; (b) G^4 quartic A-splines defined for input polygons with C^4 data at vertices. Only the polygons on the left "wing" are shown. Note the intersecting A-splines on the right "wing" are produced by having intersecting input polygons.

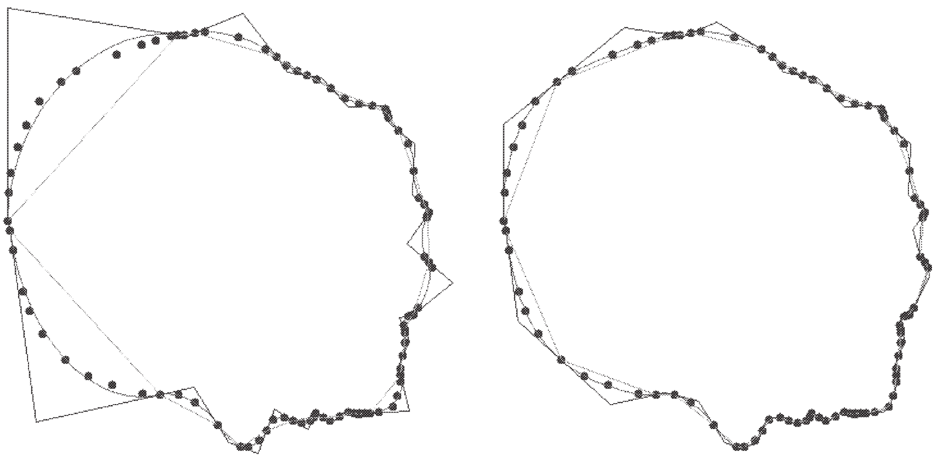


Fig. 4. G^3 cubic A-splines fit of head data with different approximation errors. The picture on the left has fewer number of pieces and has larger error than the picture on the right.

Fig. 5. G^3 cubic A-splines approximation of a stack of Magnetic Resonance Imaging volumetric cross-sectional data.

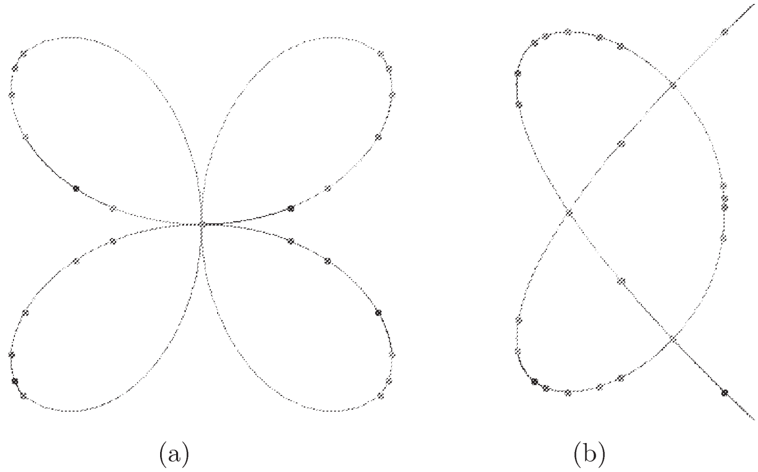


Fig. 6. A-spline approximation of implicit algebraic curves: (a) $(x^2 + y^2)^3 - 4x^2y^2 = 0$; (b) $x^3 - 3x - (1/9)(y^4 - 12y^2 + 18)$. The curve segments between consecutive vertices (dots) are all cubic degree and with G^3 continuity at the vertices.

The break points on the curves are generated by the tracing scheme in (Bajaj and Xu, 1997).

Several open problems remain. One, the faster and robust methods of A-splines display based on subdivision or integer forward differencing need to be developed. Second, applications of these A-splines with comparison to parametric B-splines, to problems in image processing, computer graphics, animation and geometric modeling need to be fully explored.

Appendix A

Proof of Theorem 3.1. Substituting $(\alpha_1, \alpha_2, \alpha_3)(t)$ into $F(\alpha_1, \alpha_2, \alpha_3)$ we have

$$\begin{aligned} B_\beta(t) :&= F((1-t)\beta, (1-t)(1-\beta), t) \\ &= \sum_{i+j+k=n} b_{ijk} \frac{n!}{i!j!k!} t^k (1-t)^{i+j} \beta^i (1-\beta)^j \\ &= \sum_{i+j+k=n} b_{ijk} B_k^n(t) B_i^{i+j}(\beta) \\ &= \sum_{k=0}^{n} b_k(\beta) B_k^n(t), \end{aligned} \tag{A.1}$$

where

$$b_k(\beta) = \sum_{i+j=n-k} b_{ijk} B_i^{n-k}(\beta), \qquad B_j^n(t) = \frac{n!}{j!(n-j)!} t^j (1-t)^{n-j}.$$

It follows from (3.1)–(3.2) that $b_0(\beta) > 0$, $b_k(\beta) \geqslant 0$, $k = 1, \dots, K-1$, $b_k(\beta) \leqslant 0$, $k = K+1, \dots, n$. If l $(0 \leqslant l \leqslant n-K-1)$ is the integer such that $b_n(\beta) = \cdots = b_{n-l+1}(\beta) = 0$; $b_{n-l}(\beta) < 0$, then $B_\beta(t)$ can be written as

$$B_\beta(t) = (1-t)^l \sum_{k=0}^{n-l} c_k(\beta) B_k^{n-l}(t),$$

where $c_0 > 0$, $c_{n-l} < 0$ and the sequence $c_0, c_1, \dots, c_{n-l}$ has one sign change. By variation diminishing property (Farin, 1990), the equation $B_\beta(t) = 0$ has exactly one root in $(0, 1)$. $\square$

Proof of Theorem 3.2. (i) Let $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$ be a singular point of $F(\alpha_1, \alpha_2, \alpha_3) = 0$, i.e.,

$$\nabla f = [\nabla\alpha_1, \nabla\alpha_2, \nabla\alpha_3] \nabla F = 0 \tag{A.2}$$

at $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$, where

$$\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right]^T, \qquad \nabla\alpha_i = \left[\frac{\partial \alpha_i}{\partial x}, \frac{\partial \alpha_i}{\partial y}\right]^T, \qquad \nabla F = \left[\frac{\partial F}{\partial \alpha_1}, \frac{\partial F}{\partial \alpha_2}, \frac{\partial F}{\partial \alpha_3}\right]^T.$$

Since the rank of the matrix $[\nabla\alpha_1, \nabla\alpha_2, \nabla\alpha_3]$ is two and $\nabla\alpha_1 + \nabla\alpha_2 + \nabla\alpha_3 = 0$, we have

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \frac{\partial F}{\partial \alpha_3}.$$

Then $B_\beta(t) = 0$ and

$$B'_\beta(t) = -\frac{\partial F}{\partial \alpha_1}\beta - \frac{\partial F}{\partial \alpha_2}(1-\beta) + \frac{\partial F}{\partial \alpha_3} = 0$$

at $(\alpha_1^*, \alpha_2^*, \alpha_3^*)$. That is, t is a double zero of $B_\beta(t)$ and a contradiction to Theorem 3.1.

(ii) Since $\alpha_2 = 0$ corresponds to $\beta = 1$ in the proof of Theorem 3.1, hence in the given case

$$B_1(t) = \sum_{k=0}^{n} b_{n-k,0k} B_k^n(t) = \sum_{k=K+1}^{n} b_{n-k,0k} B_k^n(t) = t^{K+1} G(t),$$

where

$$G(t) = \sum_{k=K+1}^{n} b_{n-k,0k} \frac{n!}{k!(n-k)!} t^{k-(K+1)} (1-t)^{n-k}$$

has no zero on $[0, 1)$ because its coefficients have same sign and the first coefficient is negative. That is, $t = 0$ is the only zero of $B_1(t)$ on $[0, 1)$ and has multiplicity $K + 1$. The second conclusion in this item can be similarly proved.

(iii) If $b_{n00} = b_{n-1,01} = b_{n-1,10} = 0$, then

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \frac{\partial F}{\partial \alpha_3} = 0$$

at p_1. Hence, by (A.2), $\nabla f = 0$. That is, p_1 is a singular point of the curve. At p_2, the same conclusion holds. $\square$

Proof of Theorem 3.3. To prove the theorem, we first prove the following fact:

If P is an inflection point of the cubic algebraic curve $f(x, y) = 0$ and $L^*(x, y) = ax + by + c = 0$ is the tangent line passing through P, then L^* separates the curve into two parts, one part is located in the half space $L^*(x, y) < 0$, the other part is located in the half space $L^*(x, y) > 0$. That is, P is the only intersection point between L and the curve.

Suppose L^* can be written as $x = ky + b$ and $P = (x^*, y^*)$. Then by the definition (see (Walker, 1978, p. 71)) of inflection point we know that y^* is a triple zero of $f(ky + b, y)$, i.e., $f(ky + b, y) = a(y - y^*)^3$ for some nonzero constant a. This means that the curve is located on both sides of L^* and the curve cannot intersect with line L^* at any other point by Bezout's theorem (Walker, 1978).

Now we prove the theorem with the aid of some geometric intuition (see Fig. 7) although it is easy to translate it into algebra. Suppose to the contrary, there are inflection points in the triangle $[p_1 p_2 v_1]$. Let $p^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*) = (1 - t^*)(\beta^*, (1 - \beta^*), 0) + t^*(0, 0, 1)$ be the

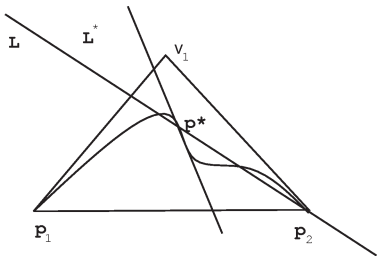


Fig. 7. L intersects with the curve four times if it has an inflection point.

first one, i.e., its β-coordinate is maximal. Now the curve in the triangle is divided into two parts by this inflection point. The first part, say C_1, corresponds to $[\beta^*, 1]$ and the second part, C_2, corresponds to $[0, \beta^*]$. Let L^* be the tangent line of the curve at this inflection point. From the fact proved above, the line L^* cannot intersect with both line segments (p_1, v_1) and (v_1, p_2). Otherwise, the curve segment cannot pass through both vertices p_1 and p_2. The only cases are that L^* intersects with either $(p_1, v_1]$ and $[p_1, p_2]$ or $(p_2, v_1]$ and $[p_1, p_2]$. Without loss of generality, we assume that L^* intersects with $(p_1, v_1]$ and $[p_1, p_2]$. In this case, the line L^* is not parallel to $[p_1, p_2]$ and $(p_1, v_1]$. It follows from the fact above that C_1 is below the line L^* since p_1 is so, and similarly C_2 is above that line. Now, let L be another line that pass through p^* and p_2. If L coincides with L^*, then a contradiction is obtained by Bezout's theorem, because L has four intersection points with the cubic. So L does not coincide with L^*. By the fact that the slope of L is smaller than the slope of L^*, we have that C_1 will intersect with line L (except point p^*). By the same reason and the fact that $[v_1, p_2]$ is tangent to the curve, we can conclude that the segment C_2 intersects with L in addition to p^* and p_2. So L intersects with our cubic algebraic curve four times. This again contradicts with Bezout's theorem. So the segment inside the triangle is convex. $\square$

Proof of Theorem 4.1. For the given parametric curve $X(t)$ or the implicit curve around the join point $p_1^{(1)}$, we compute (see Section 4.2) the two local power series. We next show that we can choose the coefficients $b_{ijk}^{(l)}$ of F_l $(l = 1, 2)$ so that the curves $F_l = 0$ fit the two local power series up to the first $2n - 3$ terms, respectively.

Suppose n is an even number. If $p_1^{(1)}$ is of nonconvex join, then the coefficients defined by (4.7)–(4.9) and (4.10)–(4.12) which make up the curves $F_1(1 - \alpha_2 - \alpha_3, \alpha_2, \alpha_3) = 0$ and $F_2(\alpha_1, 1 - \alpha_1 - \alpha_3, \alpha_3) = 0$ fit the two given power series up to the first $2n - 3$ terms. The coefficients

$$b_{n-k-1,1,k}^{(1)} \quad \text{and} \quad b_{1,n-k-1,k}^{(2)}, \quad k = 0, 1, \dots, n-2,$$

are determined from $a_i^{(l)}$, with $b_{1,0,n-1}^{(1)} < 0$ and $b_{0,1,n-1}^{(2)} < 0$ chosen to be free parameters. All these coefficients, except $b_{1,1,n-2}^{(l)}$, are independent of the data at $p_2^{(1)}$ and $p_1^{(2)}$.

If $p_1^{(1)}$ is of convex join, then

$$b_{1,0,n-1}^{(1)} = b_{0,1,n-1}^{(2)} = 0,$$

conditions (4.14)–(4.15) and (4.17)–(4.18) imply that $b_{n-k-1,1,k}^{(1)}$ and $b_{1,n-k-1,k}^{(2)}$, $k = 0, 1, \dots, n-3$, which make up the curves $F_1(1 - \alpha_2 - \alpha_3, \alpha_2, \alpha_3) = 0$ and $F_2(\alpha_1, 1 - \alpha_1 - \alpha_3\alpha_3) = 0$ fit the given power series up to the first $2n - 3$ terms, respectively. The coefficient $b_{1,1,n-2}^{(l)}$ is not involved. The case of n being an odd number is similar. $\square$

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