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C^1 modeling with A-patches from rational trivariate functions

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Abstract

We approximate a manifold triangulation in $\mathbb{R}^3$ using smooth implicit algebraic surface patches, which we call A-patches. Here each A-patch is a real iso-contour of a trivariate rational function defined within a tetrahedron. The rational trivariate function provides increased degrees of freedom so that the number of surface patches needed for free-form shape modeling is significantly reduced compared to earlier similar approaches. Furthermore, the surface patches have quadratic precision, that is they exactly recover quadratic surfaces. We give conditions under which a C^1 smooth and single sheeted surface patch is isolated from the multiple sheets. © 2001 Elsevier Science B.V. All rights reserved.

Keywords: Algebraic surface; Rational A-patch; Surface fit; Triangulation

1. Introduction

We begin with a manifold triangulation (often considered the preprocessing step of computational geometric design). We construct a C^1 smooth collection of implicit algebraic (polynomial) surface patches defined within tetrahedra, which we call A-patches (Bajaj et al., 1995b; Bajaj, 1997). This family of A-patches are real iso-contours of trivariate rational functions (ratio of polynomials) which interpolate the vertices of the input manifold triangulation and provide smooth approximations of the input shape.

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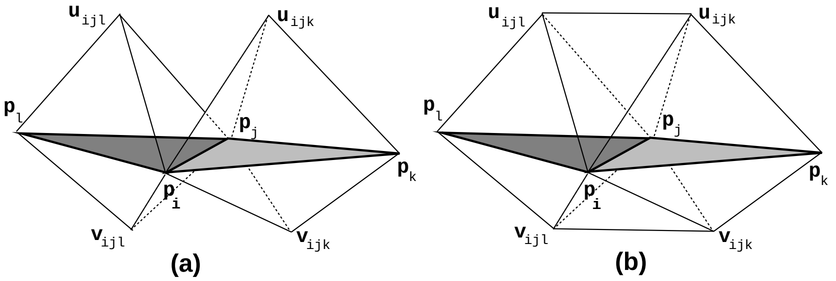

Fig. 1. (a) Face tetrahedra. (b) Face tetrahedra and edge tetrahedra.

Various approaches of using implicit surface representation in modeling geometric objects or reconstructing the image to scattered data have been described in papers (see, for example (Bajaj, 1993; Dahmen and Thamm-Schaar, 1993; Guo, 1991a; Lodha, 1992; Sederberg, 1985; Bajaj et al., 1995b)). Prior schemes using implicit surface representations interpolating vertices of a surface triangulation $\mathcal{T}$ use various simplicial hulls (see (Dahmen and Thamm-Schaar, 1993; Guo, 1991a, 1991b; Bajaj et al., 1995b; Bajaj, 1997)). It consists of the following three steps:

(a) Generate a normal for each vertex of $\mathcal{T}$ which will also be the normal of the constructed smooth surface at the vertex.
(b) Build a surrounding simplicial hull Σ (consisting of a series of tetrahedra) of the triangulation.
(c) Construct a piecewise trivariate polynomial F within that simplicial hull, and use the zero contour of F to represent the surface.

Dahmen (1989) first proposed an approach for constructing a simplicial hull of $\mathcal{T}$. In this approach, for each face $[p_i p_j p_k]$ of $\mathcal{T}$, two points u_{ijk} and v_{ijk} off each side of the face are chosen and two tetrahedra $[p_i p_j p_k u_{ijk}]$ and $[p_i p_j p_k v_{ijk}]$ (called face tetrahedra) are constructed. For each edge of $\mathcal{T}$, two tetrahedra (called edge tetrahedra) are formed that blend the neighboring face tetrahedra (see Fig. 1). The collection of these tetrahedra contains the tangent plane near the vertices and have no self-intersection. Since such simplicial hulls are nontrivial to construct for arbitrary triangulation, several improvements have been made in later publications to overcome the difficulties (see (Dahmen and Thamm-Schaar, 1993; Guo, 1991a, 1991b; Bajaj et al., 1995b)). For the construction of the surface within Σ, Dahmen (1989) used six quadric patches for each face tetrahedron and four quadric patches for each edge tetrahedron. Guo (1991a) uses a Clough–Tocher split to subdivide each face tetrahedron of the simplicial hull, hence utilizing three cubic patches per face of $\mathcal{T}$. The edge tetrahedra are subdivided into two. Dahmen and Thamm-Schaar (1993) do not split the face tetrahedra, but the edge tetrahedra is split. All of these papers provided heuristics to overcome the multiple-sheeted and singularity problem of the implicit patches. Since the multi-sheeted property may cause the constructed surface to be disconnected, Bajaj et al. (1995b) constructed A-patches that were guaranteed to be nonsingular, connected and single sheeted within each tetrahedron.

We too use the simplicial hull approach in this paper, however, we give up the requirement of being single sheeted within each tetrahedron. The idea we exploit is to

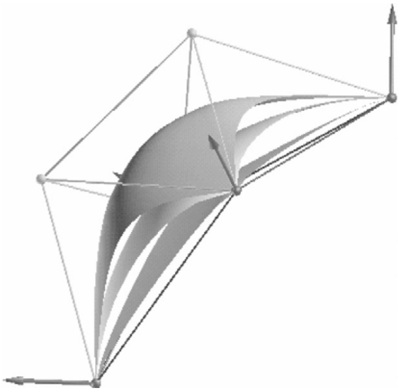

Fig. 2. Surface family.

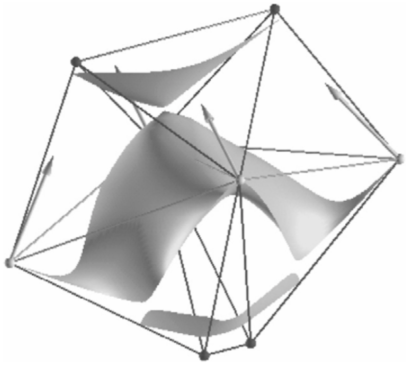

Fig. 3. Multiple sheeted surface.

separate a smooth and single sheeted surface piece from the multi-sheeted patches within each tetrahedron, to construct a smooth and connected surface. Since the single sheeted requirement of the real zero contour in tetrahedra leads to much stronger conditions on the coefficients of F than that of allowing the existence of multi-sheeted surfaces, the present scheme yields much broader candidates of allowable surface families. Fig. 2 shows such a surface family over two adjacent triangles. Fig. 3 shows the multiple sheets property of the surface.

We use A-patches derived from rational trivariate functions, that is F is piecewise rational function defined on Σ. This rational form provides extra degrees of freedom in the construction of the C^1 function F so that often we do not need to split the edge or face tetrahedra. Suppose v, e and f are the vertex, edge and face numbers of the triangulation $\mathcal{T}$. Then, if the surface considered is of genus zero, Euler's formula yields $v + f - e = 2$. It follows from this formula that $e = 3v - 6$, $f = 2v - 4$. Then the ratio of the patch numbers of our earlier scheme (Bajaj et al., 1995b) and the present one is about $8:5$. Therefore, the patch number is significantly reduced. Although the modeling function F is rational in form, it is evaluated as easily as a cubic polynomial (details in Sections 4 and 5). Furthermore, the constructed surface has the "plane recovery" property, that is, if the three normals at the vertices of a triangle are all perpendicular to the triangle, then the surface coincides with the triangle. Having this feature is important since many geometric CAGD models have planar regions adjacent to curved patches. Even further, the surface constructed could also recover quadric, i.e., possesses quadratic precision.

The paper is organized as follows. Section 2 provides notation and some necessary lemmas. Section 3 builds the simplicial hull. The construction of the trivariate rational function F and the computation of the coefficients of F are provided in Section 4. In Section 5, we present schemes to evaluate the rational patches. Examples that show the effectiveness of the schemes are presented in Section 6. The proof the regularity of the constructed A-patches is given in Appendix A.

2. Notations and preliminary details

The trivariate polynomials and rational functions used in this paper are expressed in Bernstein–Bézier (BB) form over tetrahedra. Let $p_1, p_2, p_3, p_4 \in \mathbb{R}^3$ be affine independent. Then the tetrahedron with vertices p_1, p_2, p_3, and p_4 is defined by

$$[p_1 p_2 p_3 p_4] := \left\{ p \in \mathbb{R}^3\colon\ p = \sum_{i=1}^{4} \alpha_i p_i,\ \alpha_i \geqslant 0,\ \sum_{i=1}^{4} \alpha_i = 1 \right\}.$$

For any $p = \sum_{i=1}^4 \alpha_i p_i \in [p_1 p_2 p_3 p_4]$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p. Any polynomial $F(p)$ of degree n then can be expressed as BB form over $[p_1 p_2 p_3 p_4]$ as $F(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$, $\lambda \in \mathcal{Z}_+^4$, where

$$B_\lambda^n(\alpha) = \frac{n!}{\lambda_1! \cdots \lambda_4!} \alpha_1^{\lambda_1} \cdots \alpha_4^{\lambda_4}$$

is Bernstein polynomial, $|\lambda| = \sum_{i=1}^4 \lambda_i$ with $\lambda = (\lambda_1, \ldots, \lambda_4)^{\mathrm{T}} = \sum_{i=1}^4 \lambda_i e_i$, the coefficients $b_\lambda = b_{\lambda_1\lambda_2\lambda_3\lambda_4}$ (as a subscript, we simply write λ as $\lambda_1\lambda_2\lambda_3\lambda_4$) are called weights, and $\mathcal{Z}_+^4$ stands for the set of all four dimensional vectors with nonnegative integer components. To simplify notation, we use $F(x, y, z)$, $F(p)$, $F(\alpha)$ and $F(\alpha_1, \ldots, \alpha_4)$ to denote the same trivariate function F. Lemmas 2.1 and 2.2 in the following give conditions of C^1 join of BB form polynomials.

Lemma 2.1 (Farin, 1990). *Let $F(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ be defined on the tetrahedron $[p_1 p_2 p_3 p_4]$, then*

$$b_{(n-1)e_i+e_j} = b_{ne_i} + \frac{1}{n}(p_j - p_i)^{\mathrm{T}} \nabla F(p_i), \quad j = 1, \ldots, 4;\ j \neq i, \tag{2.1}$$

where $\nabla F(p) = [\frac{\partial F(p)}{\partial x}, \frac{\partial F(p)}{\partial y}, \frac{\partial F(p)}{\partial z}]^{\mathrm{T}}$.

Lemma 2.2 (Farin, 1990). *Let $F(p) = \sum_{|\lambda|=n} a_\lambda B_\lambda^n(\alpha)$ and $G(p) = \sum_{|\lambda|=n} b_\lambda B_\lambda^n(\alpha)$ be two polynomials defined on the tetrahedra $[p_1 p_2 p_3 p_4]$ and $[p_1' p_2\ p_3 p_4]$, respectively. Then F and G are C^1 at the face $[p_2 p_3 p_4]$ iff*

$$a_{0\lambda_2\lambda_3\lambda_4} = b_{0\lambda_2\lambda_3\lambda_4}, \quad \lambda_2 + \lambda_3 + \lambda_4 = n, \tag{2.2}$$

$$b_{1\lambda_2\lambda_3\lambda_4} = \sum_{i=1}^{4} \beta_i a_{0\lambda_2\lambda_3\lambda_4+e_i}, \quad \lambda_2 + \lambda_3 + \lambda_4 = n - 1, \tag{2.3}$$

where $(\beta_1, \beta_2, \beta_3, \beta_4)^{\mathrm{T}}$ is the barycentric coordinate of p_1' about $[p_1 p_2 p_3 p_4]$.

Lemma 2.3. *Let $F(t, b) = \sum_{i=0}^3 b_i B_i^3(t) + \beta t^2 b$ with $b_0 < 0$ and $0 < \beta$. Let $F_j(t) = \sum_{i=0}^j b_i B_i^j(t)$, $j = 0, \ldots, 3$, and*

$$b_{\min}^{(1)} = \min_{t\in[0,1]} \max\left\{ -\frac{3F_2(t)}{\beta t^2},\ -\frac{F_3(t)}{\beta t^2} \right\}, \qquad b_{\min}^{(2)} = \min_{t\in[0,1]\cap\{t:F_1(t)\leqslant 0\}} \left\{ -\frac{F_3(t)}{\beta t^2} \right\}.$$

Then for any given real b, if $b > b_{\min} := \min\{b_{\min}^{(1)},\ b_{\min}^{(2)}\}$, the minimal real zero of $F(t, b)$ in $[0, 1)$ is simple (see Fig. 4).

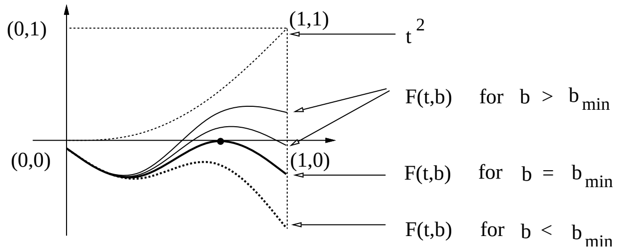

Fig. 4. Lifting the curve $\sum b_i B_i^3(t)$ on the right end by adding the term $\beta t^2 b$ so that its minimal zero in $[0, 1)$ is simple.

Proof. Suppose $b > b_{\min}^{(1)}$. Let $t_{\min} \in [0, 1]$ be the point at which the minimal value of the function that defines $b_{\min}^{(1)}$ is achieved. Then $t_{\min} > 0$ and $b > -3F_2(t_{\min})/\beta t_{\min}^2$, $b > -F_3(t_{\min})/\beta t_{\min}^2$. Since $t^2 = B_2^2(t) = B_2^3(t)/3 + B_3^3(t)$, we have

$$F_2(t_{\min}) + \beta B_2^2(t_{\min})b/3 > 0,$$
$$F_3(t_{\min}) + \beta\left[B_2^3(t_{\min})/3 + B_3^3(t_{\min})\right]b > 0.$$

Since $F_0(t_{\min}) = b_0 < 0$, then by the subdivision formula (see (Farin, 1990), p. 88) we know that the coefficients of the Bézier form of $F(t, b)$ over $[0, t_{\min}]$ have a one time sign change. By the variation diminishing property of BB form, $F(t, b)$ has exactly one real zero in $[0, t_{\min}]$. Hence, the minimal real zero of $F(t, b)$ in $[0, 1]$ is simple. If $b > b_{\min}^{(2)}$, the lemma is similarly proved. □

Note. The minimal value $b_{\min}^{(1)}$ and $b_{\min}^{(2)}$ can be obtained by computing the critical points in $(0, 1]$ of the rational functions considered, which leads to solving quadratic and cubic polynomial equations since the coefficients of the highest degree are vanishing.

Lemma 2.3 plays a central role in the construction of our A-patches for separating a single sheeted surface piece from the multiple-sheeted patches, in which the coefficients b_i and β are given and b is a free parameter (see Section 4.2).

3. Simplicial hull

For the given triangulation $\mathcal{T}$, the normals on the vertices are determined such that they point to one side of the input triangulation. We call this side the positive side. The other side is the negative side. Since the constructed C^1 surface will have the normals at the vertices, the value of these normals provide a mechanism to control the shape of the surface. Our criterion for determining these normals is to avoid producing bumpy surfaces. The estimation of surface normals from a surface triangulation is important and has several solutions, such as using interpolatory subdivision schemes (see (Dyn et al., 1990; Kobbelt, 1997)), or limit surface normal of Loop's subdivision (see (Loop, 1987)), or constraint fitting (see (Xu and Bajaj, 1997)). We use Loop's subdivision scheme.

Let $[p_i p_j]$ be an edge of $\mathcal{T}$, if $(p_j - p_i)^{\mathrm{T}} n_j \; (p_i - p_j)^{\mathrm{T}} n_i \geqslant 0$ and at least one of $(p_j - p_i)^{\mathrm{T}} n_j$ and $(p_i - p_j)^{\mathrm{T}} n_i$ is positive, then we say the edge is *positive convex*. If both

Fig. 5. (a) Positive convex edge; (b) Zero convex edge; (c) Nonconvex edge.

the numbers are zero then we say it is *zero convex*. The *negative convex* edge is similarly defined. If $(p_j - p_i)^{\mathrm{T}} n_j \ (p_i - p_j)^{\mathrm{T}} n_i < 0$, then we say the edge is *nonconvex* (see Fig. 5). Let $[p_i p_j p_k]$ be a face of $\mathcal{T}$. If its three edges are nonnegative (positive or zero) convex and at least one of them is positive convex, then we say the face $[p_i p_j p_k]$ is *positive convex*. If all the three edges are zero convex then we label the face as *zero convex*. The *negative convex* face is similarly defined. All the other cases $[p_i p_j p_k]$ are classified as *nonconvex*.

Let $\mathcal{T} = \mathcal{T}_{\text{nonzero}} \cup \mathcal{T}_{\text{zero}}$, where $\mathcal{T}_{\text{nonzero}}$ and $\mathcal{T}_{\text{zero}}$ are the collections of the faces of $\mathcal{T}$ that are not zero convex and zero convex, respectively. A *simplicial hull* of $\mathcal{T}_{\text{nonzero}}$, denoted by Σ, is a collection of nondegenerate tetrahedra which is constructed as follows:

3.1. Build face tetrahedra

Let $[p_i p_j p_k]$ be a face of $\mathcal{T}_{\text{nonzero}}$. Let $c = (p_i + p_j + p_k)/3$, n_{ijk} be the normal of face $[p_i p_j p_k]$ that points to the positive side of $\mathcal{T}$. Then choose point u_{ijk} if the face is positive convex or nonconvex and point v_{ijk} if the face is negative convex or nonconvex as follows.

$$u_{ijk} = c + t_{\max} n_{ijk}, \qquad v_{ijk} = c + t_{\min} n_{ijk}$$

with

$$t_{\max} = \max\{0, t_i + \varepsilon, t_j + \varepsilon, t_k + \varepsilon, t_{ij}, t_{jk}, t_{ik}\},$$
$$t_{\min} = \min\{0, t_i - \varepsilon, t_j - \varepsilon, t_k - \varepsilon, t_{ij}, t_{jk}, t_{ik}\},$$

where $\varepsilon > 0$ is a small number, t_l is defined by

$$(c - p_l + t_l n_{ijk})^{\mathrm{T}} n_l = 0,$$

i.e., $t_l = (p_l - c)^{\mathrm{T}} n_l / n_{ijk}^{\mathrm{T}} n_l,\ l = i, j, k$. t_l plus a positive ε guarantees that the Bézier coefficients $f_{0201}^{(ijk)}, f_{2001}^{(ijk)}, f_{0021}^{(ijk)}$ of the surface patch are positive (see Section 4.2 Step 2). t_{lm} is defined so that the Bézier coefficients $f_{0111}^{(ijk)}, f_{1101}^{(ijk)}, f_{1011}^{(ijk)}$ are nonnegative (see Section 4.2 Step 4). This leads to

$$t_{lm} = \frac{[p_l - c + \alpha(c, p_l, p_m)(p_l - p_m)]^{\mathrm{T}} n_l}{(n_l + n_m)^{\mathrm{T}} n_{ijk}} + \frac{[p_m - c + \alpha(c, p_m, p_l)(p_m - p_l)]^{\mathrm{T}} n_m}{(n_l + n_m)^{\mathrm{T}} n_{ijk}}$$

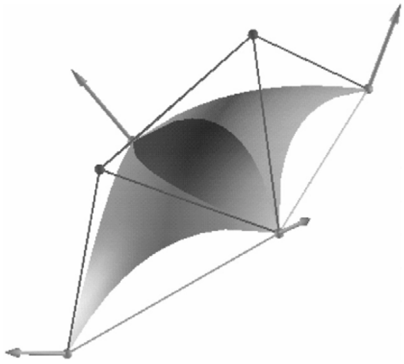

Fig. 6. Convex faces, normals, tetrahedra and surface patches.

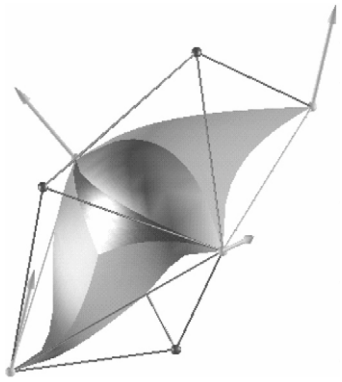

Fig. 7. Convex and nonconvex faces, normals, tetrahedra and surface patches.

$(l,m)=(i,j),(j,k),(i,k)$ and

$$\alpha(c,p_m,p_l)=\frac{[2(c-p_l)+(c-p_m)]^{\mathrm{T}}(p_m-p_l)}{\|p_m-p_l\|^2}. \tag{3.1}$$

Then the positive face tetrahedron $[p_i p_j p_k u_{ijk}]$ and/or the negative face $[p_i p_j p_k v_{ijk}]$ are formed.

For two adjacent faces $[p_i p_j p_k]$ and $[p_i p_j p_l]$, if the constructed face tetrahedra $[p_i p_j p_k u_{ijk}]$ and $[p_i p_j p_l u_{ijl}]$ or $[p_i p_j p_k v_{ijk}]$ and $[p_i p_j p_l v_{ijl}]$ intersect, we locally modify the common edge $[p_i p_j]$ by the Butterfly subdivision scheme (see (Dyn et al., 1990)) and then re-compute the normal and re-build the face tetrahedra. The local subdivision may take several steps if necessary until the adjacent face tetrahedra do not intersect.

3.2. Build edge tetrahedra

Let $[p_i p_j]$ be an edge of $\mathcal{T}$ where $[p_i p_j p_k]$ and $[p_i p_j p_l]$ are the two adjacent faces in $\mathcal{T}_{\text{nonzero}}$. If the edge is positive convex or nonconvex, the positive face tetrahedra $[p_i p_j p_k u_{ijk}]$ and $[p_i p_j p_l u_{ijl}]$ have been constructed. The positive edge tetrahedron is then $[p_i p_j u_{ijk} u_{ijl}]$. Similarly, the negative edge tetrahedron $[p_i p_j v_{ijk} v_{ijl}]$ is constructed if the edge is negative convex or nonconvex.

Figs. 6 and 7 show two adjacent triangles, vertex normals, tetrahedra as well as surface patches. The detail of the input is described in Section 6.

4. C^1 modeling by rational functions

In this section, we construct a piecewise C^1 rational function F over Σ. Its real zero contour $\{p\colon\ F(p)=0\}$ possesses a separate subset S such that $S\cup\mathcal{T}_{\text{zero}}$ (i) passes through the vertices of $\mathcal{T}$, (ii) has the given normal at each vertex, and (iii) is a smooth and single sheeted surface.

4.1. A-patches from rational functions

Let $[p_i p_j p_k] \in \mathcal{T}_{\text{nonzero}}$, then if $[p_i p_j p_k]$ is positive convex, define

$$F|_{[p_k p_i p_j u_{ijk}]} = \sum_{|\lambda|=3} f_\lambda^{(ijk)} B_\lambda^3(\alpha) + \frac{a_{1110}^{(ijk)}\alpha_2\alpha_3 + b_{1110}^{(ijk)}\alpha_1\alpha_3 + c_{1110}^{(ijk)}\alpha_1\alpha_2}{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2} B_{1110}^3(\alpha). \tag{4.1}$$

Similarly, if $[p_i p_j p_k]$ is negative convex, define

$$F|_{[p_k p_i p_j v_{ijk}]} = \sum_{|\lambda|=3} \tilde{f}_\lambda^{(ijk)} B_\lambda^3(\alpha) + \frac{\tilde{a}_{1110}^{(ijk)}\alpha_2\alpha_3 + \tilde{b}_{1110}^{(ijk)}\alpha_1\alpha_3 + \tilde{c}_{1110}^{(ijk)}\alpha_1\alpha_2}{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2} B_{1110}^3(\alpha). \tag{4.2}$$

If $[p_i p_j p_k]$ is nonconvex, define $F|_{[p_k p_i p_j u_{ijk}]}$ as the composite of three cubics that are defined on $[c p_i p_j u_{ijk}]$, $[p_k c p_j u_{ijk}]$ and $[p_k p_i c u_{ijk}]$, respectively, where $c = (p_i + p_j + p_k)/3$. $F|_{[p_k p_i p_j v_{ijk}]}$ is similarly defined. As in the the papers of Dahmen and Thamm-Schaar (1993), Guo (1991a; 1991b), Bajaj, Chen and Xu (1995b), we could use one cubic instead of three in some cases. However, using three cubics yields some nice shape advantages (see the Note in Section 4.2).

Let $[p_i p_j]$ be an edge of $\mathcal{T}$ that is not zero-convex, with $[p_i p_j p_k]$ and $[p_i p_j p_l]$ are two adjacent faces in $\mathcal{T}_{\text{nonzero}}$. If $[p_i p_j]$ is positive convex or nonconvex, define

$$F|_{[u_{ijl} p_i p_j u_{ijk}]} = \sum_{|\lambda|=3} f_\lambda^{(ijkl)} B_\lambda^3(\alpha) + \frac{a_{1101}^{(ijkl)}\alpha_1 + b_{1101}^{(ijkl)}\alpha_4}{\alpha_1 + \alpha_4} B_{1101}^3(\alpha) + \frac{a_{1011}^{(ijkl)}\alpha_1 + b_{1011}^{(ijkl)}\alpha_4}{\alpha_1 + \alpha_4} B_{1011}^3(\alpha). \tag{4.3}$$

If $[p_i p_j]$ is negative convex or nonconvex, define

$$F|_{[v_{ijl} p_i p_j v_{ijk}]} = \sum_{|\lambda|=3} \tilde{f}_\lambda^{(ijkl)} B_\lambda^3(\alpha) + \frac{\tilde{a}_{1101}^{(ijkl)}\alpha_1 + \tilde{b}_{1101}^{(ijkl)}\alpha_4}{\alpha_1 + \alpha_4} B_{1101}^3(\alpha) + \frac{\tilde{a}_{1011}^{(ijkl)}\alpha_1 + \tilde{b}_{1011}^{(ijkl)}\alpha_4}{\alpha_1 + \alpha_4} B_{1011}^3(\alpha). \tag{4.4}$$

Now we define our A-patches from which a smooth surface approximation of the triangulation $\mathcal{T}$ is constructed.

Face A-patch. Let $[p_i p_j p_k] \in \mathcal{T}_{\text{nonzero}}$. If $[p_i p_j p_k]$ is positive convex, then the face A-patch of $[p_i p_j p_k]$, denoted by F_{ijk}, is defined by

$$F_{ijk} = \{p \in \mathbb{R}^3\colon\ F(p) = 0;\ p = (1 - t_{\min})q + t_{\min} u_{ijk};\ \forall q \in [p_i p_j p_k];\ t_{\min} \in (-\infty, 0) \text{ if } F(q) > 0;\ t_{\min} \in [0, 1] \text{ if } F(q) \leqslant 0\}, \tag{4.5}$$

where $t_{\min}$ is the minimal t in absolute value among all $t's$ that satisfy the required condition. If $[p_i p_j p_k]$ is negative convex, F_{ijk} is similarly defined. If $[p_i p_j p_k]$ is nonconvex, F_{ijk} is defined by

$$\begin{aligned} F_{ijk} = \big\{p \in \mathbb{R}^3\colon\ & F(p) = 0;\ p = (1 - t_{\min})q + t_{\min}u_{ijk} \text{ if } F(q) \leqslant 0; \\ & \text{or } p = (1 - t_{\min})q + t_{\min}v_{ijk} \text{ if } F(q) > 0; \\ & t_{\min} \in [0, 1);\ \forall q \in [p_i p_j p_k]\big\}, \end{aligned} \tag{4.6}$$

where $t_{\min}$ is the minimal of all the $t's$ in $[0, 1)$ satisfying the required condition.

Edge A-patch. Let $[p_i p_j]$ be a nonzero-convex edge of $\mathcal{T}$ with $[p_i p_j p_k]$ and $[p_i p_j p_l]$ are two adjacent faces, then the edge A-patch of $[p_i p_j]$, denoted by E_{ij}, is defined by

$$\begin{aligned} E_{ij} = \big\{p \in \mathbb{R}^3\colon\ & F(p) = 0;\ p = (1 - t_{\min})q + t_{\min}u,\ \forall u \in [u_{ijk}u_{ijl}] \text{ if } F(q) \leqslant 0; \\ & \text{or } p = (1 - t_{\min})q + t_{\min}v,\ \forall v \in [v_{ijk}v_{ijl}] \text{ if } F(q) > 0; \\ & t_{\min} \in [0, 1);\ \forall q \in [p_i p_j]\big\}, \end{aligned} \tag{4.7}$$

where $t_{\min}$ is the minimal of all the $t's$ in $[0, 1)$ satisfying the required condition.

That is, F_{ijk} and E_{ij} are defined by the minimal zero of so called *shooting polynomials* in t. For face A-patches, the *shooting polynomials*, that are cubic in variable t, are

$$\begin{aligned} &\sum_{s=0}^{3} B_s^{(ijk)}(q)B_s^3(t) := F|_{[p_k p_i p_j u_{ijk}]}\big((1 - t)q + tu_{ijk}\big), \\ &\sum_{s=0}^{3} \tilde{B}_s^{(ijk)}(q)B_s^3(t) := F|_{[p_k p_i p_j v_{ijk}]}\big((1 - t)q + tv_{ijk}\big), \end{aligned} \tag{4.8}$$

where $q = \alpha_1 p_i + \alpha_2 p_j + \alpha_3 p_k \in [p_i p_j p_k]$ and

$$\begin{aligned} B_s^{(ijk)}(q) &= \sum_{\lambda_1+\lambda_2+\lambda_3=3-s} f_{\lambda_1\lambda_2\lambda_3 s}^{(ijk)} B_{\lambda_1\lambda_2\lambda_3}^{3-s}(\alpha_1, \alpha_2, \alpha_3), \quad 1 \leqslant s \leqslant 3, \\ B_0^{(ijk)}(q) &= \sum_{\lambda_1+\lambda_2+\lambda_3=3} f_{\lambda_1\lambda_2\lambda_3 0}^{(ijk)} B_{\lambda_1\lambda_2\lambda_3}^{3}(\alpha_1, \alpha_2, \alpha_3) \\ &\quad + \frac{a_{1110}^{(ijk)}\alpha_2\alpha_3 + b_{1110}^{(ijk)}\alpha_1\alpha_3 + c_{1110}^{(ijk)}\alpha_1\alpha_2}{\alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_2} B_{111}^3(\alpha_1, \alpha_2, \alpha_3) \end{aligned}$$

and $\widetilde{B}_s^{(ijk)}(q)$ is similarly defined. If $[p_i p_j p_k]$ is nonconvex, $B_s^{(ijk)}(q)$ and $\widetilde{B}_s^{(ijk)}(q)$ are similarly but defined piecewise with the rational part being zero. For edge A-patches, the *shooting polynomials* are

$$\begin{aligned} &\sum_{s=0}^{3} B_s^{(ijkl)}(q, u)B_s^3(t) := F|_{[u_{ijl} p_i p_j u_{ijk}]}\big((1 - t)q + tu\big), \\ &\sum_{s=0}^{3} \widetilde{B}_s^{(ijkl)}(q, v)B_s^3(t) := F|_{[v_{ijl} p_i p_j v_{ijk}]}\big((1 - t)q + tv\big), \end{aligned} \tag{4.9}$$

where $q = xp_i + (1-x)p_j$, $u = yu_{ijk} + (1-y)u_{ijl}$, $v = yv_{ijk} + (1-y)v_{ijl}$,

$$B_s^{(ijkl)}(q,u) = \sum_{\lambda_2+\lambda_3=3-s}\sum_{\lambda_1+\lambda_4=s} f_\lambda^{(ijkl)} B_{\lambda_2}^{3-s}(x) B_{\lambda_4}^{s}(y), \quad s = 0,1,3,$$

$$\begin{aligned} B_2^{(ijkl)}(q,u) = &\sum_{\lambda_2+\lambda_3=1}\sum_{\lambda_1+\lambda_4=2} f_\lambda^{(ijkl)} B_{\lambda_2}^{1}(x) B_{\lambda_4}^{2}(y) \\ &+ \left[x b_{1101}^{(ijkl)} + (1-x) b_{1011}^{(ijkl)}\right] y B_1^2(y) \\ &+ \left[x a_{1101}^{(ijkl)} + (1-x) a_{1011}^{(ijkl)}\right] (1-y) B_1^2(y) \end{aligned}$$

and $\widetilde{B}_s^{(ijkl)}(q,v)$ is similarly defined. Note that the coefficients $B_s^{(ijk)}$ and $B_s^{(ijkl)}$ of the shooting polynomials relate only to the Bézier coefficients of layer s (see Fig. 8).

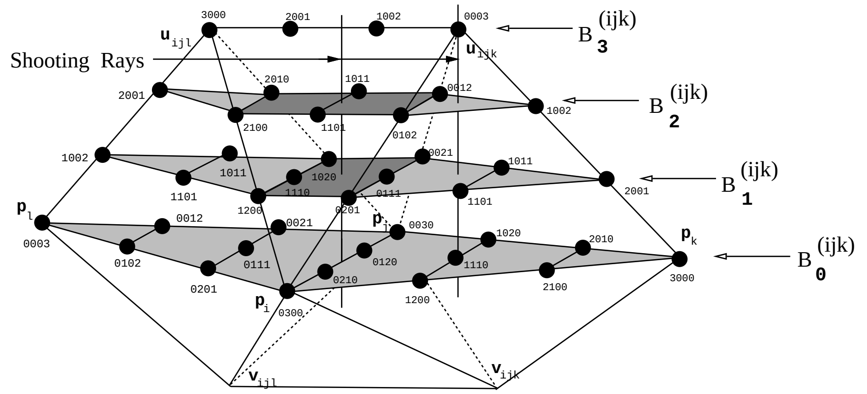

Fig. 8. Each coefficient of the shooting polynomials depends on one layer Bézier coefficients. Layers are shaded separately.

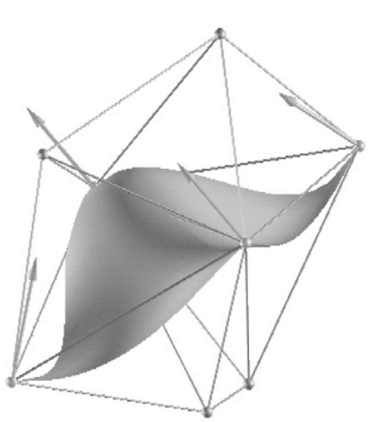

Fig. 9. Smooth join of two nonconvex face patches with edge patch.

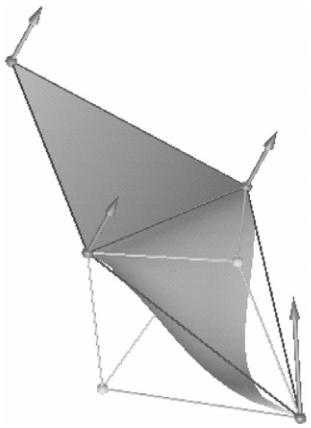

Fig. 10. Smooth join of plane with nonconvex patch.

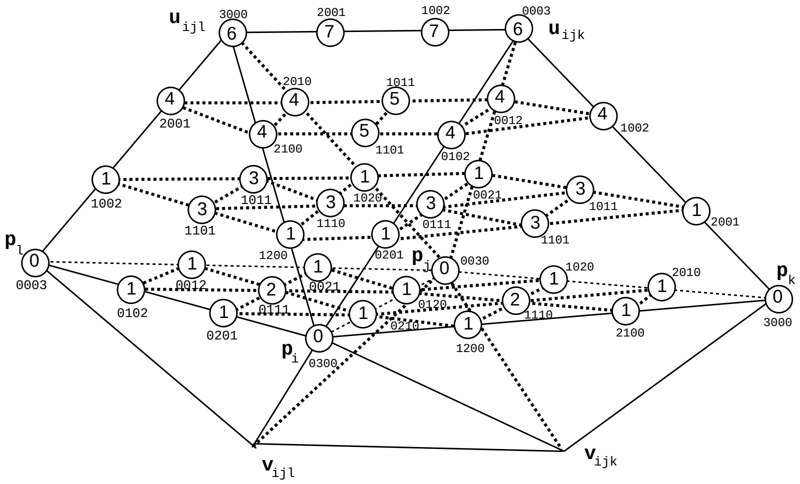

Fig. 11. Adjacent tetrahedra, the numbering of the weights of the cubic functions.

Figs. 6, 7, 9 and 10 show various cases of face A-patches and edge A-patches. See Section 6 for a detailed description.

4.2. Construction of rational functions

Now we determine the coefficients of F step by step (see Fig. 11 for the numbering of the coefficients of the polynomial pieces).

Step 1. In order to have the constructed surface interpolate the vertices of $\mathcal{T}$, we take the number 0 coefficients to be zero.

Step 2. The number 1 coefficients are determined by formula (2.1) from normals. From the construction of the face tetrahedra of Σ, we have $f_{0201}^{(ijk)} > 0,\ f_{2001}^{(ijk)} > 0,\ f_{0021}^{(ijk)} > 0.$

Step 3. If $[p_i p_j p_k]$ is positive (or negative) convex, then by interpolating the directional derivative

$$\frac{1}{2}\left[\frac{(p_k - p_j)^{\mathrm{T}}(p_i - p_j)}{\|p_i - p_j\|^2}(p_k - p_i) + \frac{(p_k - p_i)^{\mathrm{T}}(p_j - p_i)}{\|p_i - p_j\|^2}(p_k - p_j)\right]^{\mathrm{T}}(n_i + n_j)$$

at the point $\frac{1}{2}(p_i + p_j)$ where the direction is in the face $[p_i p_j p_k]$ and perpendicular to the edge $[p_i p_j]$, we can derive that

$$\begin{aligned} f_{1110}^{(ijk)} + a_{1110}^{(ijk)} = {} & \tfrac{1}{2}\big[f_{1200}^{(ijk)} + f_{1020}^{(ijk)} \\ & + \alpha(p_k, p_i, p_j) f_{0210}^{(ijk)} + (1 - \alpha(p_k, p_i, p_j)) f_{0120}^{(ijk)}\big], \end{aligned} \tag{4.10}$$

where $\alpha(p_k, p_i, p_j)$ is defined by (3.1). By making F interpolate the other two directional derivatives at the mid-point of the edge $[p_i p_k]$ and $[p_j p_k]$, respectively, we have

$$\begin{aligned} f_{1110}^{(ijk)} + c_{1110}^{(ijk)} &= \tfrac{1}{2}\big[f_{0210}^{(ijk)} + f_{2010}^{(ijk)} \\ &\quad + \alpha(p_j, p_i, p_k) f_{1200}^{(ijk)} + \big(1-\alpha(p_j, p_i, p_k)\big) f_{2100}^{(ijk)}\big], \end{aligned} \tag{4.11}$$

$$\begin{aligned} f_{1110}^{(ijk)} + b_{1110}^{(ijk)} &= \tfrac{1}{2}\big[f_{0120}^{(ijk)} + f_{2100}^{(ijk)} \\ &\quad + \alpha(p_i, p_j, p_k) f_{1020}^{(ijk)} + \big(1-\alpha(p_i, p_j, p_k)\big) f_{2010}^{(ijk)}\big]. \end{aligned} \tag{4.12}$$

Though for any given $f_{1110}^{(ijk)}$, (4.10)–(4.12) define uniquely $a_{1110}^{(ijk)}$, $b_{1110}^{(ijk)}$ and $c_{1110}^{(ijk)}$. But we take $f_{1110}^{(ijk)}$ to be the average value of the right-handed sides of (4.10)–(4.12), such that $a_{1110}^{(ijk)}$, $b_{1110}^{(ijk)}$ and $c_{1110}^{(ijk)}$ are as small as possible. If $[p_i p_j p_k]$ is nonconvex, we have three $f_{1110}^{(ijk)}$, that are determined by right-handed sides of (4.10)–(4.12) with $\alpha(p_k, p_i, p_j)$, $\alpha(p_j, p_i, p_k)$ and $\alpha(p_i, p_j, p_k)$ are replaced by $\alpha(c, p_i, p_j)$, $\alpha(c, p_i, p_k)$ and $\alpha(c, p_j, p_k)$, respectively.

Step 4. The number 3 coefficients $f_{0111}^{(ijk)}$, $f_{1101}^{(ijk)}$ and $f_{1011}^{(ijk)}$ are determined by the same approach as the number 2 but the directions are in the faces $[p_i p_j u_{ijk}]$, $[p_i p_k u_{ijk}]$ and $[p_j p_k u_{ijk}]$, respectively. From these we get

$$\begin{aligned} f_{0111}^{(ijk)} &= \tfrac{1}{2}\big[f_{0201}^{(ijk)} + f_{0021}^{(ijk)} + \alpha(u_{ijk}, p_i, p_j) f_{0210}^{(ijk)} + \big(1-\alpha(u_{ijk}, p_i, p_j)\big) f_{0120}^{(ijk)}\big], \\ f_{1101}^{(ijk)} &= \tfrac{1}{2}\big[f_{0201}^{(ijk)} + f_{2001}^{(ijk)} + \alpha(u_{ijk}, p_i, p_k) f_{1200}^{(ijk)} + \big(1-\alpha(u_{ijk}, p_i, p_k)\big) f_{2100}^{(ijk)}\big], \\ f_{1011}^{(ijk)} &= \tfrac{1}{2}\big[f_{2001}^{(ijk)} + f_{0021}^{(ijk)} + \alpha(u_{ijk}, p_j, p_k) f_{1020}^{(ijk)} + \big(1-\alpha(u_{ijk}, p_j, p_k)\big) f_{2010}^{(ijk)}\big]. \end{aligned}$$

From the construction of the face tetrahedra (see Section 3.1), we know that $f_{0111}^{(ijk)} \geqslant 0$, $f_{1101}^{(ijk)} \geqslant 0$ and $f_{1011}^{(ijk)} \geqslant 0$.

Step 5. The number 4 coefficients are free. We choose them so that the polynomial part approximates a quadratic in the least square sense. That is, we solve the equation

$$f_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(ijk)} = \frac{1}{3}\sum_{i=1}^{4} \lambda_i q_{\lambda_1\lambda_2\lambda_3\lambda_4 - e_i}, \quad \sum_{i=1}^{4} \lambda_i = 3,\ \lambda_4 = 0, 1, \tag{4.13}$$

for unknowns $q_{\lambda_1\lambda_2\lambda_3\lambda_4 - e_i}$, where q_{ijkl} represent the coefficients of the quadratic in BB form. Then take

$$f_{\lambda_1\lambda_2\lambda_3\lambda_4}^{(ijk)} = \frac{1}{3}\sum_{i=1}^{4} \lambda_i q_{\lambda_1\lambda_2\lambda_3\lambda_4 - e_i}, \quad \sum_{i=1}^{4} \lambda_i = 3,\ \lambda_4 = 2, \tag{4.14}$$

with $q_{0002} = f_{0003}^{(ijk)}$ as a free parameter. If the tetrahedron $[p_k p_i p_j u_{ijk}]$ is subdivided, then the undetermined coefficients are determined by the Clough–Tocher scheme (see (Bajaj et al., 1995b)) for detail).

Step 6. In order to make the face A-patch and edge A-patch smooth and single sheeted (see the proof in Appendix), we need to specify lower bounds B_{ijk} and B_{ijl} for the number 6 coefficients $f_{0003}^{(ijk)}$ and $f_{3000}^{(ijl)}$. Firstly, compute B_{ijk} so that the face A-patch in $[p_i p_j p_k u_{ijk}]$ is smooth and single sheeted if $f_{0003}^{(ijk)} > B_{ijk}$. Applying Lemma 2.3 to the shooting polynomial defined in (4.8) with $b_3 = 0$, $\beta = 1$ and $b = f_{0003}^{(ijk)}$, we obtain $b_{\min}(q)$ and then define B_{ijk} to be $\max_{q\in[p_i p_j p_k]} b_{\min}(q)$. This maximal value can be numerically computed. B_{ijl} is similarly computed.

Second, we set $f_{0003}^{(ijk)} = B_{ijk} + b$ and $f_{3000}^{(ijl)} = B_{ijl} + b$ with b as a parameter. Then compute number 5, 6, 7 coefficients by the C^0 and C^1 conditions (see Steps 7–9 for details). These coefficients depend linearly on the parameter b. Applying Lemma 2.3 for the shooting polynomial in (4.9) with $\beta = B_0^3(y) + \frac{2}{3}\nu_1 B_1^3(y) + \frac{2}{3}\mu_4 B_2^3(y) + B_3^3(y) + y(1-y)$, we can determine a lower bound B_{ijkl} for b so that the edge A-patch is smooth and single sheeted if $f_{0003}^{(ijk)} > B_{ijk} + B_{ijkl}$ and $f_{3000}^{(ijl)} > B_{ijl} + B_{ijkl}$.

Third, modify the bounds B_{ijk} and B_{ijl} by adding B_{ijkl} to them if $B_{ijkl} > 0$.

Step 7. The number 5 coefficients are taken to be zero. The coefficients of the rational function on the edge tetrahedron are given by the C^1 conditions (2.3). Let

$$\begin{aligned} u_{ijl} &= \mu_1 p_k + \mu_2 p_i + \mu_3 p_j + \mu_4 u_{ijk}, \quad \sum_{s=1}^{4} \mu_s = 1, \\ u_{ijk} &= \nu_1 u_{ijl} + \nu_2 p_i + \nu_3 p_j + \nu_4 p_l, \quad \sum_{s=1}^{4} \nu_s = 1. \end{aligned} \tag{4.15}$$

Then

$$\begin{aligned} b_{1101}^{(ijkl)} &= \mu_1 f_{1101}^{(ijk)} + \mu_2 f_{0201}^{(ijk)} + \mu_3 f_{0111}^{(ijk)} + \mu_4 f_{0102}^{(ijk)}, \\ b_{1011}^{(ijkl)} &= \mu_1 f_{1011}^{(ijk)} + \mu_2 f_{0111}^{(ijk)} + \mu_3 f_{0021}^{(ijk)} + \mu_4 f_{0012}^{(ijk)}, \\ a_{1101}^{(ijkl)} &= \nu_1 f_{2100}^{(ijl)} + \nu_2 f_{1200}^{(ijl)} + \nu_3 f_{1110}^{(ijl)} + \nu_4 f_{1101}^{(ijl)}, \\ a_{1011}^{(ijkl)} &= \nu_1 f_{2010}^{(ijl)} + \nu_2 f_{1110}^{(ijl)} + \nu_3 f_{1020}^{(ijl)} + \nu_4 f_{1011}^{(ijl)}. \end{aligned}$$

It follows from (4.14) that $b_{1101}^{(ijkl)}$ and $b_{1011}^{(ijkl)}$ depend linearly on $f_{0003}^{(ijk)}$. Similarly, $a_{1101}^{(ijkl)}$ and $a_{1011}^{(ijkl)}$ depend linearly on $f_{3000}^{(ijl)}$.

Step 8. The number 6 coefficients are free parameters which provide a smooth and single sheeted A-patch family if they are above the lower bounds provided in Step 6. One may also choose desirable shape surfaces from this family by suitably adjusting the free parameters. The default choice of the free parameters is to make the cubic approximate a linear function. That is, we solve the equations

$$q_{\lambda_1\lambda_2\lambda_3\lambda_4} = \frac{1}{2}\sum_{i=1}^{4} \lambda_i p_{\lambda_1\lambda_2\lambda_3\lambda_4 - e_i}, \quad \sum_{i=1}^{4} \lambda_i = 3,\ \lambda_4 = 0, 1, \tag{4.16}$$

for unknowns $p_{\lambda_1\lambda_2\lambda_3\lambda_4 - e_i}$, where $q_{\lambda_1\lambda_2\lambda_3\lambda_4}$ are defined by Eqs. (4.13) and p_{ijkl} represent the coefficients of the linear function in BB form. Then take $f_{0003}^{(ijk)}$ to be $\max\{p_{0001}, B_{ijk}\}$.

Step 9. The number 7 coefficients are given by C^1 condition (2.3). For example,

$$f_{1002}^{(ijkl)} = \mu_1 f_{1002}^{(ijk)} + \mu_2 f_{0102}^{(ijk)} + \mu_3 f_{0012}^{(ijk)} + \mu_4 f_{0003}^{(ijk)}. \tag{4.17}$$

Step 10. The remaining coefficients in the negative side of $\mathcal{T}$ are determined similar to the corresponding coefficients in the positive side of $\mathcal{T}$.

Note. As the approach of Bajaj, Chen and Xu (1995b), if $[p_i p_j p_k]$ is nonconvex, we could use one cubic instead of three if the face is not coplanar with its neighbor faces and the three inner products of the face normal and its three adjacent face normals have the same signs. In this case, the number 2 coefficient in Step 3 is freely chosen, the number 3 coefficients in Step 4 are defined by C^1 condition (2.3). We could adjust the number 2 coefficients so that the number 3 coefficients are positive. All other steps are the same. However, F constructed may not interpolate the partial derivatives at the mid-points of the edge. We strongly believe that the interpolation of properly chosen partials at the mid-points of the edges will make the constructed surface has better shape. For this reason, we prefer using the three cubic approach.

4.3. Properties of A-patches

Theorem 4.1. *For the given triangulation $\mathcal{T}$ with a constructed simplicial hull Σ, the surface defined by the union of all edge A-patches, face A-patches and zero convex faces of $\mathcal{T}$ interpolates the vertices and normals of the triangulation, and is smooth and single sheeted and topologically equivalent to $\mathcal{T}$.*

Proof. We first show that the function F is C^1 over Σ. Note that the rational function is well defined even if at the points the denominator is zero. Hence F is a well defined function on Σ. Since F is obviously smooth in the interior of each tetrahedron, we consider only the smoothness of F at the common faces of tetrahedra in Σ.

At the face $[p_i p_j u_{ijk}]$, the rational functions and their first order partial derivatives are polynomials. Hence the coefficients determined by the C^1 condition (2.3) make F be C^1 everywhere on the common face. For the common face $[p_i p_j p_k]$, since the number 1 and the number 3 coefficients are defined by interpolating C^1 data at vertices and edges of the face, hence the polynomials in the two face tetrahedra have a C^1 join. Therefore, F is C^1 there.

Now we show that the constructed surface has the required properties. We first note that the surface patch F_{ijk} or E_{ij} is smooth and single sheeted since the construction of F satisfies the conditions of Lemma 2.3 (the detail proof of this fact is given in Appendix A). Also, these surface patches interpolate corresponding vertices and have the given normals on the vertices. Second, the edge A-patch and face A-patch are continuous at the common face since the surface points there are derived from the same equation (see (4.5)–(4.7)). Furthermore, since F is C^1, the two surface patches smoothly join on the common face.

Now we show that the zero convex face joins suitably with its neighbor surface patches. Let $[p_i p_j p_k]$ be a zero convex face. Then the surface patch is the face itself. If its adjacent face, say $[p_i p_j p_l]$, is also zero-convex, then the two faces are coplanar since they share the same surface normals at the common vertices p_i and p_j. If $[p_i p_j p_l]$ is not zero-convex, then by the construction of F, we know that F_{ijl} contains the edge $[p_i p_j]$. In particular, the face $[p_i p_j p_k]$ and the surface patch F_{ijl} C^0 join at the edge. Since both the surface patches have the same three normals on the edge and the normal functions are polynomial vectors of degree two, they are uniquely defined by the three normals that are perpendicular to the face. Hence the normal function is perpendicular to the face everywhere on the edge. That is, $[p_i p_j p_k]$ and F_{ijl} have the same normals on the edge. Therefore, the two surface patches join smoothly.

Finally, since each edge and each face of Σ corresponds to one surface patch (the zero convex edge corresponds to itself), the constructed surface is topologically equivalent to $\mathcal{T}$. $\square$

The scheme proposed above makes the constructed surface have the plane recovery property. Furthermore, the scheme also has quadratic precision. This is made precise in the following theorem and corollary.

Theorem 4.2. *Suppose the three normals at the vertices of the triangle* $[p_i p_j p_k]$ *are extracted from a quadratic surface* $Q(p) = 0$ *that interpolates the three vertices. For any given* $f^{(ijk)}_{0003}$, *if the coefficients* $f^{(ijk)}_{\lambda_1\lambda_2\lambda_3 2}$ *of* $F|_{[p_k p_i p_j u_{ijk}]}$ *are defined by* (4.14) *with* $q_{0002} := f^{(ijk)}_{0003}$ *and* $q_{\lambda_1\lambda_2\lambda_3\lambda_4}(\lambda_4 = 0, 1)$ *are defined by* (4.13), *then* $F|_{[p_k p_i p_j u_{ijk}]}(p) = Q(p) + [f^{(ijk)}_{0003} - Q(u_{ijk})]\alpha_4^2$.

Note that $\alpha_4^2 = B^2_{0002}(\alpha)$ is a quadratic Bézier base function at u_{ijk}. Hence the theorem says that we have a quadratic surface family with $f^{(ijk)}_{0003}$ as parameter. Each member in this family has the same normals as $Q(p) = 0$ at the vertices. If we take the parameter $f^{(ijk)}_{0003}$ to be $Q(u_{ijk})$, then the constructed function coincides with Q.

The proof of the theorem is based on the following facts:

(a) F interpolates function values and first order partial derivatives of Q at the vertices p_i, p_j and p_k, and F interpolates directional derivatives of Q in any direction that is perpendicular to edges of the triangle at the mid-points of the three edges.

(b) The number 4 coefficients are defined by degree elevation formula.

(c) The rational function degenerates to zero.

The detailed discussion needs to distinguish the cases when the face is convex or nonconvex. We omit these details here.

Now consider the edge $[p_i p_j]$ with $[p_i p_j p_k]$ and $[p_i p_j p_l]$ as its adjacent faces. If we want the function defined on the edge tetrahedron $[u_{ijl} p_i p_j u_{ijk}]$ to recover a quadratic, we then require the functions defined on $[p_k p_i p_j u_{ijk}]$ and $[u_{ijl} p_i p_j p_l]$ to recover the same quadratic. Therefore, we have the following

Corollary 4.3. *Suppose the four normals at the vertices of the two adjacent triangles* $[p_i p_j p_k]$ *and* $[p_i p_j p_l]$ *are extracted from a quadratic surface* $Q(p) = 0$ *that passes through the four vertices. For any given* $f^{(ijk)}_{0003}$ *and* $f^{(ijl)}_{3000}$ *satisfying,* $\nu_1[f^{(ijl)}_{3000} - Q(u_{ijl})] = \mu_4[f^{(ijk)}_{0003} - Q(u_{ijk})]$ *with* ν_1 *and* μ_4 *are defined by* (4.15), *if the coefficients* $f^{(ijk)}_{\lambda_1\lambda_2\lambda_3 2}$ *and* $f^{(ijl)}_{2\lambda_2\lambda_3\lambda_4}$ *of* $F|_{[p_k p_i p_j u_{ijk}]}$ *and* $F|_{[u_{ijl} p_i p_j p_l]}$, *respectively, are defined as those in Theorem 4.2 correspondingly, then the function* $F|_{[u_{ijl} p_i p_j u_{ijk}]}$ *is quadratic and is the same as the functions defined on* $[p_k p_i p_j u_{ijk}]$ *and* $[u_{ijl} p_i p_j p_l]$.

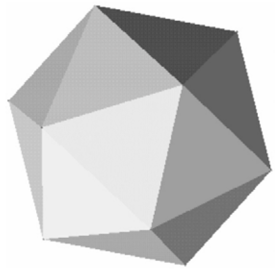

Fig. 12. Regular 20-faces polyhedron as a discretization of sphere.

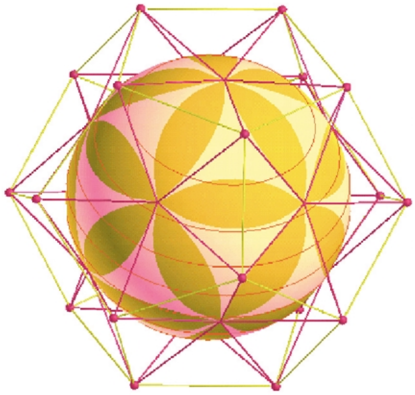

Fig. 13. Constructed sphere from the 20-faces polyhedron.

Fig. 13 shows the quadric recovery property. The input (Fig. 12) is a regular 20-faces polyhedron that is a discretization of a sphere. The curves over the surface are isophotes that show the sphere is perfectly recovered.

4.4. Closed form of the surface patch

The *shooting polynomial*, defined by (4.8) and (4.9), for evaluating the surface point is of degree three. Its coefficients depend continuously upon several parameters. For each set of the parameters, the polynomial in general has three roots. These roots are computed by the well known Cardan's formulas (see (4.18)–(4.20)). However, it is possible that for different sets of parameters different formulas are used. In this subsection, we will answer the following question: *Which formula among the three should we use?* Let the shooting polynomial be expressed in the following form

$$dB_0^3(t) + cB_1^3(t) + bB_2^3(t) + aB_3^3(t) = (1-t)^3\big(d + 3cs + 3bs^2 + as^3\big),$$

where $s = t/(1-t) \in [0, \infty)$ for $t \in [0, 1)$. Let $s = x - b/a$. Then the cubic polynomial in s could be written as $x^3 + px + q$ with $p = 3(ac - b^2)/a^2$, $q = (2b^3 - 3abc + a^2d)/a^3$. The roots of $x^3 + px + q = 0$ are given by Cardan's formulas

$$x_1 = \sqrt[3]{-q/2 + \sqrt{\Delta}} + \sqrt[3]{-q/2 - \sqrt{\Delta}}, \tag{4.18}$$

$$x_2 = \omega\sqrt[3]{-q/2 + \sqrt{\Delta}} + \omega^2\sqrt[3]{-q/2 - \sqrt{\Delta}}, \tag{4.19}$$

$$x_3 = \omega^2\sqrt[3]{-q/2 + \sqrt{\Delta}} + \omega\sqrt[3]{-q/2 - \sqrt{\Delta}}, \tag{4.20}$$

where $\Delta = (q/2)^2 + (p/3)^3$ is the discriminant of the equation and $\omega = (-1 + \mathrm{i}\sqrt{3})/2$. If $\Delta < 0$, $-q/2 \pm \sqrt{\Delta}$ are nonreal complex numbers. We define

$$\sqrt[3]{-q/2 \pm \sqrt{\Delta}} = \big|-q/2 \pm \sqrt{\Delta}\big|^{1/3}\mathrm{e}^{\pm \mathrm{i}\theta/3}, \tag{4.21}$$

where $\pm\theta = \arg(-q/2 \pm \sqrt{\Delta}) \in (-\pi, \pi]$. Taking $\pm\theta \in (-\pi, \pi]$, instead of $[0, 2\pi)$, assures that x_1 is real. It is well known that (i) if $\Delta > 0$ the equation has one real root (i.e., x_1) and two nonreal complex roots (i.e., x_2 and x_3); (ii) if $\Delta = 0$ the equation has three real roots, but at least two of them are the same ($x_2 = x_3$); (iii) if $\Delta < 0$ the equation has three different real roots.

In our application of the Cardan's formulas, the coefficients depend continuously upon some parameters. For face patches these parameters are $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ with $\alpha_i \geqslant 0$, $\sum \alpha_i = 1$ (see (4.8)). For edge patches these parameters are $\alpha := (x, y) \in [0, 1] \times [0, 1]$ (see (4.9)). However, the continuity of the coefficients does not imply that $x_j := x_j(\alpha)$ vary continuously with α, since the cubic root function defined by (4.21) is not continuous at $\Delta = 0$ and $q > 0$. Now we have the following

Result. (i) *Let* $D^+ = \{\alpha\colon\ \Delta(\alpha) > 0\}$, $D^0 = \{\alpha\colon\ \Delta(\alpha) = 0\}$. *Then for any* $\alpha \in D^+ \cup D^0$, $x_1(\alpha)$ *is the required root for the surface patch.*

(ii) *Let* $D^- = \{\alpha\colon\ \Delta(\alpha) < 0\}$ *and* $D_i^- \subset D^-$, $i = 1, \ldots, k$ $(k \geqslant 1)$, *such that each* D_i^- *is connected region,* $D_i^- \cap D_j^- = \emptyset$ *for* $i \neq j$ *and* $\bigcup D_i^- = D^-$. *Then if* $\{\alpha\colon\ q(\alpha) < 0\} \cap D_i^0 \neq \emptyset$, $x_1(\alpha)$ *is the required root on* D_i^-, *where* $D_i^0 = D^0 \cap \bar{D}_i^-$, *and* $\bar{D}_i^-$ *is the closure of* D_i^-.

Proof. (i) For the points in D^+, the equation has only one real root, i.e., x_1. For the points in D^0, $x_2 = x_3$. Since the minimal root is simple, x_1 is the required root.

(ii) From (i) we know that, at the points of $\{q < 0\} \cap D_i^0$, the required root is x_1. Now we show that the required root cannot switch to x_2 or x_3 in D_i^-. For example, if x_1 switches to x_2 or x_3 at a point α'' in D_i^-, then from the connectedness of D_i^-, there is a continuous path L in D_i^- starting with a point α' in $\{q < 0\} \cap D_i^0$ and ending with α''. Since x_1 is continuous at α' and in D_i^-, x_1 is continuous on the path L. With loss of generality (WLG), we may assume α'' is the first point on this path such that x_1 switches to x_2 or x_3. Then by the continuity of x_j in D_i^-, $x_1 = x_2$ or $x_1 = x_3$, we have a contradiction to the fact that the roots are distinct when $\Delta < 0$. □

Corollary. *If* $ab > 0$, *then there exists only one of the* x_j *that is our required root for all the* α. *Furthermore, if* $D^+ \cup D^0 \neq \emptyset$, *then* $x_1(\alpha)$ *is the required root.*

Proof. Now we show that if $ab > 0$ the set $D^0 \cap \{q(\alpha) > 0\}$ is empty. Since the required root of the shooting polynomial is simple and since $x_2 = x_3$ when $\Delta = 0$, $x_1 = 2\sqrt[3]{-q/2}$ is the required root. Since our root $t \in [0, 1)$ or $s = x - b/a \in [0, \infty)$, then

$$x_1 - b/a \geqslant 0, \quad \text{or} \quad x_1 \geqslant b/a > 0, \quad \text{or} \quad q < 0.$$

Therefore, $x_j(\alpha)$ are continuous functions. Hence only one expression could be used, since any transform from one to another will lead to the intersection of the functions. This contradicts the fact that the minimal root is simple. Furthermore, if $D^+ \cup D^0 \neq \emptyset$, from Result(i), $x_1(\alpha)$ must be the required root. □

It follows from Lemma 2.3 that $ab > 0$ always could be achieved by increasing the coefficient $f_{0003}^{(ijk)}$.

5. Display of the C^1 surfaces

5.1. The face A-patch

For each face $[p_i p_j p_k] \in \mathcal{T}_{\text{non-zero}}$, we shall produce a piecewise triangular approximation for the surface patch F_{ijk}. Let n be a given positive number, which represents the resolution of the piecewise approximation. Then the piecewise triangular approximation is defined by the naive connection of the points $f_{xyz}(x + y + z = n,\ x, y, z \geqslant 0)$. Here f_{xyz} is the intersection point of the polygonal line $[u_{ijk} q_{xyz}] \cup [q_{xyz} v_{ijk}]$ and the surface $F = 0$, where $q_{xyz} = \frac{x}{n} p_i + \frac{y}{n} p_j + \frac{z}{n} p_k$ and the intersection point is computed by solving the cubic polynomial equation $F|_{[p_k p_i p_j u_{ijk}]}((1 - t)q_{xyz} + tu_{ijk}) = 0$ if $F(q_{xyz}) \leqslant 0$ or solving a similar equation $F|_{[p_k p_i p_j v_{ijk}]}((1-t)q_{xyz} + tv_{ijk}) = 0$ if $F(q_{xyz}) > 0$, where the required root is the minimal one.

5.2. The edge A-patch

For each edge in $\mathcal{T}_{\text{non-zero}}$, we shall produce a piecewise quadrilateral approximation for the surface patch defined in the two edge tetrahedra that share the edge. Let m, n be two given positive numbers, which represent the resolution of the piecewise approximation and n should have the same value as above, let $[p_i p_j]$ be a edge of $\mathcal{T}_{\text{non-zero}}$ and $[u_{ijl} p_i p_j u_{ijk}]$ and $[v_{ijl} p_i p_j v_{ijk}]$ be the edge tetrahedra. Then the piecewise quadrilateral approximation is defined by connecting the points $e_{xy}(x = 0, \ldots, n;\ \ y = 0, \ldots, m)$. Here e_{xy} is the intersection point of the polygonal line $[u_y q_x] \cup [q_x v_y]$ and the surface $F = 0$, where

$$q_x = \frac{x}{n} p_i + \frac{n-x}{n} p_j, \quad u_y = \frac{y}{m} u_{ijl} + \frac{m-y}{n} u_{ijk}, \quad v_y = \frac{y}{m} v_{ijl} + \frac{m-y}{n} v_{ijk}$$

and the intersection point is computed by solving the cubic polynomial equation

$$F|_{[u_{ijl} p_i p_j u_{ijk}]}\big((1-t)q_x + tu_y\big) = 0 \quad \text{if } F(q_x) \leqslant 0,$$

or solving a similar equation

$$F|_{[v_{ijl} p_i p_j v_{ijk}]}\big((1-t)q_x + tv_y\big) = 0 \quad \text{if } F(q_x) > 0.$$

Again, we use the minimal root.

Note that the linear decomposition of the A-patches are identical on the common faces and edges of the hull Σ, so there are no cracks or aliasing artifacts present in the display of the final C^1 surface.

5.3. Interactive change of the surface shape

From the construction of F, we know that the coefficient $f_{0003}^{ijk} > B_{ijk}$ is free for each face $[p_i p_j p_k]$. This degree of freedom can be used to control interactively the shape of the designed surface. When f_{0003}^{ijk} approaches to the lower bound B_{ijk}, the surface is lifted towards the top vertex, while f_{0003}^{ijk} goes to infinity, the surface is depressed to the bottom triangle (see Fig. 2). Now we show that the surface could be quickly displayed when f_{0003}^{ijk} changes. Let $\sum_{s=0}^{3} B_s^{(ijk)}(q) B_s^3(t) := F|_{[p_k p_i p_j u_{ijk}]}((1-t)q + tu_{ijk})$ be the shooting polynomial for the face $[p_i p_j p_k]$ (see (4.8)). Then the main computation for evaluating a surface point is to compute the coefficients $B_s^{(ijk)}(q)$ and then find the roots of a cubic polynomial. Since $B_3^{(ijk)}(q) = f_{0003}^{ijk}$, the costs of the computation of $B_s^{(ijk)}(q)$ is zero if $B_s^{(ijk)}(q)$ are pre-computed (before the interactive control step) and kept, for $s = 0, 1, 2$ and $q = q_{xyz}$ (see Section 5.1). Since there exists a closed form for finding the roots of a cubic polynomial, the cost of finding these roots is very small.

For displaying the edge A-patch, the situation is similar, but forming

$$B_3^{(ijkl)}(q, u) = \sum_{\lambda_1+\lambda_4=3} f_\lambda^{(ijkl)} B_{\lambda_4}^3(y)$$

needs little computation (see (4.9)) when $f_{3000}^{(ijkl)}$ or $f_{0003}^{(ijkl)}$ changes.

6. Conclusions and examples

There are two main contributions of this paper. The first is the use of rational terms in the modeling function. These provide enough degrees of freedom so that: (i) we do not need to subdivide the edge patch and hence the overall number of patches is reduced; (ii) the modeling function interpolates additional data on the middle of the edges and hence leads to better shaped A-patches (see Figs. 6, 7, 9, 10, 2 and 3); (iii) the A-patch can smoothly join planar faces (see Fig. 10) and also has the quadratic precision property. The second major contribution is that multiple sheeted surfaces are allowable in the hull. This widens the class of the algebraic spline surfaces to select nicer looking shaped C^1 surface models. However, we have provided a robust scheme to select a sheet of the surface for a final C^1 surface which is topologically equal to the triangulation. The implementation of the paper and our test examples show that these two techniques work very well to make it a practical method for free form modeling. We point out that the techniques proposed in this paper can be used in the C^1 quadric A-patch introduced by Guo (1991a) and the C^2 quintic A-patch introduced by Bajaj, Chen and Xu (1995a).

Figs. 6, 7, 9 and 10 show the different join configurations of two face A-patches. In Fig. 6, the two given faces $[p_1 p_2 p_3]$ and $[p_1 p_2 p_4]$ are convex, where $p_1 = (-2, 0, 0)^T$, $p_2 = (2, 0, 0)^T$, $p_3 = (0, 4, -1)^T$ and $p_4 = (0, -4, -1)^T$. The corresponding normals are chosen as $n_1 = (-1, 0, 1)^T$, $n_2 = (1, 0, 1)^T$, $n_3 = (0, 1, 1)^T$ and $n_4 = (0, -1, 1)^T$. In Fig. 7, the vertices are the same as in Fig. 6. The normals n_3 and n_4 are replaced by $(0, 1, 1.5)^T$ and $(0, 1, 1.5)^T$. Hence the face $[p_1 p_2 p_3]$ is convex and face $[p_1 p_2 p_4]$ is nonconvex. In Fig. 9, the vertices are the same as before, but the normals are chosen as $n_1 = n_2 = (1, 0, 1)^T$, $n_3 = (0, -1, 1.5)^T$, and $n_4 = (0, 1, 1.5)^T$. Hence the two faces are nonconvex and the common edge $[p_1 p_2]$ is also nonconvex. Fig. 10 shows a smooth join of a nonconvex face A-patch with a degenerate planar patch, where $p_1 = (-2, 0, 0)^T$, $p_2 = (2, 0, 0)^T$, $p_3 = (0, 4, -1)^T$ and $p_4 = (0, -4, 0)^T$. The normals are chosen as $n_1 = n_2 = n_4 = (0, 0, 1)^T$ and $n_3 = (0, -1, 1.5)^T$. Hence, the face $[p_1 p_2 p_3]$ is nonconvex and the face $[p_1 p_2 p_4]$ is zero convex. The number 3 and number 4 coefficients of the constructed function in Figs. 6, 7, 9 and 10 are determined by the default choice (see Step 5 and Step 7 in Section 4.2). Fig. 2 shows the surface family with changes in parameters $f_{0003}^{(ijk)}$ and $f_{3000}^{(ijl)}$, and keeping

Fig. 14. Edge patches and face patches for a head.

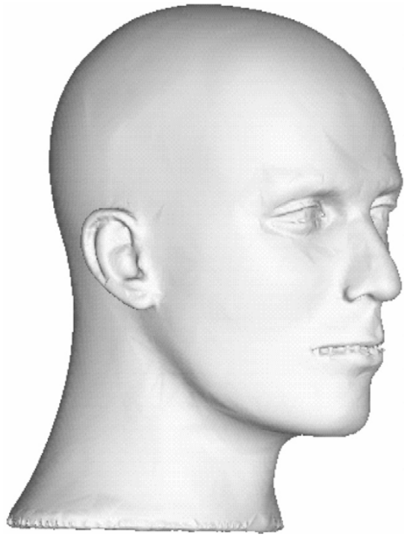

Fig. 15. Rational A-patch construction for a head.

the vertices and normals the same as in Fig. 6. The vertices and normals used in Fig. 3 is the same as those in Fig. 9. This example shows the multiple sheeted property of the zero contour. The free parameters are taken near the lower bound B_{ijk}, hence the surface is nearly singular at the point where the two sheets of the surface approximately touch. This shows also that the lower bound B_{ijk} given in Step 6 of Section 4.2 is exact. Figs. 14 and 15 show a smooth A-patch construction model of a human head. In this example, the number 3 and 4 coefficients in the construction of the function are given by the default choice of Section 4.2. In Fig. 14, edge patches and face patches are shown separately by different shading, to depict the topology of the input triangulation as well as its adaptivity. The higher curved regions have more triangles. If the input triangulation is relatively flat at an edge, then the edge patch is correspondingly degenerate. Fig. 17 is the A-patch construction for the triangulation shown in Fig. 16. Again, the edge A-patches and face A-patches are shown by different shading, and mirror each other's topology.

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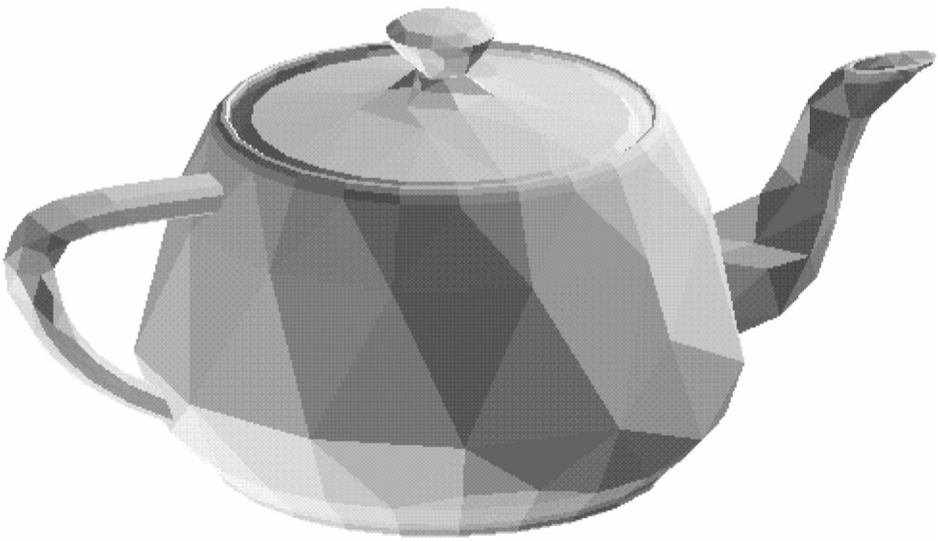

Fig. 16. The input triangulation of a teapot.

Fig. 17. The edge patches and face patches for the teapot.

Appendix A. The regularity of the surface patch

Consider first a face patch in $[p_i p_j p_k u_{ijk}]$ for a convex face $[p_i p_j p_k]$. Let $F|_{[p_k p_i p_j u_{ijk}]}$ be extended to $\mathbb{R}^3$ using its closed form (4.1) in $[p_i p_j p_k u_{ijk}]$ (we still use F to denote the extended function). Then F is C^1 in $\mathbb{R}^3$. Let $\phi_q(t) = F(p(t, q))$ be the shooting polynomial at $q \in [p_i p_j p_k]$ and $t_{\min}(q)$ be the minimal root in defining the surface point $p(q) := p(t_{\min}(q), q)$, where

$$p(t, q) = (1-t)q + tu_{ijk}. \tag{A.1}$$

Then since $t_{\min}(q)$ is a simple zero of ϕ_q,

$$\frac{\mathrm{d}\phi_q(t_{\min}(q))}{\mathrm{d}t} = \nabla F\big(p(q)\big)^{\mathrm{T}}(u_{ijk} - q) \neq 0. \tag{A.2}$$

WLG, we may assume $\nabla F(p(q))^{\mathrm{T}}(u_{ijk} - q) > 0$. Then by the C^1 continuity of F, there is an open sphere $\mathcal{N}_q$ with center $p(q)$, such that

$$\nabla F(p)^{\mathrm{T}}\big(u_{ijk} - q(p)\big) > 0 \quad \text{for } \forall p \in \mathcal{N}_q, \tag{A.3}$$

where

$$q(p) = sp + (1-s)u_{ijk}, \quad s = (c - u_{ijk})^{\mathrm{T}} n_{ijk}/(p - u_{ijk})^{\mathrm{T}} n_{ijk},$$

$c = (p_i + p_j + p_k)/3$ and n_{ijk} is the normal of the face $[p_i p_j p_k]$. $q(p)$ is the projection of p onto the plane $\langle p_i p_j p_k \rangle$ in the direction $p - u_{ijk}$. Condition (A.3) implies that F is monotonic in the direction $u_{ijk} - q(p)$ in the sphere $\mathcal{N}_q$. Let $S_q = \{F = 0\} \cap \mathcal{N}_q$. Then S_q is a smooth surface, since $\nabla F(p) \neq 0$ for $p \in \mathcal{N}_q$. Hence each surface point $p \in S_q$ define a simple zero $t(p)$ of the corresponding shooting polynomial $\phi_{q(p)}(t)$. It follows from (A.1) that

$$t(p) = (c - p)^{\mathrm{T}} n_{ijk}/(c - u_{ijk})^{\mathrm{T}} n_{ijk}. \tag{A.4}$$

Our goal is to prove that there exists a neighborhood $\mathcal{N}_q' \subset \mathcal{N}_q$ of $p(q)$ such that $t(p) = t_{\min}(q(p))$ for $p \in \mathcal{N}_q' \cap S_q$. If this is not true, there exists a sequence $p^{(l)} \in S_q$, such that $\|p^{(l)} - p(q)\| \to 0$ (this and (A.4) imply that $t(p^{(l)})$ converge to $t_{\min}(q)$), and $t_{\min}(q(p^{(l)})) \neq t(p^{(l)})$. Suppose $t_{\min}(q) > 0$ (the discussion is similar for the other cases). WLG, we may assume $t_{\min}(q(p^{(l)}))$ converge to t' and $t_{\min}(q(p^{(l)})) < t(p^{(l)})$. Since $F(p(q(p^{(l)}))) = 0$, the continuity of F implies that $F(p(t', q)) = 0$. If $t' < t_{\min}(q)$, a contradiction is yielded since $t_{\min}(q)$ is minimal zero. Hence $t' = t_{\min}(q)$. Then by Rolle's theorem, there is a $\xi^{(l)} \in (t_{\min}(q(p^{(l)})), t(p^{(l)})$, such that

$$0 = \frac{F(p^{(l)}) - F(p(q(p^{(l)})))}{t(p^{(l)}) - t_{\min}(q(p^{(l)}))} = \nabla F\big(p(\xi^{(l)}, q\big(p^{(l)}\big))\big)^{\mathrm{T}}\big[u_{ijk} - q\big(p^{(l)}\big)\big].$$

From the continuity, we have $\nabla F(p(t_{\min}(q), q)^{\mathrm{T}}[u_{ijk} - q] = 0$, a contradiction to (A.2). Therefore, the neighborhood $\mathcal{N}_q'$ does exist. Let $S_q' = S_q \cap \mathcal{N}_q'$. Then the surface patch S_q' is defined by the minimal root of the shooting polynomial. Let $N_q = \{q(p)\colon\ p \in S_q'\}$. Then by (A.3) N_q is an open neighborhood of q. Since all the N_q form an open covering of $[p_i p_j p_k]$, then by Heine–Borel theorem (see (Royden, 1968) p. 42), there exists a finite number of $N_{q^{(l)}}$ such that their union covers $[p_i p_j p_k]$. For any point in $N_{q^{(m)}} \cap N_{q^{(n)}}$, the shooting polynomial is the same and hence the surface point is the same. Therefore, the composite surface is connected.

If the face $[p_i p_j p_k]$ is nonconvex, we could divide the face into several connected closed regions D_l^+ and D_l^-, such that $F \geqslant 0$ on D_l^+, and $F \leqslant 0$ on D_l^-. After carrying out the proof above for each of the regions, we have a connected surface over each region. On the common boundaries (at where $F = 0$) of these regions, the surface points coincide with boundary points, hence the composite surface is connected. For the edge surface, the proof of the connectness is similar.

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