C^1 Modeling with Hybrid Multiple-sided A-patches

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Abstract

We propose a new scheme for modeling a smooth interpolatory surface, from a surface discretization consisting of triangles, quadrilaterals and pentagons, by algebraic surface patches which are subsets of real zero contours of trivariate rational functions defined on a collection of tetrahedra and pyramids. The rational form of the modeling function provides enough degrees of freedom so that the number of the surface patches is significantly reduced, and the surface has quadratic recover property.

Key words: Implicit surface; rational A-patch; surface fit; tetrahedra; pyramids

1 Introduction

The problem considered in the present paper is to construct a smooth interpolatory surface from a surface discretization $\mathcal{L}$ by piecewise implicit surface patches. The discretization $\mathcal{L}$ of the surface consists of triangles, quadrilaterals and pentagons. The constructed surface passes through the vertices of the discretization and has the given normals at the vertices. This solution uses piecewise rational functions defined on a hull that consists of tetrahedra and pyramids (see Fig 1.1).

Several approaches to using implicit surface representation in modeling geometric objects have been proposed in papers (see for examples, ([1], [3], [6], [9], [11], [14]). Most of the schemes use various simplicial hulls over surface triangulation and polynomial functions (see [2], [3], [6], [9], [10]). They in general consist of the following three steps: **a**. Generate a normal for each vertex of $\mathcal{L}$ which will also be the normal of the constructed smooth surface at the vertex. **b**. Build a surrounding simplicial hull $\sum$ (consisting of a series of tetrahedra) of the triangulation. **c**. Construct a piecewise trivariate polynomial F within that simplicial hull, and use the zero contour of F to represent the surface. Dahmen [5] first

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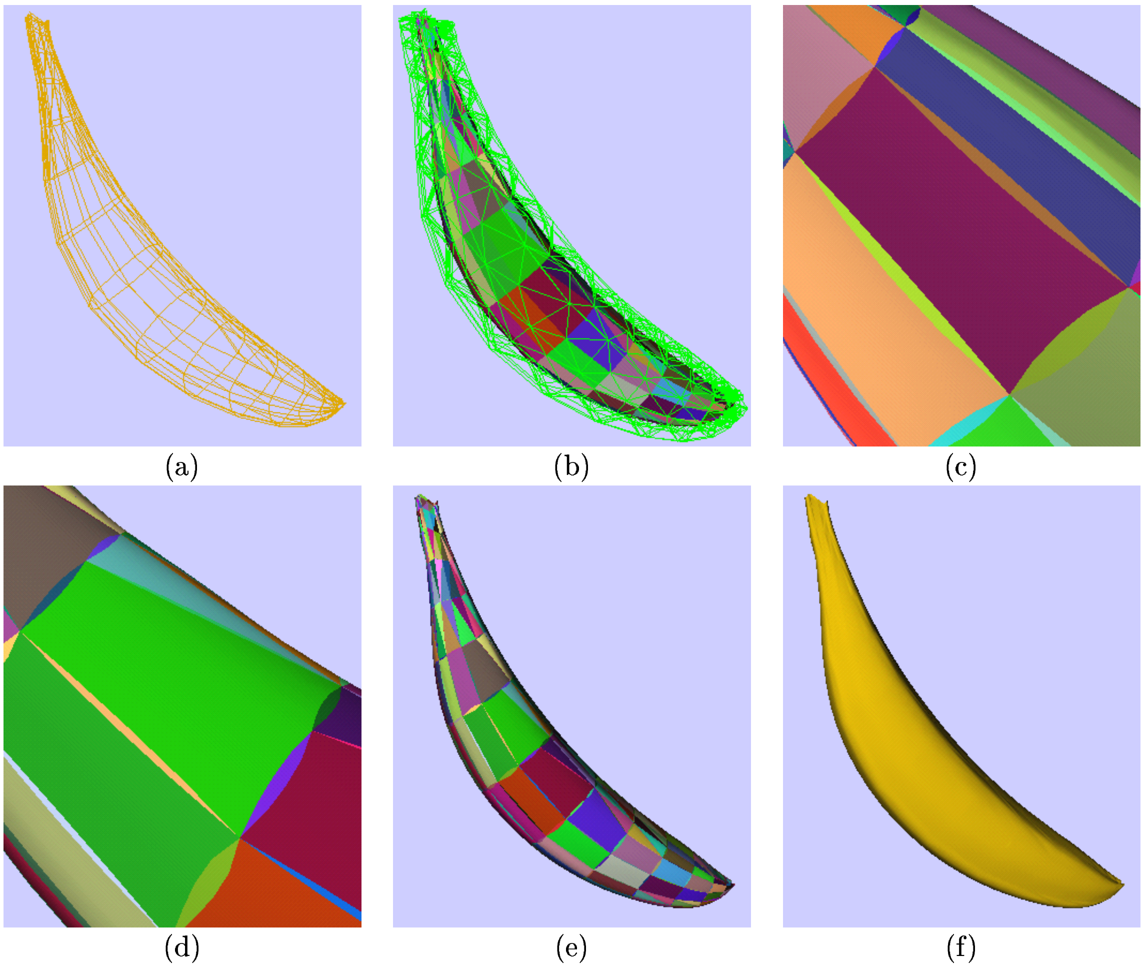


Fig 1.1: (a) Input mesh consists of three, four and five sided polygons; (b) Constructed hull; (c) A zoom in four-sided patch; (d) A zoom in five-sided patch; (e) and (f) Patches over tetrahedra and pyramids

proposed an approach for constructing a simplicial hull of $\mathcal{L}$. In this approach, for each face $[p_i p_j p_k]$ of $\mathcal{L}$, two points u_{ijk} and v_{ijk} off each side of the face are chosen and two tetrahedra $[p_i p_j p_k u_{ijk}]$ and $[p_i p_j p_k u_{ijk}]$ (called face tetrahedra) are constructed. For each edge of $\mathcal{L}$, two tetrahedra (called edge tetrahedra) are formed that blend the neighboring face tetrahedra. The collection of these tetrahedra contains the tangent plane near the vertices and have no self-intersection. Since such simplicial hulls are nontrivial to construct for arbitrary triangulation, several improvements have been made in later publications to overcome the difficulties (see [3], [6], [9], [10]). For the construction of the surface within $\sum$, Dahmen [5] used six quadric patches for each face tetrahedron and four quadric patches for each edge tetrahedron. Guo [9] uses a Clough-Tocher split to subdivide each face tetrahedron of the simplicial hull, hence utilizing six cubic patches per face of $\mathcal{L}$. The edge tetrahedra are subdivided into two. Dahmen and Thamm-Schaar [6] do not split the face tetrahedra, but the edge tetrahedra is split. All of these papers provided heuristics to overcome the multiple-sheeted and singularity problem of the implicit patches. Since the multi-sheeted property may cause the constructed surface to be disconnected, Bajaj et al [3] constructed A-patches that were guaranteed to be nonsingular, connected and single sheeted within each tetrahedron. Xu et al [17] use rational functions in constructing F so that the edge patches and convex face patches do not need to be split.

All the works mentioned above construct smooth implicit surface patches for the given surface triangulation. The more general parametric spline fitting problem of constructing a mesh of finite elements that interpolate or approximate multivariate data is discussed in [4]. One approach to creating multi–sided patches has been by introducing base points into rational parametric functions. Base points are parameter values for which the homogeneous coordinates (x,y,z,w) are mapped to (0,0,0,0) by the rational parameterization. Gregory's patch [8] is defined using a special collection of rational basis functions that evaluate to 0/0 at vertices of the parametric domain and thus introduce base points in the resulting parameterization. Warren [16] uses base points to create parameterizations of four-, five-, and six–sided surface patches using rational Bézier surfaces defined over triangular domains. Setting a triangle of weights to zero at one corner of the domain triangle produces a four–sided patch that is the image of the domain triangle. [13, 12] present generalizations of biquadratic and bicubic B-spline surfaces that are capable representing surfaces of arbitrary topology by placing restrictions on the connectivity of the control mesh, relaxing C^1 continuity to G^1 (geometric) continuity, and allowing n-sided finite elements. This generalized view considers the spline surface to be a collection of possibly rational polynomial maps from independent n-sided polygonal domains, whose union possesses continuity of some number of geometric invariants, such as tangent planes. This more general view allows patches to be sewn together to describe free form surfaces in more complex ways.

In this paper, we shall construct 3,4,5 sided A-patches from rational functions. That is F is a piecewise rational function defined on a hull that consists of tetrahedra and pyramids. The construction method of F on edge tetrahedra is the same as our earlier scheme [17]. However, the method of face patch construction is new. Although the modeling function F is rational in form, it is evaluated as easy as cubic (see section 4 and 5). Furthermore, the surface constructed has plane recovery property. That is, if the normals at the vertices of a face are perpendicular to the face, then the surface coincides with the face. Having this feature is important since many geometric objects have planar portion. Even further, the surface constructed could recover quadratics.

The paper is organized as follows. Section 2 introduces notations and facts that lay the foundation of the present scheme. Section 3 builds the finite-element hull. The modeling function F and the computation of the parameters in F are described in section 4. In Section 5, we present schemes to evaluate 3–5 sided patches. Examples that show the effectiveness of the schemes are presented in Section 6. All the proofs are given in the Appendix.

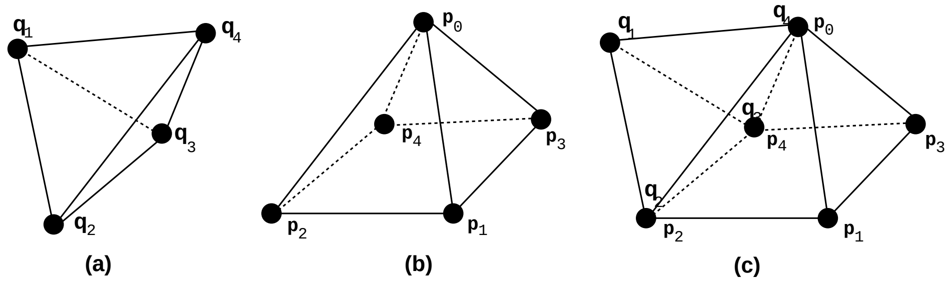


Fig 2.1: (a) Tetrahedron; (b) Pyramid; (c) Joined tetrahedron and pyramid

2 Bases

This section introduces the notations and the basic knowledges used in this paper. All the proofs of the introduced facts are given in the Appendix.

Let p_1 and p_2 be two different points in $I\!\!R^3$. We use $[p_1p_2]$ to denote the line segment that has end-points p_1 and p_2. Let $p_1, \cdots, p_k$ be k $(k > 3)$ different points in $I\!\!R^3$. Then we use $\langle p_1 \cdots p_k \rangle$ to denote the polygon consisting of $[p_1p_2], \cdots, [p_{k-1}p_k]$, $[p_kp_1]$. Further, we use the following notations

$$[p_1p_2p_3] = \{p = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 : \ \alpha_i \in [0,1], \alpha_1 + \alpha_2 + \alpha_3 = 1\}$$
$$[p_1p_2p_3p_4] = \{p = \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 : \ \alpha_i \in [0,1], \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1\}$$
$$(p_1p_2p_3p_4) = \{p = t[sp_1 + (1-s)p_2] + (1-t)[sp_3 + (1-s)p_4] : \ (s,t) \in [0,1]^2\}$$
$$[p_0p_1 \cdots p_4] = \{p = up_0 + (1-u)q : \ u \in [0,1], \ q \in (p_1p_2p_3p_4)\}$$

That is, $[p_1p_2p_3]$ is a triangle, $(p_1p_2p_3p_4)$ is ruled surface, $[p_1p_2p_3p_4]$ is a tetrahedron and $[p_0p_1p_2p_3p_4]$ is a pyramid.

1. BB Form on Tetrahedra. The trivariate polynomials defined in a tetrahedron are expressed in Bernstein-Bézier (BB) form in this paper. Let $q_1, q_2, q_3, q_4 \in I\!\!R^3$ (see Fig. 2.1(a)) be affine independent. Then any $p \in I\!\!R^3$ could be written as is

$$p = (x,y,z)^T = \sum_{i=1}^{4} \alpha_i q_i, \qquad \sum_{i=1}^{4} \alpha_i = 1 \tag{2.1}$$

$\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p. Any polynomial $F_T(p)$ of degree n then can be expressed as BB form over $[q_1q_2q_3q_4]$ as

$$F_T(p) = \sum_{i+j+k+l=n} a_{ijkl} \ B^n_{ijkl}(\alpha) \tag{2.2}$$

where $B^n_{ijkl}(\alpha) = \frac{n!}{!j!k!l!} \alpha_1^i \alpha_2^j \alpha_3^k \alpha_4^l$ is the Bernstein polynomial. Lemma 2.1 in the following gives conditions of C^1 join of two BB form polynomials defined on two adjacent tetrahedra.

Lemma 2.1 ([7]) *Let* $F_T(p) = \sum_{i+j+k+l=n} a_{ijkl} B^n_{ijkl}(\alpha)$ *and* $G_T(p) = \sum_{i+j+k+l=n} b_{ijkl} B^n_{ijkl}(\alpha)$ *be two polynomials defined on two tetrahedra* $[q_1q_2q_3q_4]$ *and* $[q_1'q_2q_3q_4]$*, respectively. Then*
(i) F_T *and* G_T *are* C^0 *at the face* $[q_2q_3q_4]$ *iff* $a_{0jkl} = b_{0jkl}$ *for any* $j+k+l=n$
(ii) F_T *and* G_T *are* C^1 *at the face* $[q_2q_3q_4]$ *iff they are* C^0 *and*

$$b_{1,j,k,l} = \beta_1 a_{1,j,k,l} + \beta_2 a_{0,j+1,k,l} + \beta_3 a_{0,j,k+1,l} + \beta_4 a_{0,j,k,l+1}, \quad j+k+l=n-1 \tag{2.3}$$

where $\beta = (\beta_1, \beta_2, \beta_3, \beta_4)^T$ *is defined by the relation* $q_1' = \beta_1 q_1 + \beta_2 q_2 + \beta_3 q_3 + \beta_4 q_4, \quad |\beta| = 1.$

Degree Elevation ([7]). A polynomial $\sum_{i+j+k+l=n-1} a_{ijkl} B^{n-1}_{ijkl}(\alpha)$ of degree $n-1$ could be written as a polynomial $\sum_{i+j+k+l=n} b_{ijkl} B^n_{ijkl}(\alpha)$ of degree n with

$$b_{ijkl} = \frac{i}{n} a_{i-1,jkl} + \frac{j}{n} a_{i,j-1,kl} + \frac{k}{n} a_{ij,k-1,l} + \frac{l}{n} a_{ijk,l-1} \tag{2.4}$$

2. BB Form on Pyramid. The BB form polynomial of degree n on a pyramid (see Fig. 2.1(b)) $[p_0p_1p_2p_3p_4]$ is defined by

$$F_P(x,y,z) := f(u,s,t) := \sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i} b_{ijk} B^n_i(u) B^{n-i}_j(s) B^{n-i}_k(t) \tag{2.5}$$

where $(x,y,z) \in [p_0p_1p_2p_3p_4]$ and $(u,s,t) \in [0,1]^3$ are related by

$$(x,y,z)^T = up_0 + (1-u)\{t[sp_1 + (1-s)p_2] + (1-t)[sp_3 + (1-s)p_4]\}. \tag{2.6}$$

Since transform (2.6) is not linear, a polynomial in (u,s,t) may not be written as a polynomial in (x,y,z). However, a polynomial in (x,y,z) of total degree n could always be written as the same degree polynomial in (u,s,t). Since transform (2.6) is not invertible at the point p_0, where $(1,s,t)$ map to p_0 for any (s,t), the polynomial F_P may not be smooth at the point p_0 even though F_P is any time differentiable in the local system (u,s,t). Fortunately, we do not use the smoothness of F_P at p_0 in this paper. The following theorem gives the conditions of C^k join between two polynomials that are defined on an adjacent tetrahedron and pyramid (see Fig. 2.1(c)), respectively.

Theorem 2.1 *Let* $F_T(x,y,z)$ *and* $F_P(x,y,z)$ *be defined as (2.2) and (2.5) on* $[q_1q_2q_3q_4]$ *and* $[p_0p_1p_2p_3p_4]$*, respectively, with* $q_2 = p_2, q_3 = p_4$ *and* $q_4 = p_0$*. Then* F_T *and* F_P *are* C^K *join on the interface* $[q_2q_3q_4] \setminus \{q_4\}$ *if and only if*

$$b_{ijk} = \sum_{\substack{i_1+i_2+j_1+j_2+j_3+k_1+k_2+k_3+l_1+l_2=n-i \\ i_1+i_2+j_1+j_2+k_1+k_2+l_1+l_2=j \\ i_1+j_1+k_1+l_1+j_3=k}} \frac{C^k_{j_3,k-j_3} C^{i_2+j_2+k_2+l_2+k_3}_{k_3,i_2+j_2+k_2+l_2}}{C^{n-i}_{j_3+k_3,n-i-j_3-k_3}}$$
$$* a_{i_1+i_2,j_1+j_2+j_3,k_1+k_2+k_3,l_1+l_2+i} B^{k-j_3}_{i_1,j_1,k_1,l_1}(a) B^{j-k+j_3}_{i_2,j_2,k_2,l_2}(b) \tag{2.7}$$

for $j = 0, \cdots, K$, $i = 0, \cdots, n-j$, *and* $k = 0, \cdots, n-i$, *where* $a = (a_1, a_2, a_3, a_4)^T$ *and* $b = (b_1, b_2, b_3, b_4)^T$ *are defined by*

$$p_1 = \sum_{i=1}^{4} a_i q_i, \quad \sum_{i=1}^{4} a_i = 1, \quad p_3 = \sum_{i=1}^{4} b_i q_i, \quad \sum_{i=1}^{4} b_i = 1 \tag{2.8}$$

Taking $K = 1$ in Theorem 2.1, we have the following corollary.

Corollary 2.1 F_T *and* F_P *that are defined in Theorem 2.1 are* C^1 *at* $[q_2q_3q_4] \setminus \{q_4\}$ *iff*

$$\begin{aligned} b_{i,0,k} &= a_{0,k,n-i-k,i}, \quad i = 0, \cdots, n, \quad k = 0, \cdots, n-i \\ b_{i,1,k} &= \frac{k}{n-i} \sum_{l=1}^{4} a_l a_{e_l+(0,k-1,n-i-k,i)} + \frac{n-i-k}{n-i} \sum_{l=1}^{4} b_l a_{e_l+(0,k,n-i-k-1,i)}, \\ &\qquad i = 0, \cdots, n-1, \quad k = 0, \cdots, n-i \end{aligned} \tag{2.9}$$

where $e_l \in I\!R^4$ *is the unit vector in the* l*-th direction,* $l = 1, \cdots, 4$.

Degree Elevation. A polynomial $\sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} \sum_{k=0}^{n-i-1} a_{ijk} \; B_i^{n-1}(u) B_j^{n-i-1}(s) B_k^{n-i-1}(t)$ of degree $n-1$ could be written as a polynomial $\sum_{i=0}^{n} \sum_{j=0}^{n-i} \sum_{k=0}^{n-i} b_{ijk} \; B_i^n(u) B_j^{n-i}(s) B_k^{n-i}(t)$ of degree n with

$$\begin{aligned} b_{ijk} = \frac{i}{n} a_{i-1,jk} &+ \frac{jk}{n(n-i)} a_{i,j-1,k-1} + \frac{(n-i-k)j}{n(n-i)} a_{i,j-1,k} \\ &+ \frac{(n-i-j)k}{n(n-i)} a_{ij,k-1} + \frac{(n-i-j)(n-i-k)}{n(n-i)} a_{ijk} \end{aligned} \tag{2.10}$$

3. C^1 of Cubics around an Edge. Let $q_1, q_2, \cdots, q_k$ be the given points around a line segment $[p_2p_3]$ such that q_{i-1} and q_{i+1} lie on different sides of the plane $[q_ip_2p_3]$ and all the tetrahedra $[q_iq_{i+1}p_2p_3]$ enclose the edge $[p_2p_3]$ as interior (see Figure 2.2(a)). Hence the five points $q_{i-1}, q_i, q_{i+1}, p_2, p_3$ are related by either

$$q_i = \alpha_1^i q_{i-1} + \alpha_2^i q_{i+1} + \alpha_3^i p_2 + \alpha_4^i p_3, \quad \sum_{j=0}^{4} \alpha_j^i = 1 \tag{2.11}$$

if $q_{i-1}, q_{i+1}, p_2, p_3$ are affine independent, or

$$0 = \alpha_1^i q_{i-1} + \alpha_2^i q_{i+1} + \alpha_3^i p_2 + \alpha_4^i p_3, \quad \sum_{j=0}^{4} \alpha_j^i = 0 \tag{2.12}$$

if $q_{i-1}, q_{i+1}, p_2, p_3$ are affine dependent, where $\alpha_1^i \neq 0$ and $\alpha_2^i \neq 0$. Let F_i be the cubic polynomial in BB-form on the tetrahedron $[q_iq_{i+1}p_2p_3]$ that satisfy C^0 condition. Let x_i

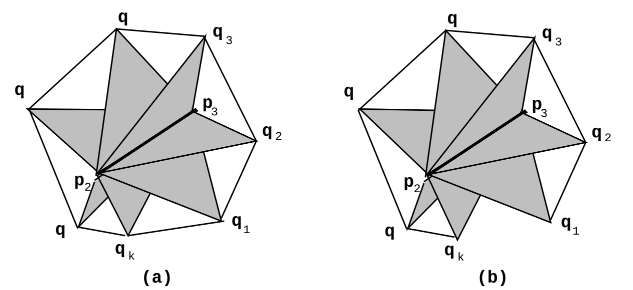


Fig 2.2: Tetrahedra around Edges. (a) Closed; (b) Open

be the Bézier coefficients on the center of $[q_i p_2 p_3]$. Then the C^1 condition at the interface $[q_i p_2 p_3]$ is either

$$x_i = \alpha_1^i x_{i-1} + \alpha_2^i x_{i+1} + \alpha_3^i b_2 + \alpha_4^i b_3 \tag{2.13}$$

if $q_{i-1}, q_{i+1}, p_2, p_3$ are affine independent, or

$$0 = \alpha_1^i x_{i-1} + \alpha_2^i x_{i+1} + \alpha_3^i b_2 + \alpha_4^i b_3 \tag{2.14}$$

otherwise, where b_2 and b_3 are the Bézier coefficients on the edge $[p_2 p_3]$. Then we have

Theorem 2.2 *If the tetrahedra around the edge $[p_2 p_3]$ enclose the edge, then (i) there are $k-2$ independent equations among the k C^1 conditions (2.13)–(2.14) with k unknowns around the edge; (ii) any two adjacent equations of them can be deleted; (iii) any two unknowns x_m, x_n can be chosen as free parameters if q_m, q_n, p_2, p_3 are affine independent.*

For the open case (see Figure 2.2(b)), we can treat it as closed case with the first and last equations being deleted. Hence we have by Theorem 2.2 that

Corollary 2.2 *If the tetrahedra around the edge $[p_2 p_3]$ do not enclose the edge, then (i) the $k-2$ C^1 conditions (2.13)–(2.14) with k unknowns around the edge is independent; (ii) any two unknowns x_m, x_n can be chosen as free parameters if q_m, q_n, p_2, p_3 are affine independent.*

Theorem 2.3 *Let $\Delta = \cup_{i=1}^k [q_i q_{i+1} p_2 p_3]$. Let $S_3^1(\Delta)$ be the collection of functions that are C^1 on Δ and cubics on each tetrahedron of Δ. Then*

$$\dim S_3^1(\Delta) = 4k + 10$$

if $[q_i p_2 p_3]$, $i = 1, \cdots, k$, lie on at least three different planes, and

$$\dim S_3^1(\Delta) = 4k + 12$$

if $[q_i p_2 p_3]$, $i = 1, \cdots, k$, lie on two different planes.

Note that the index of q_{i+1} is out off the range $1, \cdots, k$ when $i = k$. We assume in this paper that it is modulo by k. This convention is used throughout the paper without indication.

4. Miscellaneous. If a trivariate function F could be expressed as Bernstein polynomial form on a line segment $[p_1p_2]$. That is, $F|_{[p_1p_2]}(p) = \sum_{i=0}^{n} b_i B_i^n(t)$ with $p = (1-t)p_1 + tp_2$. Then

$$b_0 = F(p_1), \quad b_1 = b_0 + \frac{1}{n}(p_2 - p_1)^T \nabla F(p_1) \tag{2.15}$$

3 Finite Element Hull

Suppose we are given a surface discretization $\mathcal{L}$ consisting of triangles, quadrilaterals and pentagons with attached normal on each vertex. We assume that the surface is double sided and all the normals on the vertices point to one side of the discretization. We call this side as positive. The other side is negative. Since we do not assume the vertices of any quadrilateral or pentagon are coplanar, we do not call the quadrilateral or pentagon as face, but *polygon*.

Let $[p_ip_j]$ be an edge of $\mathcal{L}$, if $(p_j-p_i)^T n_j \ (p_i-p_j)^T n_i \geq 0$ and at least one of $(p_j-p_i)^T n_j$ and $(p_i - p_j)^T n_i$ is positive, then we say the edge $[p_ip_j]$ is *positive convex* (see [3]). If both the numbers are zero then we say it is *zero convex*. The *negative convex* edge is similarly defined. If $(p_j - p_i)^T n_j \ (p_i - p_j)^T n_i < 0$, then we say the edge is *non-convex*. Let F be a polygon of $\mathcal{L}$. If all its edges are nonnegative (positive or zero) convex and at least one of them is positive convex, then we say the polygon is *positive convex*. If all its edges are zero convex then we label the polygon as *zero convex*. The *negative convex* polygon is similarly defined. All the other cases are labeled as *non-convex*.

Let $\mathcal{L} = \mathcal{L}_{non-zero} \cup \mathcal{L}_{zero}$, where $\mathcal{L}_{non-zero}$ and $\mathcal{L}_{zero}$ are the collections of non-zero convex polygons and zero convex polygons of $\mathcal{L}$, respectively.

Now we construct a *finite-element-hull*, denoted as $\mathcal{H}$, that consists of tetrahedra and pyramids on $\mathcal{L}_{non-zero}$ such that each polygon of $\mathcal{L}_{non-zero}$ is contained in $\mathcal{H}$ and tangent plane at each vertex of $\mathcal{L}_{zero}$ is contained in $\mathcal{H}$.

a. Build Tetrahedra for Convex Triangle. Let $\langle p_1p_2p_3\rangle$ be a convex triangular polygon of $\mathcal{L}_{non-zero}$. Let $c = (p_1 + p_2 + p_3)/3$, n be the normal of face $[p_1p_2p_3]$ that points to the positive side of $\mathcal{L}$. Then choose a *top vertex* u if the polygon is positive convex or a *bottom vertex* v if the polygon is negative convex as follows: $u = c+tn$, or $v = c-tn$. Then *positive face tetrahedron* $[up_1p_2p_3]$ or *negative face tetrahedron* $[vp_1p_2p_3]$ are formed (see Fig. 3.1(a)), where $t > 0$ is a properly chosen number such that the tangent planes at the vertices are contained in the tetrahedra constructed.

b. Build Pyramids for Convex Quadrilaterals. Let $\langle p_1p_2p_3p_4\rangle$ be a convex quadrilateral of $\mathcal{L}_{non-zero}$. Let $c = (p_1 + p_2 + p_3 + p_4)/4$, n be the normal of the ruled surface $(p_1p_2p_3p_4)$ at c that points to the positive side of $\mathcal{L}$. Then choose a *top vertex* u if the polygon is positive convex or a *bottom vertex* v if the polygon is negative convex as follows:

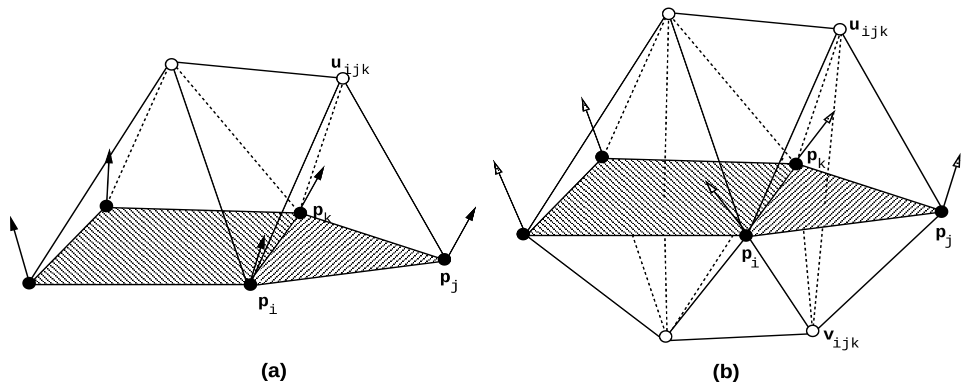


Fig 3.1: Finite-element Hull. (a) Convex case; (b) Non-convex case

$u = c + tn$, or $v = c - tn$. Then *positive face pyramid* $[up_1p_2p_3p_4]$ or *negative face pyramid* $[vp_1p_2p_3p_4]$ are formed (see Fig. 3.1(a)), where $t > 0$ is a properly chosen number such that the tangent planes at the vertices are contained in the pyramids constructed.

c. Build Tetrahedra for Non-convex Polygons. Let $\langle p_1p_2\cdots p_k\rangle$ $(3 \le k \le 5)$ be a non-convex polygon of $\mathcal{L}_{non-zero}$. Let $c = (p_1 + \cdots + p_k)/k$. Define a normal n as the average of the normals of the triangle faces $[p_ip_{i+1}c]$, $i = 1, \cdots, k$. Then both top vertex u and bottom vertex v are chosen as $u = c + tn$, or $v = c - tn$, and k tetrahedra $[uvp_ip_{i+1}]$, $i = 1, \cdots, k$, are formed (see Fig. 3.1(b)). Here $t > 0$ is defined so that the tangent planes at the vertices are contained in the tetrahedra constructed.

d. Build Tetrahedra for Edges. Let $[p_1p_2]$ be an edge of $\mathcal{L}$ where F_l and F_r are the two adjacent polygons in $\mathcal{L}_{non-zero}$. If the top vertices u_l and u_r exist for F_l and F_r (see Fig. 3.1), respectively, then the *positive edge tetrahedron* is $[u_lu_rp_1p_2]$. Similarly, the *negative edge tetrahedron* $[v_lv_rp_1p_2]$ is constructed if the bottom vertices v_l and v_r exist for F_l and F_r (see Fig. 3.1), respectively.

4 C^1 Modeling of Surface by Rational A-patches

In this section, we shall construct a piecewise C^1 rational function F over $\mathcal{H}$ whose zero contour $\{p : F(p) = 0\}$ possesses a separate subset S such that $S \cup \mathcal{L}_{zero}$ (i) passes through the vertices of $\mathcal{L}$, (ii) has the given normal at each vertex, and (iii) is a smooth surface. We further require that the function F has quadratic recovery property.

4.1 Modeling Functions

First we give the forms of the modeling functions over the finite elements. The parameters in these functions will be specified later in this section.

1. Function on tetrahedron for a convex triangular polygon. Let $\langle p_1p_2p_3\rangle \in$

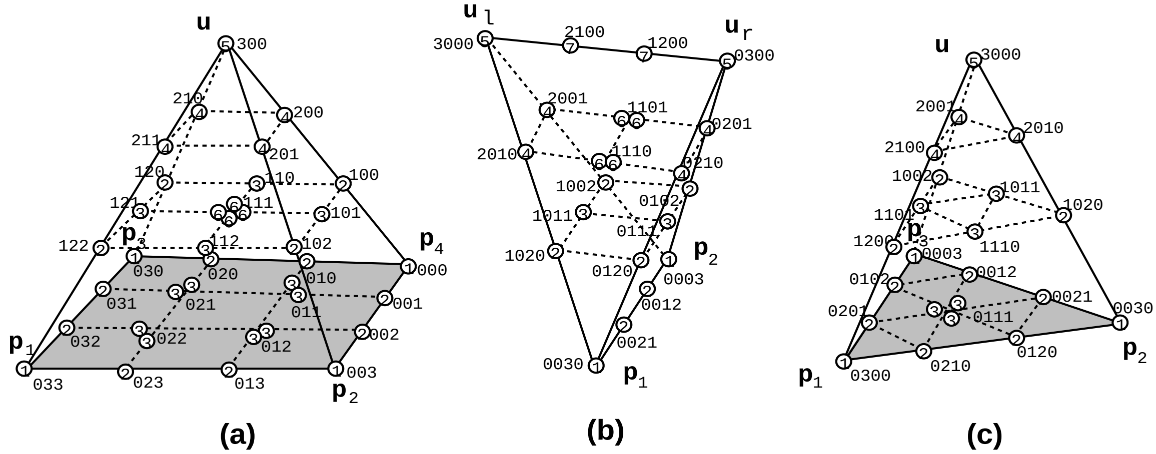


Fig 4.1: Weight indices on finite-element. (a) Polynomial of degree 3 on pyramid with rational terms; (b) Cubic on edge tetrahedron with rational terms; (c) Cubic on face tetrahedron with rational terms; All the weights are numbered (in circle) which will be referred in the Algorithms

$\mathcal{L}_{non-zero}$ be any one convex triangular polygon. If the positive face tetrahedron $[up_1p_2p_3]$ exists, we define (see Fig. 4.1(c) for the indices of the coefficients)

$$
\begin{aligned}
F|_{[up_1p_2p_3]} &= \sum_{i+j+k+l=3} t_{ijkl} B^3_{ijkl}(\alpha) \\
&+ \frac{t^{(3)}_{0111}\alpha_2\alpha_3 + t^{(2)}_{0111}\alpha_2\alpha_4 + t^{(1)}_{0111}\alpha_3\alpha_4}{\alpha_2\alpha_3 + \alpha_2\alpha_4 + \alpha_3\alpha_4} B^3_{0111}(\alpha) \qquad (4.1)
\end{aligned}
$$

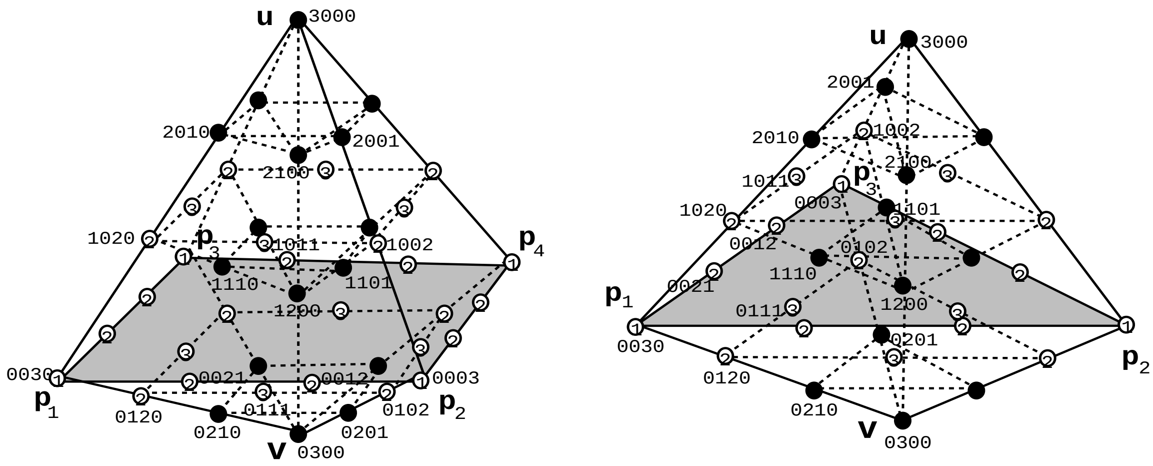
$F|_{[vp_1p_2p_3]}$ is similarly defined if the negative face tetrahedron $[vp_1p_2p_3]$ exists. In the following we only give the expressions of the functions on positive elements. The functions on the negative elements are in the same forms. To distinguish the difference, we place a tilde on the corresponding coefficients.

2. Function on pyramid for convex quadrilateral. Let $\langle p_1p_2p_3p_4\rangle$ be a convex quadrilateral of $\mathcal{L}$ and $[up_1p_2p_3p_4]$ be the pyramid. Then define (see Fig. 4.1(a) for the indices of the coefficients)

$$
\begin{aligned}
F|_{[up_1p_2p_3p_4]} = &\sum_{i=0}^{3}\sum_{j=0}^{3-i}\sum_{k=0}^{3-i} p_{ijk} B^3_i(u) B^{3-i}_j(s) B^{3-i}_k(t) \\
&+ \left[\left(p^{(l)}_{022}w^{(l)}_{lb} + p^{(b)}_{022}w^{(b)}_{lb}\right) B^3_2(s) + \left(p^{(r)}_{012}w^{(r)}_{rb} + p^{(b)}_{012}w^{(b)}_{rb}\right) B^3_1(s)\right] B^3_0(u) B^3_2(t) \\
&+ \left[\left(p^{(l)}_{021}w^{(l)}_{lt} + p^{(t)}_{021}w^{(t)}_{lt}\right) B^3_2(s) + \left(p^{(r)}_{011}w^{(r)}_{rt} + p^{(t)}_{011}w^{(t)}_{rt}\right) B^3_1(s)\right] B^3_0(u) B^3_1(t) \\
&+ \left(p^{(l)}_{111}w_l + p^{(r)}_{111}w_r + p^{(t)}_{111}w_t + p^{(b)}_{111}w_b\right) B^3_1(u) B^2_1(s) B^2_1(t) \qquad (4.2)
\end{aligned}
$$

where

$$
w^{(l)}_{lb} = \frac{v_l}{v_l + v_b}, \quad w^{(b)}_{lb} = \frac{v_b}{v_l + v_b}, \quad w^{(r)}_{rb} = \frac{v_r}{v_r + v_b}, \quad w^{(b)}_{rb} = \frac{v_b}{v_r + v_b},
$$



(a) **(b)**

Fig 4.2: Weight indices on finite-element. (a) Four cubics on tetrahedra for quadrilateral; (b) Three cubics on tetrahedra for triangle

$$w_{lt}^{(l)} = \frac{v_l}{v_l + v_t}, \quad w_{lt}^{(t)} = \frac{v_t}{v_l + v_t}, \qquad w_{rt}^{(r)} = \frac{v_r}{v_r + v_t}, \quad w_{rt}^{(t)} = \frac{v_t}{v_r + v_t},$$
$$w_l = \frac{v_l}{v_l + v_r + v_t + v_b}, \qquad w_r = \frac{v_r}{v_l + v_r + v_t + v_b},$$
$$w_t = \frac{v_t}{v_l + v_r + v_t + v_b}, \qquad w_b = \frac{v_b}{v_l + v_r + v_t + v_b},$$

with

$$v_l(s,t) = s^2t^2(1-t)^2, \qquad v_r(s,t) = (1-s)^2t^2(1-t)^2$$
$$v_t(s,t) = s^2(1-s)^2(1-t)^2, \qquad v_b(s,t) = s^2(1-s)^2t^2$$

In Fig. 4.1(a), the coefficients $p_{111}^{(l)}, p_{111}^{(r)}, p_{111}^{(t)}$ and $p_{111}^{(b)}$ are numbered as 6. The other coefficients of rational terms are number as 3.

3. Function on tetrahedra for non-convex 3,4-sided polygon and 5-sided polygon. Let $\langle p_1 \cdots p_K \rangle$ be a non-convex polygon, $3 \le K \le 5$. Then K tetrahedra $[uvp_ip_{i+1}]$ have been constructed. On each tetrahedron, a cubic is used (see Fig. 4.2(a) and (b) for the indices of the coefficients):

$$F|_{[uvp_mp_{m+1}]} = \sum_{i+j+k+l=3} t_{ijkl}^{(m)} B_{ijkl}^3(\alpha), \quad m = 1, \cdots, K. \tag{4.3}$$

4. Function on edge tetrahedron. Let $[p_1p_2]$ be a non-zero-convex edge of $\mathcal{L}$ and $[u_lu_rp_1p_2]$ be the positive edge tetrahedron. Then define (see Fig. 4.1(b) for the indices of the coefficients)

$$
\begin{aligned}
F|_{[u_l u_r p_1 p_2]} &= \sum_{i+j+k+l=3} e_{ijkl} B^3_{ijkl}(\alpha) \\
&+ \frac{e^{(l)}_{1101}\alpha_1 + e^{(r)}_{1101}\alpha_2}{\alpha_1+\alpha_2} B^3_{1101}(\alpha) + \frac{e^{(l)}_{1110}\alpha_1 + e^{(r)}_{1110}\alpha_2}{\alpha_1+\alpha_2} B^3_{1110}(\alpha)
\end{aligned} \tag{4.4}
$$

4.2 Construction of Rational A-patches

Now we shall determine the parameters of F step by step (see Fig. 4.1, 4.2).

Total Algorithm. *Specifying the weights*
Step 1. In order to have the surface constructed contains the vertices of $\mathcal{L}$, we take the number 1 weights to be zero.
Step 2. The number 2 weights are determined by formula (2.15) from the normals.
Step 3. The number 3 weights in triangle interfaces are defined by interpolating the perpendicular directional derivative. For example, the number 3 weight on the triangle interface $[p_1 p_2 u]$ (see Fig. 4.1(c)) is defined by interpolating the directional derivative $\frac{1}{2}\left[\frac{(u-p_j)^T(p_1-p_2)}{\|p_1-p_2\|^2}(u-p_1) + \frac{(u-p_1)^T(p_2-p_1)}{\|p_1-p_2\|^2}(u-p_2)\right]^T(n_1+n_2)$ at the point $\frac{1}{2}(p_1+p_2)$ where the direction is in the face $[p_1 p_2 u]$ and perpendicular to the edge $[p_1 p_2]$. We can derive that

$$
t_{1110} = \frac{1}{2}\left[t_{1200} + t_{1020} + \alpha(u,p_1,p_2)t_{0210} + (1-\alpha(u,p_1,p_2))t_{0120}\right]
$$

where $\alpha(u,p_1,p_2)$ is given by

$$
\alpha(u,p_1,p_2) = \frac{[2(u-p_2)+(u-p_1)]^T(p_1-p_2)}{\|p_1-p_2\|^2}
$$

The three rational coefficients $t^{(1)}_{0111}$, $t^{(2)}_{0111}$ and $t^{(3)}_{0111}$ are defined as above by interpolating directional perpendicular derivatives at the mid-point of the edge $[p_2 p_3]$, $[p_1 p_3]$ and $[p_1 p_2]$, respectively. Set $t_{0111} = \frac{1}{3}\left(t^{(1)}_{0111} + t^{(2)}_{0111} + t^{(3)}_{0111}\right)$ and then reset the values of $t^{(1)}_{0111}$, $t^{(2)}_{0111}$ and $t^{(3)}_{0111}$ by reducing the value t_{0111}.

The number 3 weights, which are the coefficients of rational terms, on the quadrilaterals are determined by C^1 condition (2.3) or (2.9). Then set the corresponding polynomial coefficients as

$$
\begin{aligned}
p_{012} &= \frac{1}{2}\left(p^{(b)}_{012} + p^{(r)}_{012}\right), \quad p_{011} = \frac{1}{2}\left(p^{(t)}_{011} + p^{(r)}_{011}\right), \\
p_{022} &= \frac{1}{2}\left(p^{(l)}_{022} + p^{(b)}_{022}\right), \quad p_{021} = \frac{1}{2}\left(p^{(l)}_{021} + p^{(t)}_{021}\right).
\end{aligned}
$$

Then reset the rational coefficients by reducing the value of corresponding polynomial coefficients.
Step 4. The remaining weights on the finite elements are specified by the following sub-algorithms.

Sub-Algorithm 1. *Compute the weights on convex face tetrahedra*

The number 4 and 5 weights are free (see Fig. 4.1(c)). We assign the function value $F(u)$ and gradient $\nabla F(u)$ as parameters. Then $p_{3000} = F(u)$ and, by (2.15),

$$t_{2000+e_{i+1}} = F(u) + \frac{1}{3}(p_i - u)^T \nabla F(u), \quad i = 1, 2, 3.$$

The use of degrees of freedom.

Parameters $F(u)$ and $\nabla F(u)$ could be used to control the shape interactively. The default choice is we make the polynomial part of F defined by (4.1) approximate a quadratic. It follows from (2.4), we have a linear system with 14 unknowns and 20 equations. Since the coefficient matrix of the system is not full rank, we add a set of equations by making F approximate a linear function. Solving this system in the least square sense, we get the parameters.

Sub-Algorithm 2. *Compute the weights on convex pyramid*

The four number 4 weights and one number 5 weight are free (see Fig. 4.1(a)). We assign the function value $F(u)$ and gradient $\nabla F(u)$ as parameters. Then $p_{300} = F(u)$ and, by (2.15),

$$p_{211} = F(u) + \frac{1}{3}(p_1 - u)^T \nabla F(u), \quad p_{201} = F(u) + \frac{1}{3}(p_2 - u)^T \nabla F(u),$$
$$p_{210} = F(u) + \frac{1}{3}(p_3 - u)^T \nabla F(u), \quad p_{200} = F(u) + \frac{1}{3}(p_4 - u)^T \nabla F(u).$$

Note that defining the number 4 and 5 weights in this way reduces the degrees of freedom from five to four. The gain of this degree reduction is that the function defined by (4.2) is guarantee to be C^1 at u.

Now we consider the computation of coefficients $p_{111}^{(l)}, p_{111}^{(r)}, p_{111}^{(t)}$ and $p_{111}^{(b)}$ of rational terms. These coefficients could be computed separately. Suppose the pyramid considered is $[u_l p_1 p_2 p_3 p_4]$ and u_r is the top vertex of the element adjacent to the interface $[u_l p_2 p_4]$. Then $p_{111}^{(l)}$ is computed as follows. Let

$$p_1 = \alpha_1 u_l + \alpha_2 u_r + \alpha_3 p_2 + \alpha_4 p_4, \quad p_3 = \beta_1 u_l + \beta_2 u_r + \beta_3 p_2 + \beta_4 p_4.$$

Then by Corollary 2.1, we have

$$p_{112} = \alpha_1 e_{2010} + \alpha_2 e_{1110} + \alpha_3 e_{1020} + \alpha_4 e_{1011} = \alpha_1 p_{201} + \alpha_2 e_{1110} + \alpha_3 p_{102} + \alpha_4 p_{101},$$
$$p_{110} = \beta_1 e_{2001} + \beta_2 e_{1101} + \beta_3 e_{1011} + \beta_4 e_{1002} = \beta_1 p_{200} + \beta_2 e_{1101} + \beta_3 p_{101} + \beta_4 p_{100},$$

and

$$\begin{aligned} p_{111}^{(r)} &= \frac{1}{2}\left(\alpha_1 e_{2001} + \alpha_2 e_{1101} + \alpha_3 e_{1011} + \alpha_4 e_{1002} + \beta_1 e_{2010} + \beta_2 e_{1110} + \beta_3 e_{1020} + \beta_4 e_{1011}\right) \\ &= \frac{1}{2}\left[\left(\beta_1 - \frac{\alpha_1\beta_2}{\alpha_2}\right) p_{201} + \left(\alpha_1 - \frac{\alpha_2\beta_1}{\beta_2}\right) p_{200} + \frac{\beta_2}{\alpha_2} p_{112} + \frac{\alpha_2}{\beta_2} p_{110} \right. \\ &\quad \left. + \left(\beta_3 - \frac{\alpha_3\beta_2}{\alpha_2}\right) p_{102} + \left(\alpha_3 - \frac{\alpha_2\beta_3}{\beta_2} + \beta_4 - \frac{\alpha_4\beta_2}{\alpha_2}\right) p_{101} + \left(\alpha_4 - \frac{\alpha_2\beta_4}{\beta_2}\right) p_{100}\right]. \end{aligned}$$

Other coefficients of rational terms are similarly computed. Set $p_{111} = \frac{1}{4}(p_{111}^{(l)} + p_{111}^{(r)} + p_{111}^{(t)} + p_{111}^{(b)})$ and then reset the $p_{111}^{(l)}, p_{111}^{(r)}, p_{111}^{(t)}$ and $p_{111}^{(b)}$ by reducing their values by p_{111}.

The use of degrees of freedom.

Parameters $F(u)$ and $\nabla F(u)$ could be used to control the shape interactively. The default choice is we make the polynomial part of F defined by (4.2) approximate a quadratic. Let $\sum_{i+j+k+l=n} a_{ijkl} B_{ijkl}^n(\alpha)$ be a polynomial of degree n over the tetrahedron $[up_1p_2p_3]$. Then we could express it as a polynomial $\sum_{I=0}^n \sum_{J=0}^{n-I} \sum_{K=0}^{n-I} B_I^n(u) B_J^{n-I}(s) B_K^{n-I}(t)$ of degree n over the pyramid $[up_1p_2p_3p_4]$. Similar to the proof of Theorem 2.1, we can derive that

$$b_{IJK} = \sum_{l=I}^{n} \sum_{i+j+k=n-l} a_{ijkl} c_{ijk}^{IJK} \tag{4.5}$$

with

$$c_{ijk}^{IJK} = \sum_{\lambda=\max\{0,K-j,J-k\}}^{\min\{i,K,J\}} \frac{C_{\lambda,K-\lambda,J-\lambda,n-I-J-K+\lambda}^{n-I}}{C_{K,n-I-K}^{n-I} C_{J,n-I-J}^{n-I}} B_{i-\lambda,j-K+\lambda,k-J+\lambda,L-I}^{n-I-J-K+\lambda}(a_1, a_2, a_3, a_4),$$

where (a_1, a_2, a_3, a_4) is defined by $p_4 = a_1 u + a_2 p_1 + a_3 p_2 + a_4 p_4$, $\sum_{i=1}^4 a_i = 1$. Hence, a quadratic over $[up_1p_2p_3]$ could be expressed as a polynomial of degree 3 over $[up_1p_2p_3p_4]$ using (4.5) first and then (2.10). Approximating this quadratic by the polynomial part of F defined by (4.2) we lead to a linear system with 14 unknowns (10 for the coefficients of the quadratic, 4 for $F(u)$ and $\nabla F(u)$) and 30 equations. Solving this system in the least square sense, we get the parameters.

Sub-Algorithm 3. *Compute the weights on tetrahedra for non-convex polygon.*

Consider a $K-$sided polygon $\langle p_1 \cdots p_K \rangle$ for $3 \leq K \leq 5$. The number 1,2,3 weights have been determined (see Fig. 4.2(a) and (b)). The other weights labeled as $\bullet$ are defined by the C^1 condition. Under the C^0 condition

$$t_{ij0k}^{(s)} = t_{ijk0}^{(s+1)}, \quad i+j+k=3, \quad s=1,2,\cdots,K, \tag{4.6}$$

there are $3(K+1)+1$ weights undefined. From Theorem 2.3 we know that the dimension of $S_3^1(\Delta)$ is $4K+10$. Since the function under construction interpolates positions and gradients at the vertices p_i for $i = 1, \cdots K$, and interpolates two directional derivatives at the midpoints of the edges $[p_ip_{i+1}]$, that is, it satisfies $6K$ interpolation conditions, the remaining degree of freedom is $10-2K$. Now we take $F(v)$ and $\nabla F(v)$ as free parameters and express other weights in terms of these parameters and derive a system of $2K-6$ equations that the parameters $F(v)$ and $\nabla F(v)$ satisfy. That is, we express

$$t_{ijkl}^{(s)} = \alpha_{ijkl}^{(s)} F(v) + \beta_{ijkl}^{(s)} \nabla F(v) + \gamma_{ijkl}^{(s)}, \quad \alpha_{ijkl}^{(s)}, \gamma_{ijkl}^{(s)} \in I\!R, \; \beta_{ijkl}^{(s)} \in I\!R^3. \tag{4.7}$$

It is obvious that for the number 1,2,3 weights $t_{ijkl}^{(s)}$, $\alpha_{ijkl}^{(s)} = 0$, $\beta_{ijkl}^{(s)} = 0$ and $\gamma_{ijkl}^{(s)} = t_{ijkl}^{(s)}$. It follows from (2.15) that

$$t_{1200}^{(1)} = F(v) + \frac{1}{3}(u-v)^T \nabla F(v), \quad t_{0210}^{(s)} = F(v) + \frac{1}{3}(p_s - v)^T \nabla F(v), \quad s = 1, \cdots, K.$$

That is, $\alpha_{0210}^{(s)} = 1$, $\beta_{0210}^{(s)} = \frac{1}{3}(p_s - v)^T$, $\gamma_{0210}^{(s)} = 0$ and $\alpha_{1200}^{(1)} = 1$, $\beta_{1200}^{(1)} = \frac{1}{3}(u - v)^T$, $\gamma_{1200}^{(1)} = 0$. Let

$$u = \alpha_1^{(s)} p_s + \alpha_2^{(s)} p_{s+1} + \alpha_3^{(s)} p_{s-1} + \alpha_4^{(s)} v, \quad s = 1, \cdots, K.$$

Then

$$t_{1110}^{(s)} = \alpha_1^{(s)} t_{0120}^{(s)} + \alpha_2^{(s)} t_{0111}^{(s)} + \alpha_3^{(s)} t_{0111}^{(s-1)} + \alpha_4^{(s)} t_{0210}^{(s)}, \quad s = 1, \cdots, K. \tag{4.8}$$

Since $t_{0120}^{(s)}$, $t_{0111}^{(s)}$, and $t_{0111}^{(s-1)}$ are all number 2 and 3 weights, we have

$$t_{1110}^{(s)} = \alpha_1^{(s)} \gamma_{0120}^{(s)} + \alpha_2^{(s)} \gamma_{0111}^{(s)} + \alpha_3^{(s)} \gamma_{0111}^{(s-1)} + \alpha_4^{(s)} (\frac{1}{3}(p_s - v)^T \nabla F(v) + F(v))$$

Hence we have the same form expression for $\alpha_{1110}^{(s)}, \beta_{1110}^{(s)}$ and $\gamma_{1110}^{(s)}$. For example,

$$\begin{aligned}
\alpha_{1110}^{(s)} &= \alpha_1^{(s)} \alpha_{0120}^{(s)} + \alpha_2^{(s)} \alpha_{0111}^{(s)} + \alpha_3^{(s)} \alpha_{0111}^{(s-1)} + \alpha_4^{(s)} \alpha_{0210}^{(s)} = \alpha_4^{(s)}, \\
\beta_{1110}^{(s)} &= \alpha_1^{(s)} \beta_{0120}^{(s)} + \alpha_2^{(s)} \beta_{0111}^{(s)} + \alpha_3^{(s)} \beta_{0111}^{(s-1)} + \alpha_4^{(s)} \beta_{0210}^{(s)} = \tfrac{1}{3} \alpha_4^{(s)} (p_s - v)^T.
\end{aligned}$$

Furthermore, we have

$$t_{2100}^{(s)} = \alpha_1^{(s)} t_{1110}^{(s)} + \alpha_2^{(s)} t_{1110}^{(s+1)} + \alpha_3^{(s)} t_{1110}^{(s-1)} + \alpha_4^{(s)} t_{1200}^{(s)}, \tag{4.9}$$

and

$$\begin{aligned}
\alpha_{2100}^{(s)} &= \alpha_1^{(s)} \alpha_{1110}^{(s)} + \alpha_2^{(s)} \alpha_{1110}^{(s+1)} + \alpha_3^{(s)} \alpha_{1110}^{(s-1)} + \alpha_4^{(s)} \alpha_{1200}^{(s)} \\
&= \alpha_1^{(s)} \alpha_4^{(s)} + \alpha_2^{(s)} \alpha_4^{(s+1)} + \alpha_3^{(s)} \alpha_4^{(s+1)} + \alpha_4^{(s)}, \\
\beta_{2100}^{(s)} &= \alpha_1^{(s)} \beta_{1110}^{(s)} + \alpha_2^{(s)} \beta_{1110}^{(s+1)} + \alpha_3^{(s)} \beta_{1110}^{(s-1)} + \alpha_4^{(s)} \beta_{1200}^{(s)} \\
&= \tfrac{1}{3}[\alpha_1^{(s)} \alpha_4^{(s)} (p_s - v)^T + \alpha_2^{(s)} \alpha_4^{(s+1)} (p_{s+1} - v)^T \\
&+ \alpha_3^{(s)} \alpha_4^{(s+1)} (p_{s-1} - v)^T + \alpha_4^{(s)} (u - v)^T].
\end{aligned}$$

In the case of a planar polygon, $\alpha_4^{(s)} = \alpha_4^{(s+1)}$ for all s and we get

$$\begin{aligned}
\alpha_{2100}^{(s)} &= \alpha_4^{(s)} (\alpha_1^{(s)} + \alpha_2^{(s)} + \alpha_3^{(s)} + 1) \\
&= \alpha_4^{(s)} (2 - \alpha_4^{(s)}) \\
\beta_{2100}^{(s)} &= \tfrac{1}{3} \alpha_4^{(s)} [\alpha_1^{(s)} (p_s - v)^T + \alpha_2^{(s)} (p_{s+1} - v)^T + \alpha_3^{(s)} (p_{s-1} - v)^T + (u - v)^T] \\
&= \tfrac{1}{3} \alpha_4^{(s)} [-v^T (\alpha_1^{(s)} + \alpha_2^{(s)} + \alpha_3^{(s)} + 1) + \alpha_1^{(s)} p_s^T + \alpha_2^{(s)} p_{s+1}^T + \alpha_3^{(s)} p_{s-1}^T + u^T] \\
&= \tfrac{1}{3} \alpha_4^{(s)} [-v^T (2 - \alpha_4^{(s)}) + u^T - \alpha_4^{(s)} v^T + u^T] \\
&= \tfrac{2}{3} \alpha_4^{(s)} (u - v)^T.
\end{aligned}$$

It follows from Theorem 2.2 that (4.9) has $K - 2$ independent equations and they define the same weight $t_{2100}^{(1)} = t_{2100}^{(2)} = \cdots = t_{2100}^{(K)}$. Therefore, we have the following equations for $F(v)$ and $\nabla F(v)$

$$\alpha_{2100}^{(s)} F(v) + \beta_{2100}^{(s)} \nabla F(v) + \gamma_{2100}^{(s)} = \alpha_{2100}^{(s+1)} F(v) + \beta_{2100}^{(s+1)} \nabla F(v) + \gamma_{2100}^{(s+1)}, \tag{4.10}$$

for $s = 1, \cdots K-3$, Similarly, we have

$$t_{2010}^{(s)} = \alpha_1^{(s)} t_{1020}^{(s)} + \alpha_2^{(s)} t_{1011}^{(s)} + \alpha_3^{(s)} t_{1011}^{(s+2)} + \alpha_4^{(s)} t_{1110}^{(s)}, \quad s = 1, \cdots, K, \tag{4.11}$$

$$t_{3000}^{(s)} = \alpha_1^{(s)} t_{2010}^{(s)} + \alpha_2^{(s)} t_{2010}^{(s+1)} + \alpha_3^{(s)} t_{2010}^{(s+2)} + \alpha_4^{(s)} t_{2100}^{(s)}, \quad s = 1, \cdots, K, \tag{4.12}$$

and $F(v)$ and $\nabla F(v)$ satisfy the following equations

$$\alpha_{3000}^{(s)} F(v) + \beta_{3000}^{(s)} \nabla F(v) + \gamma_{3000}^{(s)} = \alpha_{3000}^{(s+1)} F(v) + \beta_{3000}^{(s+1)} \nabla F(v) + \gamma_{3000}^{(s+1)}, \tag{4.13}$$

for $s = 1, \cdots K-3$. Hence all the weights are defined and all $\alpha_{ijkl}^{(s)}$, $\beta_{ijkl}^{(s)}$ and $\gamma_{ijkl}^{(s)}$ can be computed from (4.8)–(4.13).

The use of freedoms. Interactive shape control by giving $F(v)$ and $\nabla F(v)$ under the restrictions (4.10) and (4.13). The default choice is to make the K cubics approximate quadratics. By using the degree elevation formula, we need to solve the following equations

$$\frac{i}{3} t_{i-1,jkl}^{(s)} + \frac{j}{3} t_{i,j-1,kl}^{(s)} + \frac{k}{3} t_{ij,k-1,l}^{(s)} + \frac{l}{3} t_{ijk,l-1}^{(s)} - \alpha_{ijkl}^{(s)} F(v) - \beta_{ijkl}^{(s)} \nabla F(v) = \gamma_{ijkl}^{(s)}, \tag{4.14}$$

for $i+j+k+l = 3, \quad s = 1, \cdots, K$, in the least squares sense for the unknowns $t_{ijkl}^{(s)}$, $F(v), \nabla F(v)$ under the C^0 condition (4.6) for the K cubics and C^0 condition

$$t_{ijk0}^{(m)} = t_{ij0k}^{(m+1)}, \quad i+j+k = 2, \quad m = 1, 2, 3$$

and the constraints (4.10) and (4.13). System (4.14) has $(4K+3)$ unknowns $t_{ijkl}^{(s)}$ for $i+j+k+l = 2$ and 4 unknowns $F(v), \nabla F(v)$, and has $10K+4$ equations.

Sub-Algorithm 4. *Compute the weights on edge tetrahedra*

Suppose the edge tetrahedron considered is $[u_l u_r p_1 p_2]$ (see Fig. 4.1(b)). The weights e_{1110} and e_{1101} are set to zero. The number 6 coefficients $e_{1110}^{(l)}, e_{1110}^{(r)}, e_{1101}^{(l)}$ and $e_{1101}^{(r)}$ are determined by the C^1 condition. If the right neighbor, that is adjacent to $[u_r p_1 p_2]$, of the edge tetrahedron is tetrahedron $[u_r p_1 p_2 p_3]$, and if we express $u_l = \alpha_1 u_r + \alpha_2 p_1 + \alpha_3 p_2 + \alpha_4 p_3$ with $\sum_{i=1}^4 \alpha_i = 1$, we have

$$e_{1110}^{(r)} = \alpha_1 f_{2100} + \alpha_2 f_{1200} + \alpha_3 f_{1110} + \alpha_4 f_{1101},$$
$$e_{1101}^{(r)} = \alpha_1 f_{2001} + \alpha_2 f_{1101} + \alpha_3 f_{1011} + \alpha_4 f_{1002}.$$

If the right neighbor of $[u_l u_r p_1 p_2]$ is pyramid $[u_r p_1 p_2 p_3 p_4]$, then let

$$u_l = \alpha_1 u_r + \alpha_2 p_1 + \alpha_3 p_2 + \alpha_4 p_3, \quad u_l = \beta_1 u_r + \beta_2 p_1 + \beta_3 p_2 + \beta_4 p_4$$

with $\sum_{i=1}^4 \alpha_i = \sum_{i=1}^4 \beta_i = 1$. Then we have

$$e_{1110}^{(r)} = \alpha_1 p_{211} + \alpha_2 p_{122} + \alpha_3 p_{112} + \alpha_4 p_{121},$$
$$e_{1101}^{(r)} = \beta_1 p_{201} + \beta_2 p_{112} + \beta_3 p_{102} + \beta_4 p_{101}.$$

The weights $e_{1110}^{(l)}$ and $e_{1101}^{(l)}$ are similarly computed.

Theorem 4.1. *For the given discretization $\mathcal{L}$ of a surface with a built finite-element hull $\mathcal{H}$ on it, the surface defined by the union of all edge A-patches, face A-patches and zero convex faces of $\mathcal{L}$ interpolates the vertices of the discretization and has the normals at the vertices, and it is smooth and topologically equivalent to $\mathcal{L}$.*

The scheme proposed above makes the constructed surface have the plane recovery property. Even further, the scheme can recover quadratic. That is if the normal at the vertices of a polygon are extracted from a quadratic surface $Q(p) = 0$ that passes through the vertices of the polygon, and furthermore if the free weights are defined by approximating a quadratic, then $F(p) = Q(p)$. Similarly, if the normals at the vertices of an edge and the vertices of the two adjacent polygons are extracted from a quadratic surface $Q(p) = 0$ that passes through these vertices, and if the free weights on the neighbor polygon elements are defined by approximating a quadratic surface, then $F(p) = Q(p)$ on the edge tetrahedron.

The proof of the quadratic recovery property is based on the following facts: (a). F interpolates function values and first order partial derivatives of Q at the vertices, and F interpolates directional derivatives of Q in any directions that perpendicular to edges at the mid-points of the edges. (b). The free weights are defined by the degree elevation formula. (c). The rational function is degenerate to zero. The detailed discussion needs to distinguish the cases when the polygon is convex or non-convex. We omit the detail here.

5 Evaluate the Surfaces

Since the patches for edges and convex triangles are defined in the same way as in [17], we can evaluate these patches using the scheme in [17]. In the following, we ignore these cases.

A. Evaluate the Triangular Face A-patch

For each triangular nonconvex polygon in $\mathcal{L}_{no-zero}$, we shall produce a piecewise triangular approximation for the surface patch $F = 0$. Let $\langle p_1p_2p_3\rangle \in \mathcal{L}_{no-zero}$ be one triangular polygon and u and v be top and bottom vertices. Let N be a given positive number, which represents the resolution of the piecewise approximation. Then the piecewise triangular approximation is defined by the naive connection of the points $s_{xyz}(x + y + z = N,\ x, y, z \geq 0)$. Here s_{xyz} is the intersection point of the surface $F = 0$ with the polygonal line $[uq_{xyz}] \cup [q_{xyz}v]$, where $q_{xyz} = \frac{x}{N}p_1 + \frac{y}{N}p_2 + \frac{z}{N}p_3$ and the intersection point is computed by solving the cubic polynomial equation $F\left((1-t)q_{xyz} + tu\right) = 0$ if $F(q_{xyz}) \leq 0$ or solving a similar equation $F\left((1-t)q_{xyz} + tv\right) = 0$ if $F(q_{xyz}) > 0$, where the required root is the minimal one.

Since q_{xyz} is in one of the tetrahedra $[uvp_ip_{i+1}]$ with $i = 1, \cdots, 3$, q_{xyz} could be expressed in the following form:

$$q = \beta_1^{(i)}u + \beta_2^{(i)}v + \beta_3^{(i)}p_i + \beta_4^{(i)}p_{i+1} \tag{5.1}$$

Then we can derive, from (4.3), that

$$F((1-t)q + tu) = \sum_{s=0}^{3}\left(\sum_{\lambda_1=s}^{3} C_{\lambda_1 s}^{(i)}\right) B_s^3(t) \tag{5.2}$$

with $C^{(i)}_{\lambda_1 s} = \sum_{\lambda_2+\lambda_3+\lambda_4=3-\lambda_1} t^{(i)}_{\lambda_1\lambda_2\lambda_3\lambda_4} B^{3-s}_{\lambda_1-s,\lambda_2\lambda_3\lambda_4}(\beta^{(i)}_1, \beta^{(i)}_2, \beta^{(i)}_3, \beta^{(i)}_4)$. Similarly, we have

$$F((1-t)q+tv) = \sum_{s=0}^{3}\left(\sum_{\lambda_2=s}^{3} \tilde{C}^{(i)}_{\lambda_2 s}\right) B^3_s(t) \tag{5.3}$$

with $\tilde{C}^{(i)}_{\lambda_2 s} = \sum_{\lambda_1+\lambda_3+\lambda_4=3-\lambda_2} t^{(i)}_{\lambda_1\lambda_2\lambda_3\lambda_4} B^{3-s}_{\lambda_1\lambda_2-s,\lambda_3\lambda_4}(\beta^{(i)}_1, \beta^{(i)}_2, \beta^{(i)}_3, \beta^{(i)}_4)$.

B. Evaluate the Quadrilateral Face A-patch

For each quadrilateral polygon in $\mathcal{L}_{no-zero}$, we shall produce a piecewise quadrilateral approximation for the surface patch. Let N be two given positive numbers, which represent the resolution of the piecewise approximation and it should have the same value as above, let $\langle p_1p_2p_3p_4\rangle$ be a quadrilateral of $\mathcal{L}_{no-zero}$ and u and v(if exist) be the top and bottom vertices of $\langle p_1p_2p_3p_4\rangle$. Then the piecewise quadrilateral approximation is defined by connecting the points $p_{xy}(x=0,\cdots,N; y=0,\cdots,N)$. Here p_{xy} is the intersection point of the polygonal line $[uq_{xy}]\cup[q_{xy}v]$ and the surface $F=0$, where

$$q_{xy} = \frac{y}{N}\left[\frac{x}{N}p_1 + \frac{N-x}{N}p_2\right] + \frac{N-y}{N}\left[\frac{x}{N}p_3 + \frac{N-x}{N}p_4\right] \tag{5.4}$$

and the intersection point is computed by solving the cubic equation $F((1-t)q_{xy}+tu)=0$ if $F(q_{xy})\le 0$ or solving a similar equation $F((1-t)q_{xy}+tv)=0$ if $F(q_{xy})>0$. Again, we use the minimal root.

If the polygon is convex, (4.2) gives explicit expression for $F((1-t)q_{xy}+tu)$.

If the polygon is non-convex, q_{xy} is in one of the tetrahedra $[uvp_ip_{i+1}]$ with $i=1,\cdots,4$. Using (5.4), q_{xy} could be expressed as (5.1), and (5.2) and (5.3) could be used again.

C. Evaluate the Pentagon Face A-patch

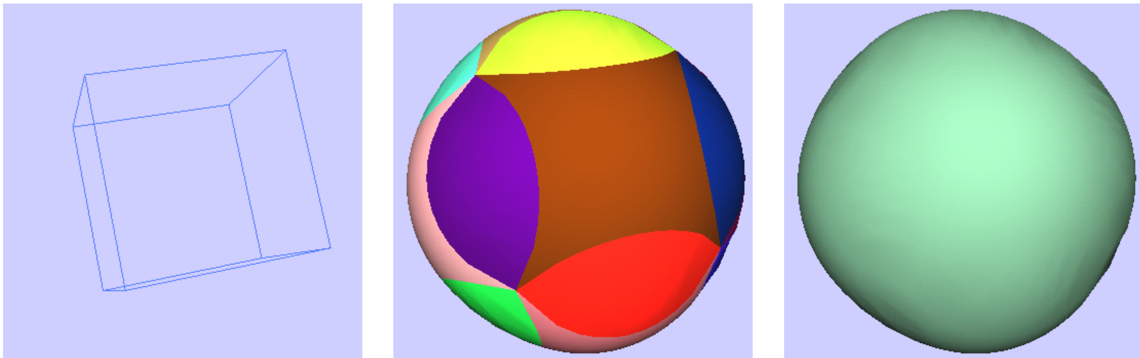
Let $\langle p_1\cdots,p_5\rangle$ be a 5-sided polygon and u and v be the top and bottom vertices. Then The pentagon face A-patch is evaluated by evaluating 5 patches defined by $F|_{[uvp_ip_{i+1}]}(p)=0$ for $i=1,\cdots 5$. Then the piecewise triangular approximation of $F_{[uvp_ip_{i+1}]}(p)=0$ is defined by connecting the points $s^{(i)}_{xyz}(x+y+z=N,\ x,y,z\ge 0)$. Here $s^{(i)}_{xyz}$ is the intersection point of the polygonal line $[uq^{(i)}_{xyz}]\cup[q^{(i)}_{xyz}v]$ and the surface $F=0$, where

$$q^{(i)}_{xyz} = \frac{x}{N}p_i + \frac{y}{N}p_{i+1} + \frac{z}{N}c, \quad c = \frac{1}{5}(p_1+\cdots+p_5).$$

The intersection point can be computed by solving the cubic polynomial equation $F((1-t)q^{(i)}_{xyz}+tu)=0$ if $F(q^{(i)}_{xyz})\le 0$ or solving a similar equation $F((1-t)q^{(i)}_{xyz}+tv)=0$ if $F(q^{(i)}_{xyz})>0$, where the required root is the minimal one. Express $q^{(i)}_{xyz}$ as (5.1), and (5.2) and (5.3) could be used to define the cubic equations.

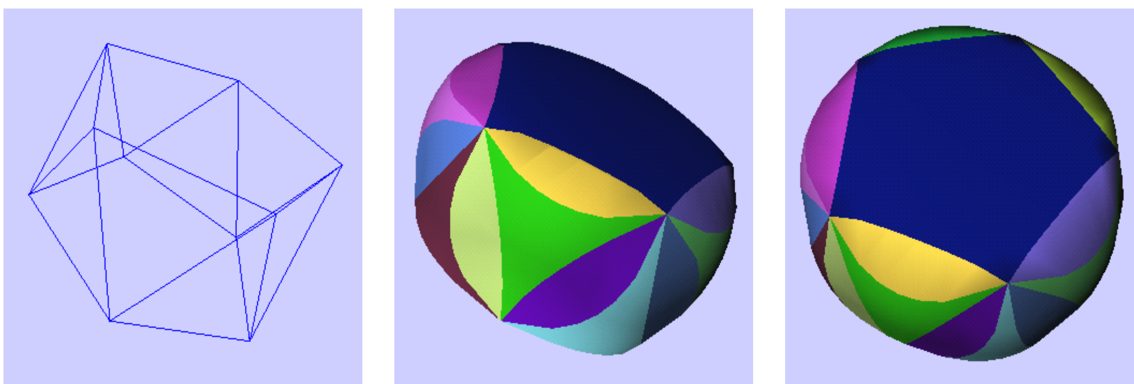
6 Conclusions and Examples

We have presented a new combination of algorithms for modeling a smooth interpolatory surface from a surface discretization by edge A-patches and 3, 4, 5-sided face A-patches. Our



(a) (b) (c)

Fig 6.1: (a) Input mesh consists of four sided polygons; (b) and (c) Patches over tetrahedra using the nonconvex algorithm



(a) (b) (c)

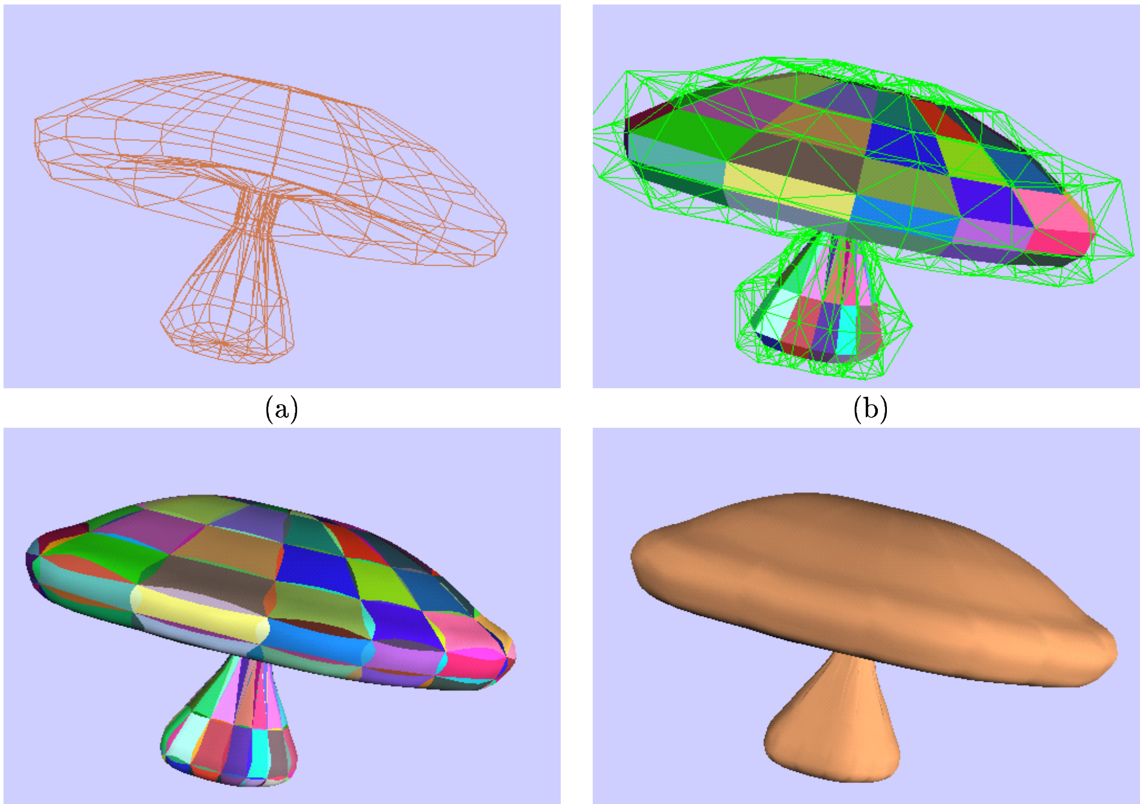
Fig 6.2: (a) Input mesh consists of three and five sided polygons; (b) and (c) Patches over tetrahedra

main contributions consist of introducing A-patches in pyramids, and the new 3, 4, 5-sided face A-patches. The advantages of this approach include reducing significantly the number of surface patches required, and having the quadratic recover property. The implementation shows that the approach we have taken is successful, as seen in figures 6.1, 6.2, and 6.3.

References

[1] C. Bajaj. The Emergence of Algebraic Curves and Surfaces in Geometric Design. In R. Martin, editor, Directions in Geometric Computing, pages 1 – 29. Information Geometers Press, 1993.

[2] C. Bajaj. Implicit surface patches. In J. Bloomenthal, editor, *Introduction to Implicit Surfaces*, pages 98 – 125. Morgan Kaufman Publishers, 1997.



(a) (b)

(c) (d)

Fig 6.3: (a) Input mesh consists of three and four sided polygons; (b) Constructed hull (c) and (d) Patches over tetrahedra and pyramids

[3] C. Bajaj, J. Chen, and G. Xu. Modeling with Cubic A-Patches. *ACM Transactions on Graphics*, 14(2):103–133, 1995.

[4] C. Bajaj and S. Evans. Splines and geometric modeling. In Jacob E. Goodman and Joseph O'Rourke, editors, *CRC Handbook of Discrete and Computational Geometry*. CRC Press, 1997. 52 chapters, approx. xiv + 987 pages; projected publication: July 1997.

[5] W. Dahmen. Smooth piecewise quadratic surfaces. In T. Lyche and L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design*, pages 181–193. Academic Press, Boston, 1989.

[6] W. Dahmen and T-M. Thamm-Schaar. Cubicoids: modeling and visualization. *Computer Aided Geometric Design*, 10:89–108, 1993.

[7] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide, Second Edition*. Academic Press Inc., 1990.

[8] J. A. Gregory. C^1 rectangular and non–rectangular surface patches. In R. E. Barnhill and W. Boehm, editors, Computer Aided Geometric Design, pages 25–33. North–Holland, 1983.

[9] B. Guo. *Modeling Arbitrary Smooth Objects with Algebraic Surfaces.* PhD thesis, Computer Science, Cornell University, 1991.

[10] B. Guo. Surface generation using implicit cubics. In N.M. Patrikalakis, editor, *Scientific Visualization of Physical Phenomena*, pages 485–530. Springer-Verlag,Tokyo, 1991.

[11] S. Lodha. *Surface Approximation with Low Degree Patches with Multiple Representations.* PhD thesis, Computer Science, Rice University, 1992.

[12] C. Loop and T. DeRose. Generalized B–spline Surfaces of Arbitrary Topology. *Computer Graphics*, 24(4):347–356, August 1990.

[13] C. Loop and T. DeRose. A Multisided Generalization of Bézier Surfaces. *ACM Transactions on Graphics*, 8(3):205–234, July 1989.

[14] T.W. Sederberg. Piecewise algebraic surface patches. *Computer Aided Geometric Design*, 2:53–59, 1985.

[15] R. Wang. The Dimension and Basis of Spaces of Multivariate Splines. *Journal of Comp. Appl. Math.*, 12:163–177, 1985.

[16] J. Warren. Creating Multisided Rational Bezier Surfaces Using Base Points. *ACM Transactions on Graphics*, 11(2):127–139, 1992.

[17] G. Xu, C. Bajaj, and H. Huang. C^1 *Modeling with A-patches from Rational Trivariate Functions. Computer Aided Geometric Design*, 18, 2001.

7 Appendix

The proof of Theorem 2.1. First of all, we show that the polynomial F_T on the tetrahedron $[q_1q_2q_3q_4]$ could be expressed as the form (2.5) in the pyramid $[p_0p_1p_2p_3p_4]$. To achieve this, we need to establish a relation between $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ and $(u, s, t)^T$. From (2.1), (2.6) and (2.8), we have

$$\begin{aligned}
&[q_1 - q_4, q_2 - q_4, q_3 - q_4][\alpha_1, \alpha_2, \alpha_3]^T \\
&= (1-u)[q_1 - q_4, q_2 - q_4, q_3 - q_4] \begin{bmatrix} a_1st + b_1s(1-t) \\ a_2st + b_2s(1-t) + (1-s)t \\ a_3st + b_3s(1-t) + (1-s)(1-t) \end{bmatrix}
\end{aligned}$$

¿From this equality and $\alpha_4 = 1 - \alpha_1 - \alpha_2 - \alpha_3$ we get

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} a_1(1-u)st + b_1(1-u)s(1-t) \\ a_2(1-u)st + b_2(1-u)s(1-t) + (1-s)t \\ a_3(1-u)st + b_3(1-u)s(1-t) + (1-s)(1-t) \\ a_4(1-u)st + b_4(1-u)s(1-t) + u \end{bmatrix} \tag{7.1}$$

Substitute (7.1) into F_T, we get its expression in the pyramid $[p_0p_1p_2p_3p_4]$:

$$\tilde{F}_T = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\sum_{k=0}^{n-i} \tilde{b}_{ijk} B_i^n(u) B_j^{n-i}(s) B_k^{n-i}(t)$$

where $\tilde{b}_{ijk}$ is defined by the right-handed said of (2.7) for $i = 0, \cdots, n$; $j = 0, \cdots, n-i$; $k = 0, \cdots, n-i$. Now we prove that, F_T and F_P are C^K at $[q_2q_3q_4] \setminus \{q_4\}$ if and only if

$$b_{ijk} = \tilde{b}_{ijk}, \quad j = 0, \cdots, K; \quad i = 0, \cdots, n-j; \quad k = 0, \cdots, n-i \tag{7.2}$$

This conclusion is true because of the following facts: (a). The extension $\tilde{F}_T$ of F_T to the pyramid $[p_0p_1p_2p_3p_4]$ is obviously C^K join F_T at $[q_2q_3q_4]$. (b). The partial derivatives of order K' of F_P on $[q_2q_3q_4] \setminus \{q_4\}$ depend upon only the coefficients b_{ijk} for $j = 0, \cdots, K'$ and $K' \le K$. Therefore, the agreement condition (7.2) implies that the partial derivatives order K' of F_P would be the same as the ones of F_T on $[q_2q_3q_4] \setminus \{q_4\}$. This is what we want. $\diamond$

The proof of Theorem 2.2. In the matrix form, (2.11) and (2.12) can be written as

$$\begin{bmatrix} q_1 & q_2 & \cdots & q_k & p_2 & p_3 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = 0 \tag{7.3}$$

where

$$A = \begin{bmatrix} \delta_1 & \alpha_1^2 & 0 & \cdots & \alpha_2^k \\ \alpha_2^1 & \delta_2 & \alpha_1^3 & \cdots & 0 \\ 0 & \alpha_2^2 & \delta_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_1^1 & 0 & 0 & \alpha_2^{k-1} & \delta_k \end{bmatrix}, \quad B = \begin{bmatrix} \alpha_3^1 & \alpha_3^2 & \alpha_3^3 & \cdots & \alpha_3^k \\ \alpha_4^1 & \alpha_4^2 & \alpha_4^3 & \cdots & \alpha_4^k \end{bmatrix}$$

where

$$\delta_i = \begin{cases} -1, & \text{if } q_{i-1}, q_{i+1}, p_2, p_3 \text{ are affine independent} \\ 0, & \text{if } q_{i-1}, q_{i+1}, p_2, p_3 \text{ are affine dependent} \end{cases}$$

Since $\alpha_1^i \neq 0$, $\alpha_2^i \neq 0$, the matrix A has rank at least $k-2$. Since $\begin{bmatrix} q_1 & q_2 & \cdots & q_k & p_2 & p_3 \\ 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix}$ has rank 4. So the rank of $[A^T \ B^T]$ is at most $k+2-4 = k-2$. Hence the rank of A and $[A^T \ B^T]$ is $k-2$. It follows from (2.13) and (2.14) that the unknown x_i satisfy the equation

$$[x_1 \ \cdots \ x_k]A + [b_2 \ b_3]B = 0 \tag{7.4}$$

Therefore, by the fact that $rank A = rank[A^T \; B^T] = k-2$, equation (7.4) is solvable and the set of the solutions is a two dimensional manifold. That is, the equation (7.4) has $k-2$ independent equations. Now we need to figure out which two equations can be removed and which two unknowns can be treated as free parameters. ¿From the definition of the matrix A, the rank of A' is $k-2$, where A' is yielded from A by deleting any two adjacent columns. Let $1 \le i < j \le k$ be two integers, such that deleting the ith and jth rows from A' lead to a rank $k-2$ square matrix A''. Then (7.3) can be written as

$$\begin{bmatrix} q_1 & \cdots & q_{i-1} & q_{i+1} & \cdots & q_{j-1} & q_{j+1} & \cdots & q_k \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix} A'' + \begin{bmatrix} q_i & q_j & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} B'' = 0$$

or

$$\begin{bmatrix} q_1 & \cdots & q_{i-1} & q_{i+1} & \cdots & q_{j-1} & q_{j+1} & \cdots & q_k \\ 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} q_i & q_j & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} C \tag{7.5}$$

for $C = -B''A''^{-1}$. Let q_m, q_n, p_2, p_3 be affine independent. Then by (7.5), there exist $C' \in I\!R^{4\times 4}$ such that

$$\begin{bmatrix} q_m & q_n & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} q_i & q_j & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} C'$$

and C' is full rank. Hence

$$\begin{bmatrix} q_i & q_j & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} q_m & q_n & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} C'' \tag{7.6}$$

with $C'' = C'^{-1}$. Substitute (7.6) into (7.5), we have

$$\begin{bmatrix} q_1 & \cdots & q_k \\ 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} q_m & q_n & p_2 & p_3 \\ 1 & 1 & 1 & 1 \end{bmatrix} D$$

with $D = C''C'$. This implies that

$$\begin{bmatrix} x_1 & \cdots & x_k \end{bmatrix} = \begin{bmatrix} x_m & x_n & b_2 & b_3 \end{bmatrix} D.$$

Therefore, we have proved the theorem. $\diamond$

The proof of Theorem 2.3. Let P_i be cubic functions on $[q_i q_{i+1} p_2 p_3]$ for $i = 1, \cdots, k$. Let $l_i(x,y,z) = 0$ be the plane containing the triangle $[q_i p_2 p_3]$. Then the C^1 join of $P_i(x,y,z)$ and $P_{i+1}(x,y,z)$ on the interface $[q_{i+1} p_2 p_3]$ is expressed by the following equation

$$P_{i+1}(x,y,z) = P_i(x,y,z) + r_i(x,y,z) l_i(x,y,z)^2, \quad i = 1, \cdots, k,$$

where $r_i(x,y,z)$ is a linear polynomial. Summing up these equalities, we have

$$\sum_{i=1}^{k} r_i(x,y,z) l_i(x,y,z)^2 = 0. \tag{7.7}$$

Then similar to the argument for dimension of the bivariate splines (see [15]), we have that

$$\dim S_3^1(\Delta) = 20 + 4k - \tau, \tag{7.8}$$

where τ is the rank of the coefficient matrix of (7.7). Now we compute the rank of this matrix. Let $\alpha_i(x,y,z) = l_i(x,y,z)$ for $i = 1, 2$. Let $\alpha_3(x,y,z)$ and $\alpha_4(x,y,z)$ be other two linear functions such that $\alpha_1, \alpha_2, \alpha_3$ and α_4 are linear independent. Then the functions in the set $\{\alpha_1^a \alpha_2^b \alpha_3^c \alpha_4^d : a+b+c+d = n\}$ are the basis of the space of the polynomial of degree n. Since l_i could be expressed by l_1 and l_2, r_i could be expressed by $\alpha_1, \alpha_2, \alpha_3$ and α_4:

$$l_i = a_i\alpha_1 + b_i\alpha_2, \quad i = 3, \cdots, k; \qquad r_i = \sum_{j=1}^{4} e_{ij}\alpha_j, \quad i = 1, \cdots, k.$$

where a_i and b_i are known constants and e_{ij} are unknowns. If $[q_i p_2 p_3]$ lies at least three different planes, we know that there is at least one $a_i b_i$ being not zero. If $[q_i p_2 p_3]$ lies two different planes, all $a_i b_i$ are zero. Substituting these into equation (7.7), we get following coefficient matrix for unknown vector $[e_{11}, \cdots, e_{14}, e_{21}, \cdots, e_{24}, \cdots, e_{k1}, \cdots, e_{k4}]$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1^2 & 0 & 0 & 0 & a_2^2 & 0 & 0 & 0 & \cdots \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & a_1^2 & 0 & 0 & c_2 & a_2^2 & 0 & 0 & \cdots \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1^2 & 0 & 0 & 0 & a_2^2 & 0 & \cdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_1^2 & 0 & 0 & 0 & a_2^2 & \cdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_1^2 & c_1 & 0 & 0 & b_2^2 & c_2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_1^2 & 0 & 0 & 0 & b_2^2 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_1^2 & 0 & 0 & 0 & b_2^2 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & b_1^2 & 0 & 0 & 0 & b_2^2 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & \cdots
\end{bmatrix},$$

where $c_1 = 2a_1b_1$, $c_2 = 2a_2b_2$. It is obvious that if there is at least one a_ib_i being not zero, then the rank of the matrix above is 10. That is, $\tau = 10$. If all a_ib_i are zero, then $\tau = 8$. Hence the theorem follows from (7.8). $\diamond$

The proof of Theorem 4.2. We first show that the function F is C^1 over $\mathcal{H}$. Note that the rational function is well defined even at the points where the denominator is zero. Hence F is a well defined function on $\mathcal{H}$. Since F is obviously smooth in the interior of the each tetrahedron and pyramid, we consider only the smoothness of F at the interfaces of the finite elements.

At the interface $[p_i p_j u_{ijk}]$, the related rational functions and their first order partial derivatives are polynomials. Hence the coefficients determined by the C^1 condition (2.3) or (2.9) make F C^1 at the interface. Therefore, F is C^1 at there.

Now we show the constructed surface has the required properties. Let S_{ij} denote the edge patch for the edge $[p_i p_j]$, $S_{ij\cdots l}$ denote face patch for the polygon $\langle p_i p_j \cdots p_l \rangle$. We note firstly that each of the surface patches is smooth. Also, these surface patches interpolate corresponding vertices and have the given normals on the vertices. Secondly, the edge A-patch and face A-patch are continuous at the interface since the surface points at there are derived from the same equation. Further, since F is C^1, the two surface patches join smoothly on the interface.

Now we show that the zero convex polygons join smoothly with their neighbor surface patches. Let $\langle p_i p_j p_k \rangle$ be a zero convex triangular polygon. Then the surface patch is the face $[p_i p_j p_k]$. If its adjacent polygon, say $\langle p_i p_j p_l \rangle$, is also zero-convex, then the two polygons are coplanar since they share the same surface normals at the common vertices p_i and p_j. If $\langle p_i p_j p_l \rangle$ is non-zero-convex, then by the construction of F, we know that S_{ijl} contains the edge $[p_i p_j]$. That is, the polygon $\langle p_i p_j p_k \rangle$ and surface patch S_{ijl} C^0 join at the edge. Since both of the surfaces have three same normals on the edge and the normal function is a polynomial vector of degree two, they are uniquely defined by the three normals that are perpendicular to the face. Hence the normal function is perpendicular to the face everywhere on the edge. That is, $\langle p_i p_j p_k \rangle$ and S_{ijl} have the same normals on the edge. Therefore, the two surface patches join smoothly. Similar conclusions can be proved for the zero convex quadrilateral.

Finally, since each edge and each polygon of $\mathcal{H}$ corresponds to one surface patch (the zero convex edge corresponds to itself), hence the constructed surface S is topologically equivalent to $\mathcal{L}$. $\diamond$