1 Introduction

Consider the following problem: you are given a closed curve $\gamma(t)$ in the plane representing a thin elastic ribbon that resists stretching and bending. We will assume that the arc-length parameterization of γ is its rest state, so that the strain of the curve in the tangent direction is $\|\gamma'\|$, and that the curve has some (not necessarily flat) rest curvature $\kappa^0(s)$. We are told that there is some unknown pressure difference ΔP between the area inside and outside γ, and that γ is in static equilibrium; we want to calculate this pressure difference ΔP.

Here is how I would approach discretizing and solving the problem, using the DDG techniques from the course.

2 Step 1: Identify the Smooth Principles

We can think of the pressure difference ΔP as the Lagrange multiplier of an area-preservation constraint on the region enclosed by γ; this suggests that the guiding smooth principle we will need for solving the problem is the principle of virtual work: if γ is in static equilibrium under pressure difference $\lambda = \Delta P$, then the virtual work done on the system by the elastic forces and the area constraint must be zero for any variation in γ or λ. In other words, (γ, λ) extremize the Lagrangian

$$L(\gamma^*, \lambda^*) = E_{\text{elastic}}(\gamma^*) + \lambda^*[A(\gamma^*) - A(\gamma)]$$

where $E_{\text{elastic}}(\gamma^*)$ is the curve's elastic energy and $A(\gamma^*)$ is the area enclosed by γ^*.

Why did we start from this variational principle, instead of writing down a bunch of force equations or PDEs and discretizing those? This is a central theme in DDG and there are several motivating reasons:

1. Expressions for energies tend to be integrated quantities which are straightforward to discretize. Solving the variational problem gives you a recipe for always getting the right formulas for the (more complicated) forces;
2. To discretize a PDE, you have to make a lot of choices about how to discretize each term, and these choices may or may not be self-consistent. Discretizing the Lagrangian involves making fewer, and more intuitive, choices;
3. Solutions that come from variational principles automatically satisfy several important symmetries, such as Noether's theorem. This is particularly important when doing dynamics, where discrete equations of motion that come from Hamilton's principle are guaranteed to conserve momentum, the symplectic form, etc. if the discrete energies have the appropriate symmetries.

3 Step 2: Write Down the Needed Smooth Terms

In this case the equation we need is for the elastic energy in the Lagrangian above. If k_s and k_b are stretching and bending moduli, the Lagrangian is

$$L(\gamma^*, \lambda^*) = k_s \int (\|\gamma^*_t(s)\| - 1)^2 \, ds + k_b \int (\kappa^*(s) - \kappa^0(s))^2 \, ds + \lambda \, [A(\gamma^*) - A(\gamma)]$$

where $\kappa^*(s)$ is the curvature of γ^*. Notice that these energies are evaluated on a deformed curve γ^* but integrated over the rest, arc-length parameterization s.

4 Step 3: Discretize the Smooth Terms

First, we must choose how to represent our curve. One easy approach is to discretize it as a set of N vertices $v_i \in \mathbb{R}^2$ connected by straight segments. Call $e_{i+1/2} = v_{i+1} - v_i$ the edge vector between vertices i and $i+1$, and $L_{i+1/2}$ the rest length of edge $e_{i+1/2}$. We can define two kinds of discrete functions F over γ: those that assign a scalar F_i to each vertex v_i and those that assign a scalar to each edge $e_{i+1/2}$. We need to discretize the L^2 inner product over γ^0 for each of these. For the former, we do this as we did in class using Voronoi areas:

$$\langle F, G\rangle = \sum_i F_i G_i A_i, \qquad A_i = \frac{L_{i-1/2} + L_{i+1/2}}{2}.$$

The inner product for edge-based functions is even simpler:

$$\langle F, G\rangle = \sum_i F_{i+1/2} G_{i+1/2} L_{i+1/2}.$$

We now have all of the pieces needed to discretize the stretching energy:

$$E_s = k_s \left\langle \frac{\|e_{i+1/2}\|}{L_{i+1/2}} - 1, \frac{\|e_{i+1/2}\|}{L_{i+1/2}} - 1 \right\rangle = k_s \sum_i \frac{L_{i+1/2}}{L_{i+1/2}^2} (\|e_{i+1/2}\| - L_{i+1/2})^2 = k_s \sum_i \frac{(\|e_{i+1/2}\| - L_{i+1/2})^2}{L_{i+1/2}}.$$

A few remarks are in order at this point. First, we should carefully keep track of what choices we've made about how to discretize the problem, although so far all have been very natural:

- How to represent the geometry of γ;
- How to represent functions over γ and their inner product;
- How to discretize tangent strain $\|\gamma'\|$.

Note also that E_s looks exactly like a spring energy, where each edge is a spring of rest length $L_{i+1/2}$, and because we discretized the energy by first defining a discrete inner product, it scales correctly with both length and refinement of the curve. A good sign!

Next, we need to discretize the bending term. In class (and the notes) we discussed several options for this. I will propose using the discrete vertex-based curvature

$$\kappa_i = \frac{4\tan(\theta/2)}{\|e_{i-1/2}\| + \|e_{i+1/2}\|},$$

where θ is the angle between consecutive edges $e_{i-1/2}$ and $e_{i+1/2}$. Using 2θ instead of $4\tan(\theta/2)$ is also common (notice that the choices are equivalent to second order), but I prefer the above for two reasons: first, it diverges as $\theta \to \pm\pi$ which makes it useful for simulations where you don't want the curve to pass through itself; and second, the "simplicity' of θ is illusory as computing θ (and, eventually, its gradient) requires dealing with inverse trigonometric functions, whereas

$$\tan(\theta/2) = \frac{\sin\theta}{1+\cos\theta} = \frac{e_{i-1/2} \times e_{i+1/2}}{\|e_{i-1/2}\|\|e_{i+1/2}\| + e_{i-1/2}\cdot e_{i+1/2}},$$

where $\times$ in the numerator is the signed two-dimensional cross-product.

We now have discrete bending energy:

$$E_b = k_b \langle \kappa_i - \kappa_i^0, \kappa_i - \kappa_i^0 \rangle = k_b \sum_i (\kappa_i - \kappa_i^0)^2 A_i.$$

The last piece we need is the area constraint: the area of the polygon γ is simply

$$A(v) = \sum_i \frac{v_{i+1} \times v_i}{2},$$

as can be seen by drawing a line from each vertex to the origin and computing the total signed area of the resulting triangles, or, as we did in class, by applying Stokes's theorem to the area integral

$$\int 1\, dA = \int \frac{1}{2} \nabla \cdot (x, y)\, dA = \frac{1}{2} \int_\gamma (x, y) \cdot \hat{n}\, ds$$

and computing this integral over each edge.

5 Step 4: Apply the Discretized Principle

Our discrete Lagrangian is $L(v^*, \lambda^*) = E_s(v^*) + E_b(v^*) + \lambda^* [A(v^*) - A(v)]$. Now we apply the discrete principle of virtual work: the discrete system is in static equilibrium if and only if the equations

$$\frac{\partial}{\partial v_i^*} L = 0$$
$$\frac{\partial}{\partial \lambda^*} L = 0$$

hold simultaneously. In the smooth cases, the principle of virtual work requires the calculus of variations: the discrete calculus of variations is just elementary calculus! More explicitly, the equations are

$$\frac{\partial}{\partial v_j^*} \left[k_b \sum_i \left(\frac{4(v_i^* - v_{i-1}^*) \times (v_{i+1}^* - v_i^*)}{\left(\|v_i^* - v_{i-1}^*\| + \|v_{i+1}^* - v_i^*\|\right)\left(\|v_i^* - v_{i-1}^*\|\|v_{i+1}^* - v_i^*\| + (v_i^* - v_{i-1}^*) \cdot (v_{i+1}^* - v_i^*)\right)} - \kappa_i^0 \right)^2 A_i + k_s \sum_i \frac{(\|v_{i+1}^* - v_i^*\| - L_{i+1/2})^2}{L_{i+1/2}} + \lambda \sum_i \frac{v_{i+1}^* \times v_i^*}{2} \right] = 0$$

$$\sum_i \frac{v_{i+1}^* \times v_i^*}{2} - A(v) = 0$$

and we want them to hold for $v^* = v$ and λ^* equal to our unknown pressure difference ΔP. The derivative in the first equation is elementary but laborious (a perfect task for Mathematica), with the bending term by far the most involved. Note that for each index j almost all terms in the sums vanish.

The second equation will be automatically satisfied, but we will still have $2N$ equations and only one unknown(λ^*. This is because for the vast majority of curves γ, it is not true that a single pressure difference will perfectly balance the elastic forces at every point. If we *know* this must be true, we could in theory solve only one of the equations above to find λ^*; since there might be some error, though, both due to discretization and due to incorrect assumptions about static equilibrium, it is better to solve the simultaneous system for λ^* in the least squares sense: after differentiating and plugging in $v^* = v$ the first system of equations can be written as $\lambda^* a = b$, where a and b are known $2N$-vectors, and so

$$\Delta P \approx \frac{a \cdot b}{\|a\|^2}.$$