Rotations and Projective Space

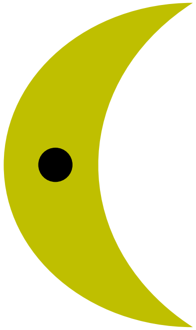
Rotations (in 2D)

Spins **points** about origin; **vectors** about their tails

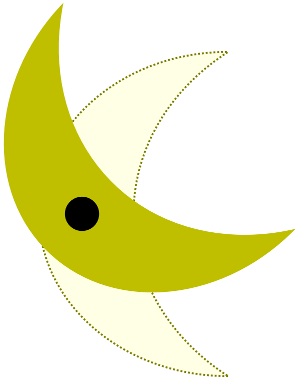
Rotation vs Orientation

Rotation is a **transformation**

Rotation of **rest pose** is **orientation**

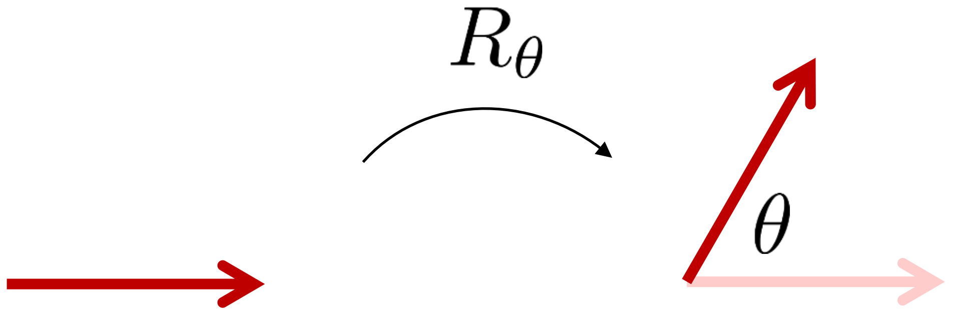


rest pose



Rotations (in 2D)

In 2D rotations are simple: parameterized by single **angle** θ



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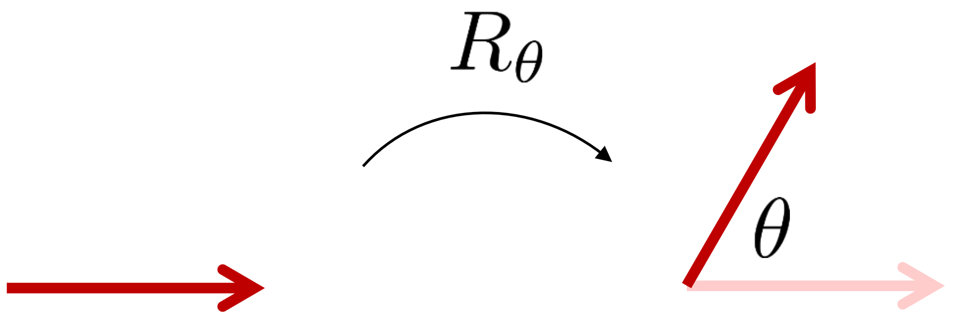
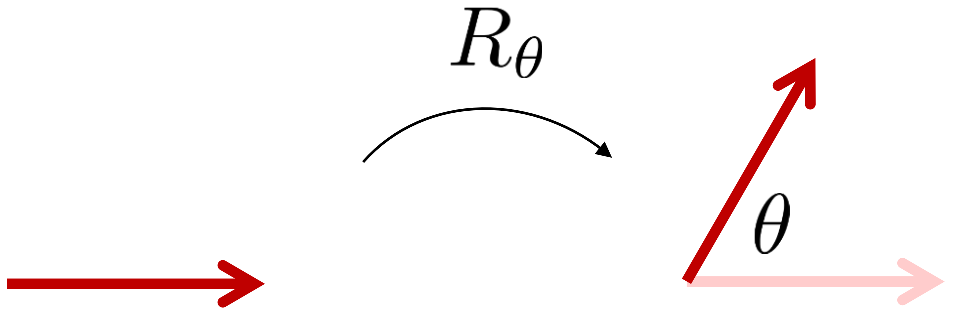
- "one dimensional"



Rotations (in 2D)

In 2D rotations are simple: parameterized by single **angle** θ

- "one dimensional"
- periodic

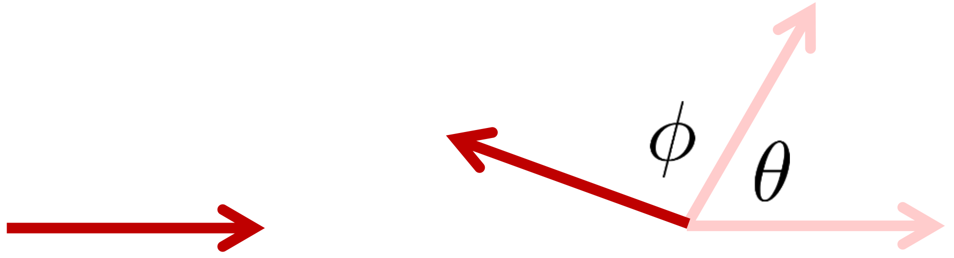


Rotation Group

2D rotations form a **group** called SO(2)

- compositions of rotations are rotations

$$R_\phi R_\theta = R_{\phi+\theta}$$



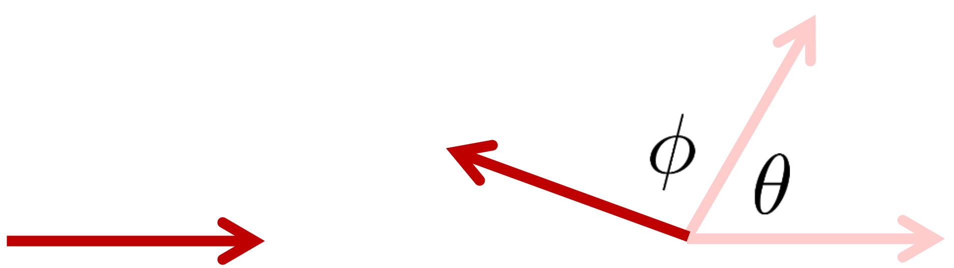
Rotation Group

2D rotations form a **group** called SO(2)

- compositions of rotations are rotations

$$R_\phi R_\theta = R_{\phi+\theta}$$

- rotations have inverses $R_{-\theta}$



Rotation Matrices (in 2D)

Rotations are linear (why)?

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Rotations are linear (why)?

Rotations can be represented by matrix (why)?

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Rotations are linear (why)?

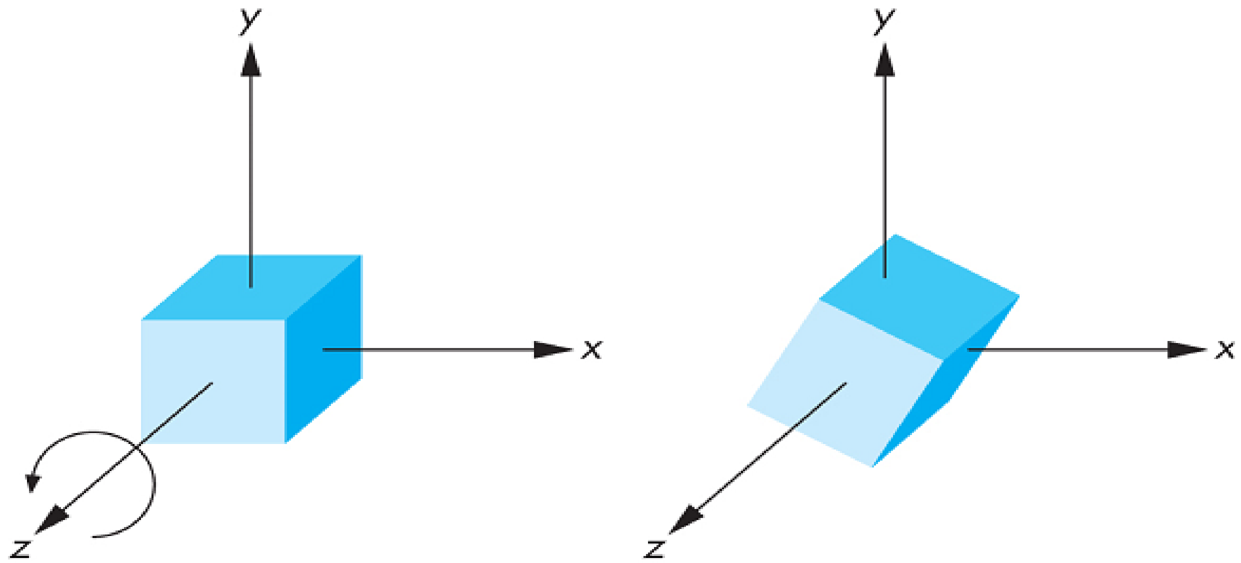
Rotations can be represented by matrix (why)?

Formula: $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

3D Rotations and Orientations

3D rotations still linear, still form group SO(3)

- **rotations don't commute!!**

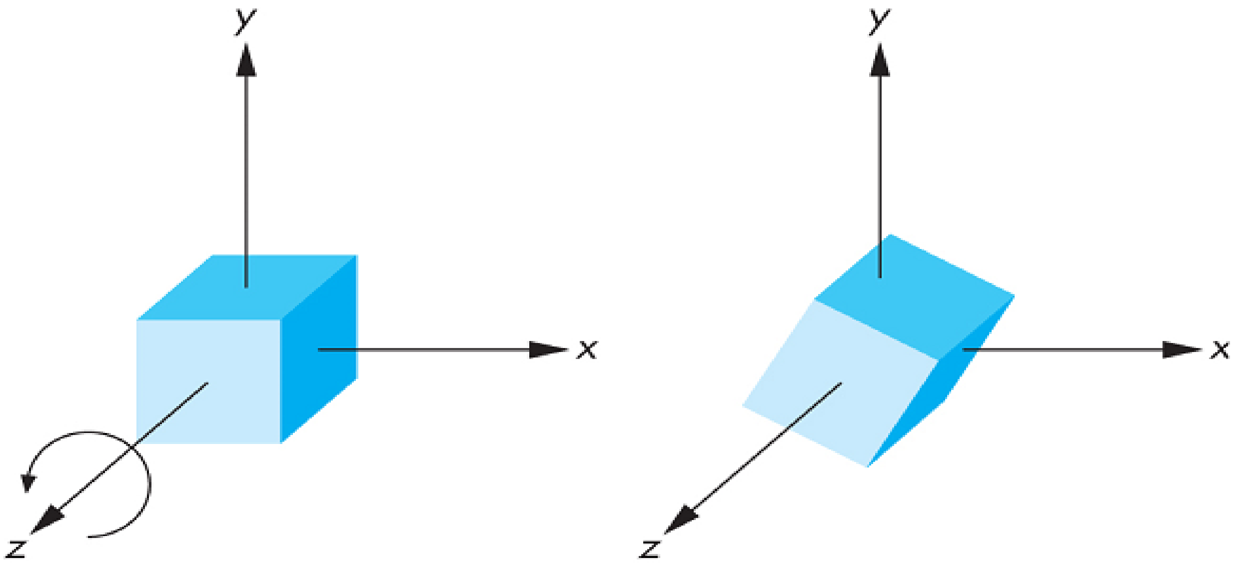


3D Rotations and Orientations

3D rotations still linear, still form group SO(3)

- **rotations don't commute!!**

But compared to 2D, very tricky



What We Want To Do With Rots

1. Write them down (represent them)

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What We Want To Do With Rots

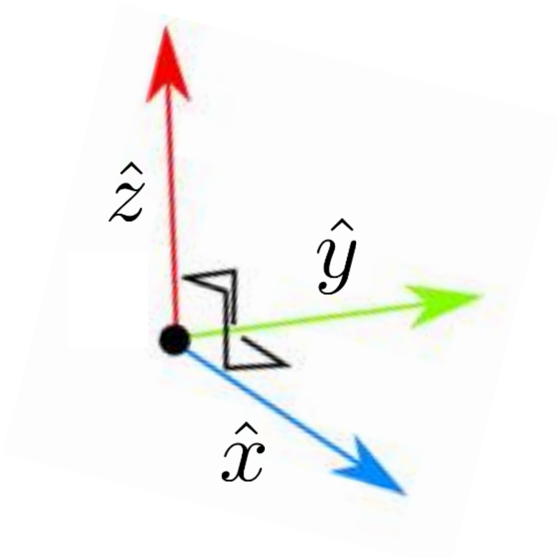
1. Write them down (represent them)

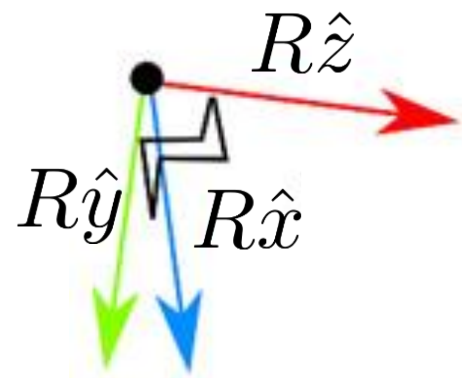
2. Use them to rotate points/vectors

3. Compose them

4. Interpolate them
    - interpolate: smoothly blend

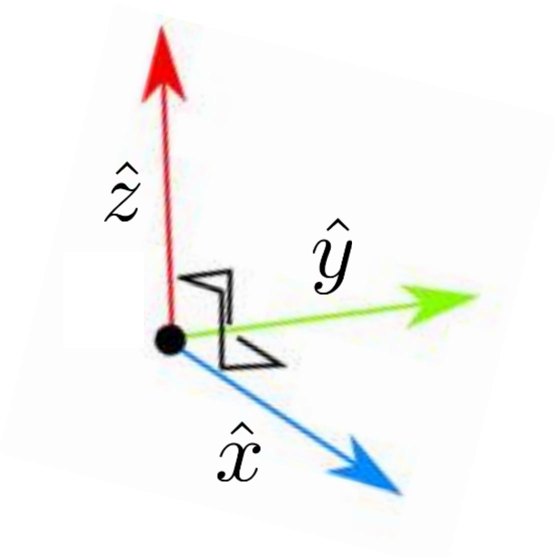
Representation 1: Frames

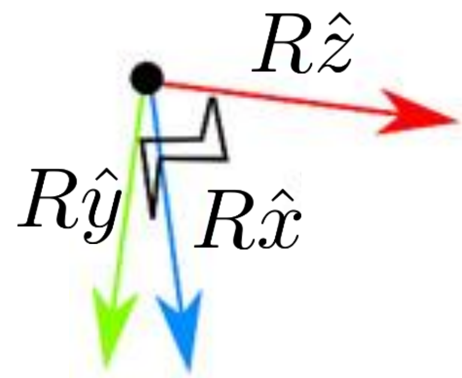




Three **orthogonal**, **unit** vectors $R\hat{x}, R\hat{y}, R\hat{z}$

Representation 1: Frames





Three **orthogonal**, **unit** vectors $R\hat{x}, R\hat{y}, R\hat{z}$

Actually only need two: $R\hat{z} = R\hat{x} \times R\hat{y}$

Representation 1: Frames

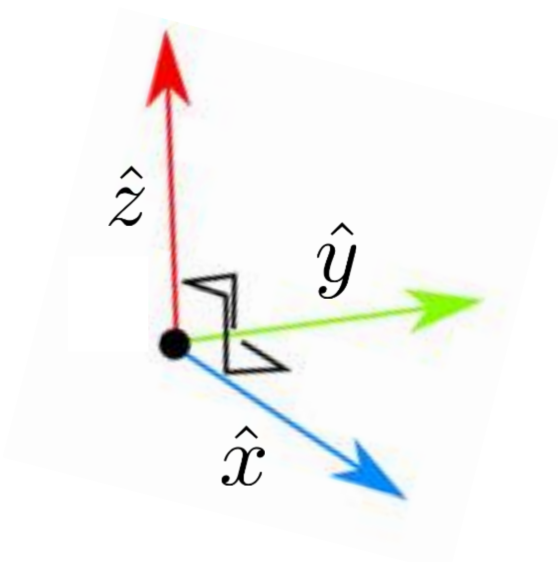
Pros:

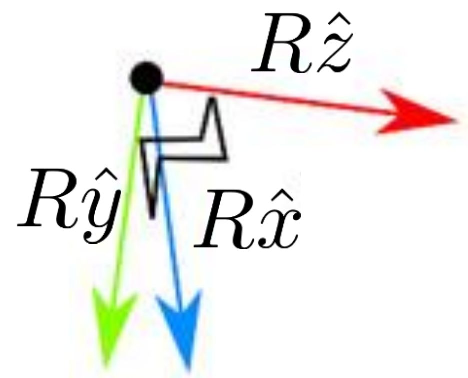
- intuitive

Cons:

- needs six numbers
- complicated orthonormality condition
- composition: unclear
- interpolation: no chance

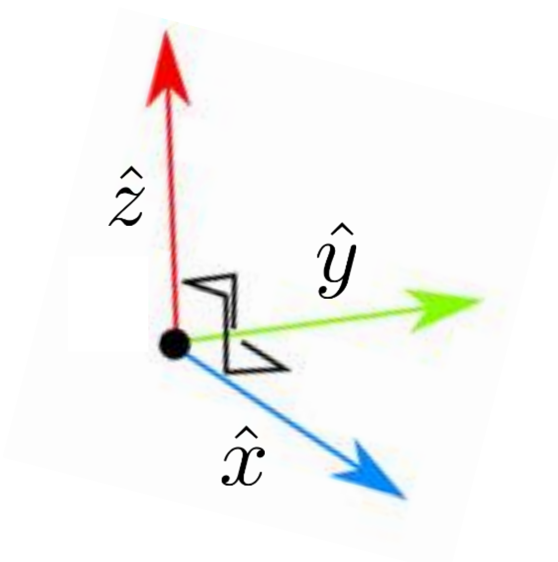
Representation 2: Matrices

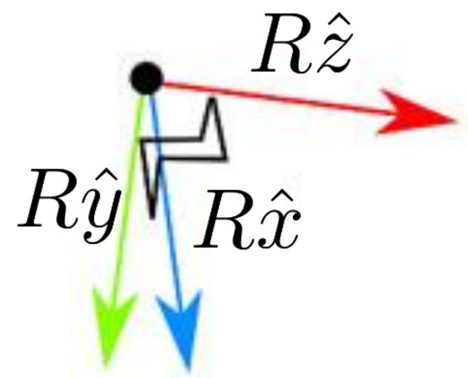




$$R = \left[\begin{array}{c|c|c} R\hat{x} & R\hat{y} & R\hat{z} \end{array} \right]$$

Representation 2: Matrices





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columns unit vectors

columns orthogonal

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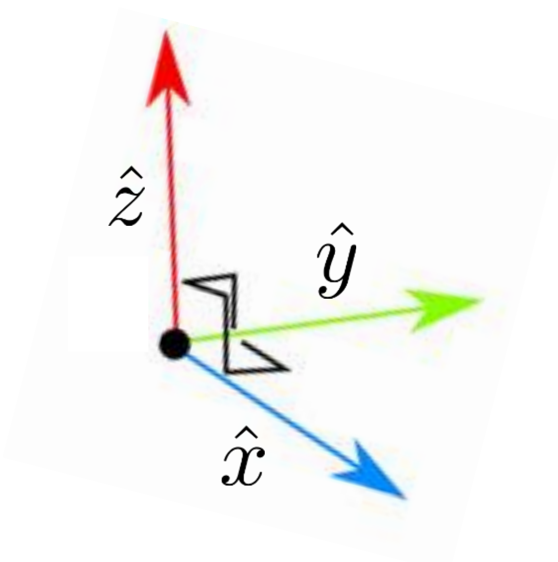

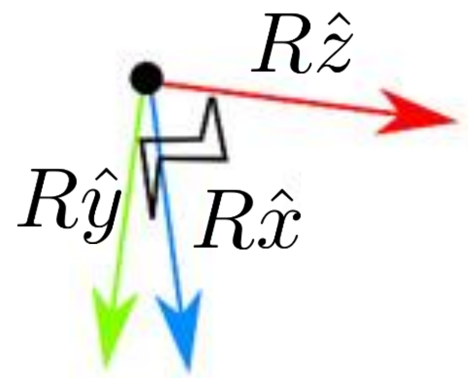

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columns unit vectors

columns orthogonal

$$R^T R = I$$

Representation 2: Matrices

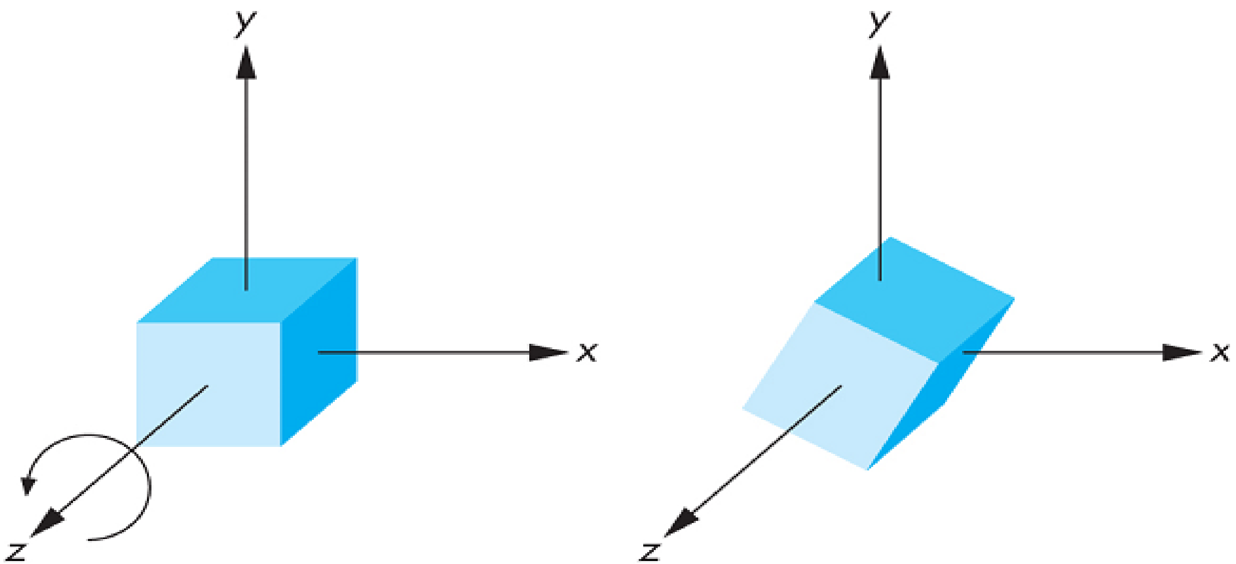
Pros

- composition easy
- applying rotation easy

Cons

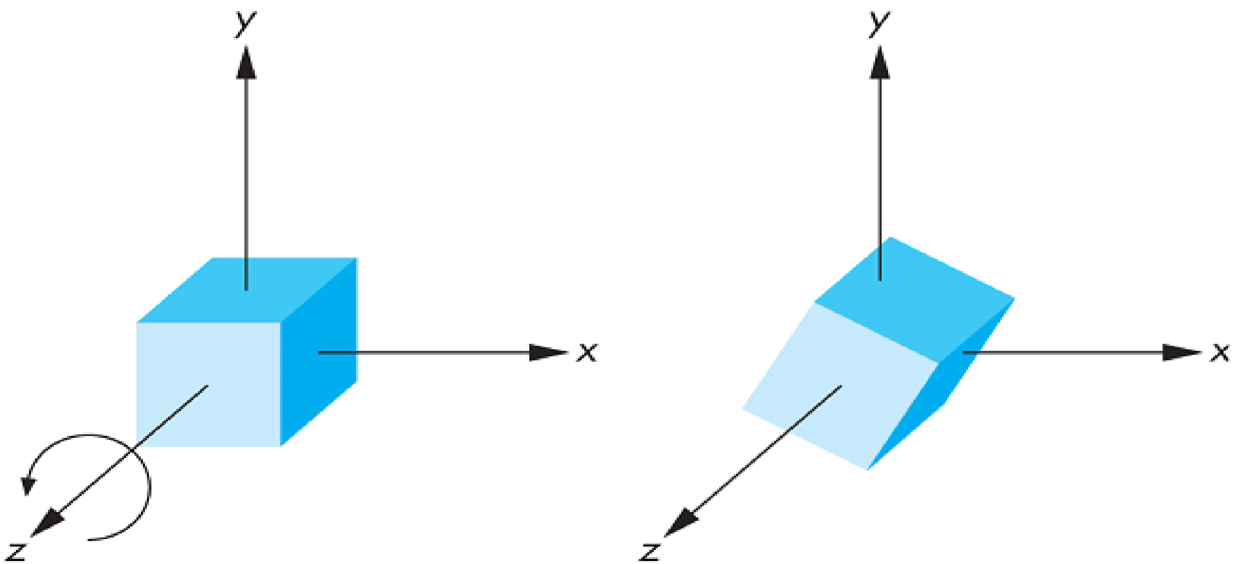
- needs **nine** numbers
- complicated condition $R^T R = I$
- interpolation: no chance

Special Case: Rotation about Axis



$$R_\theta^z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

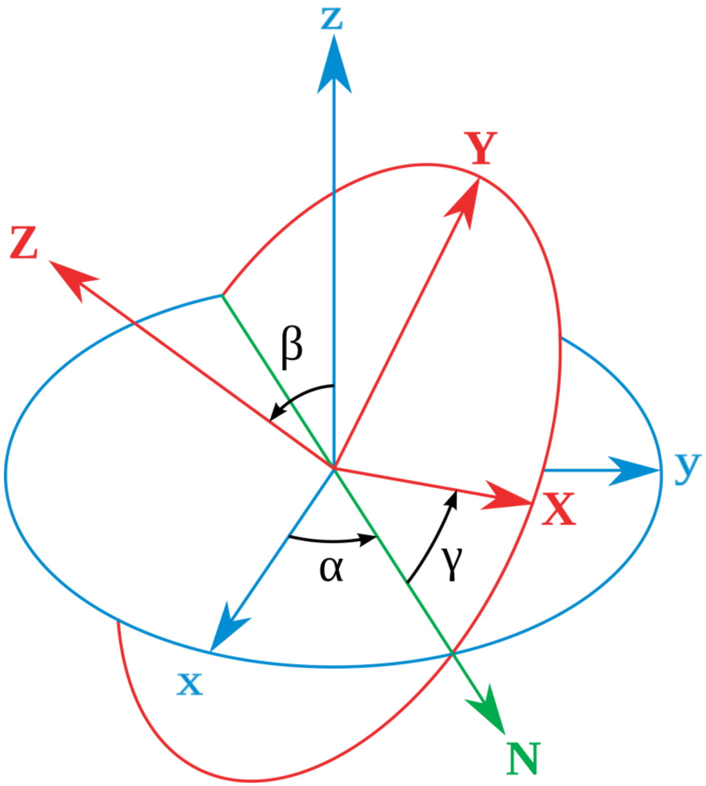
Special Case: Rotation about Axis



$$R_\theta^z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Idea: $R = R_\alpha^x R_\beta^y R_\gamma^z$

Representation 3: Euler Angles



$$R = R^z_\alpha R^x_\beta R^z_\gamma$$

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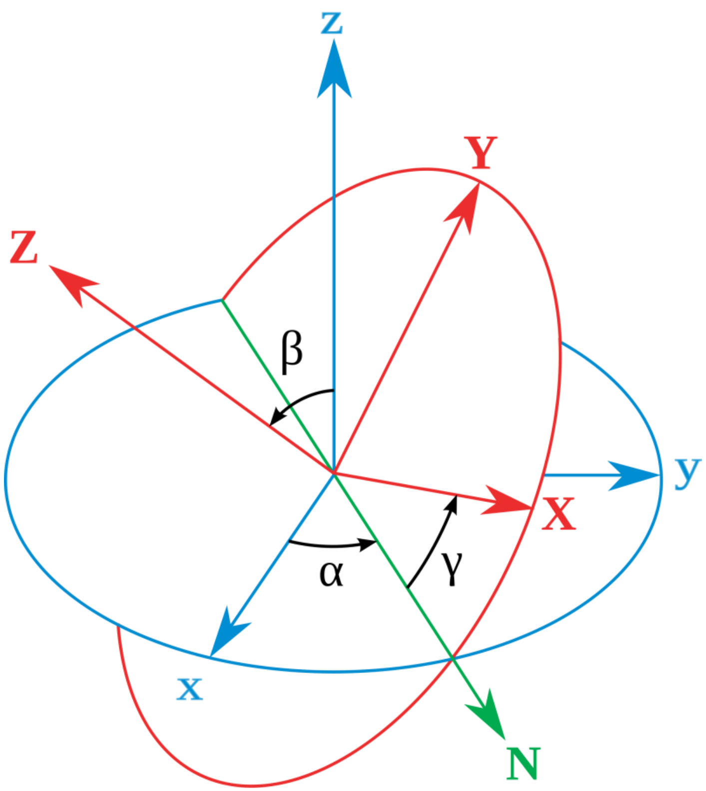

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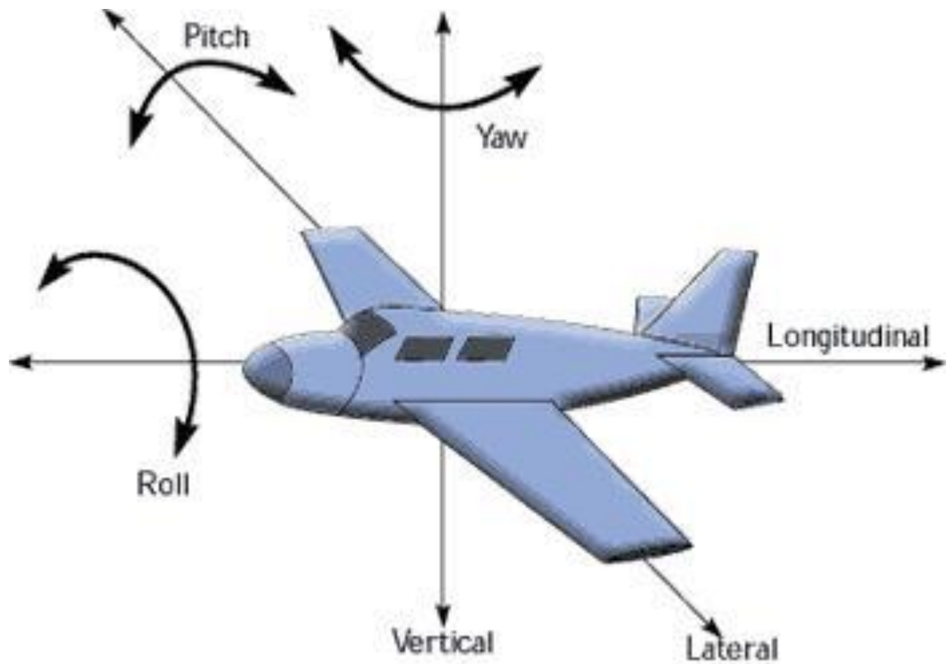


common in aviation

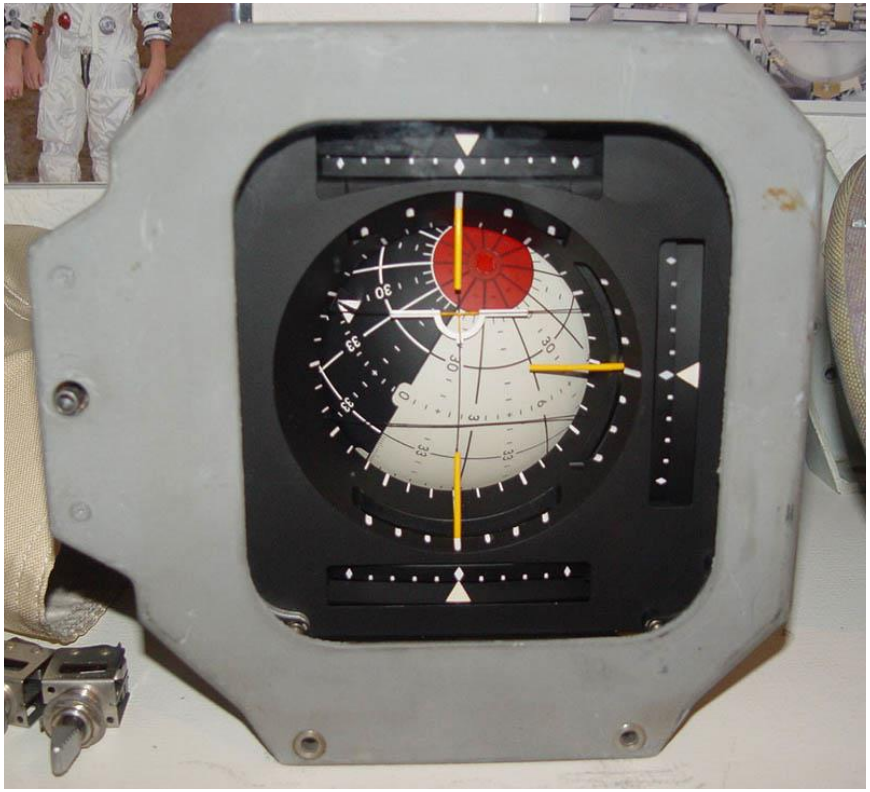
order matters!

Gimbal Lock

When first rotation aligns other axes

Rotations get “stuck”

Apollo 11 Gimbal Lock



Representation 3: Euler Angles

Pros:

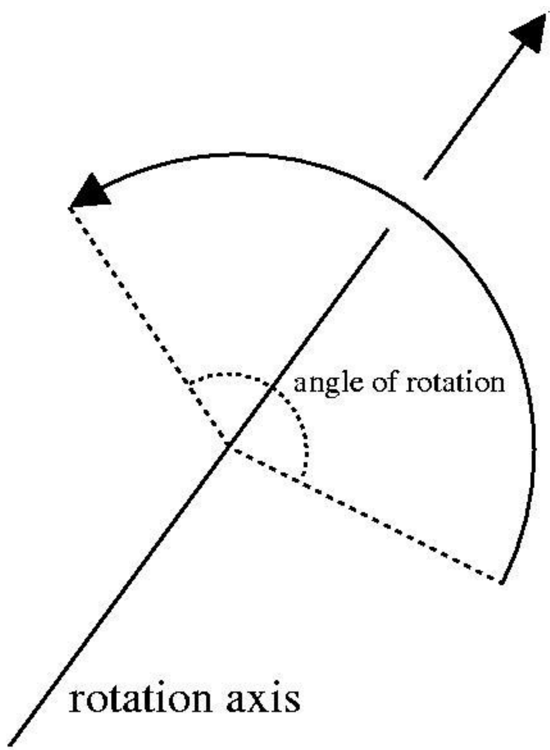
- just three numbers
- easy to convert to matrix

Cons

- complicated to apply/compose
- interpolation: gimbal lock

Representation 4: Axis-angle

Geometry fact: every rotation fixes an axis



represent rotation as:

- axis $\hat{a}$
- angle θ

Representation 4: Axis-angle

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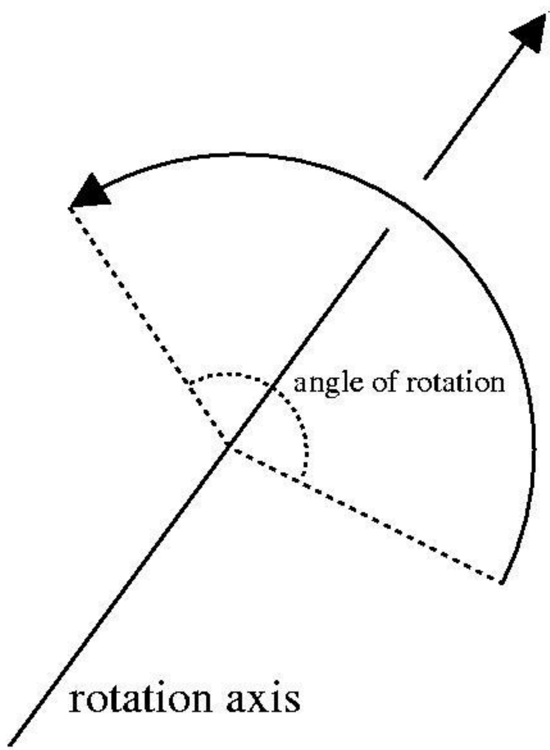

represent rotation as:

- axis $\hat{a}$
- angle θ

…or “axis-angle” $\vec{\theta} = \theta\hat{a}$

Representation 4: Axis-angle

Converting to matrix **not** simple

Rodrigues Rotation Formula:

$$R = I + \sin\theta \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} + (1 - \cos\theta) \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}^2$$

Representation 4: Axis-angle

Pros:

- very intuitive
- only three numbers

Cons:

- complicated + **slow** to compose/apply
- interpolation: complicated

Representation 4: Quaternions

What is a quaternion?

- like complex numbers, but **three** imaginary dimensions

$$a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$$
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

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$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

- cyclic relationship

$$\mathbf{ij} = \mathbf{k}$$
$$\mathbf{jk} = \mathbf{i}$$
$$\mathbf{ki} = \mathbf{j}$$

Representation 4: Quaternions

Turns out rotations can be represented by **unit** quaternions

$$a^2 + b^2 + c^2 + d^2 = 1$$

Rotation about axis (v_x, v_y, v_z):

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}\right)$$

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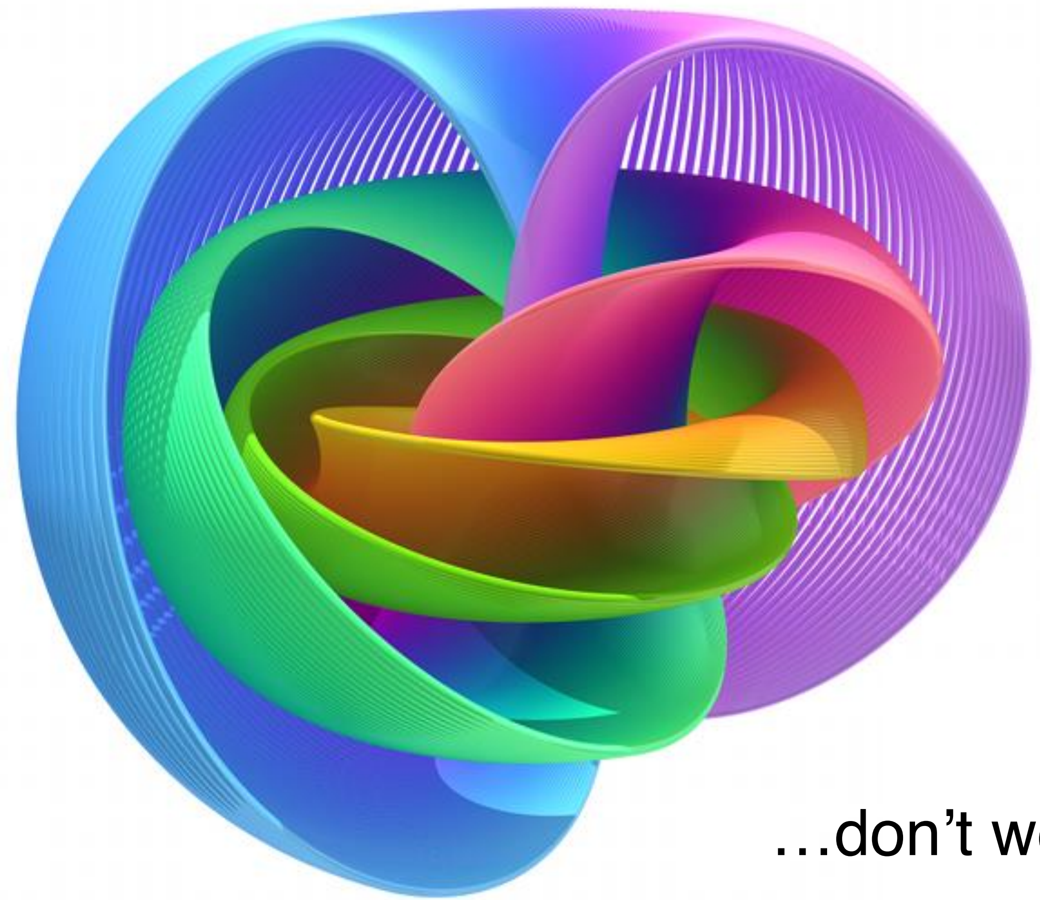
Acts on points/vectors by conjugation:

$$q\left(p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k}\right)\bar{q}$$

Why Are Quaternions Rotations?

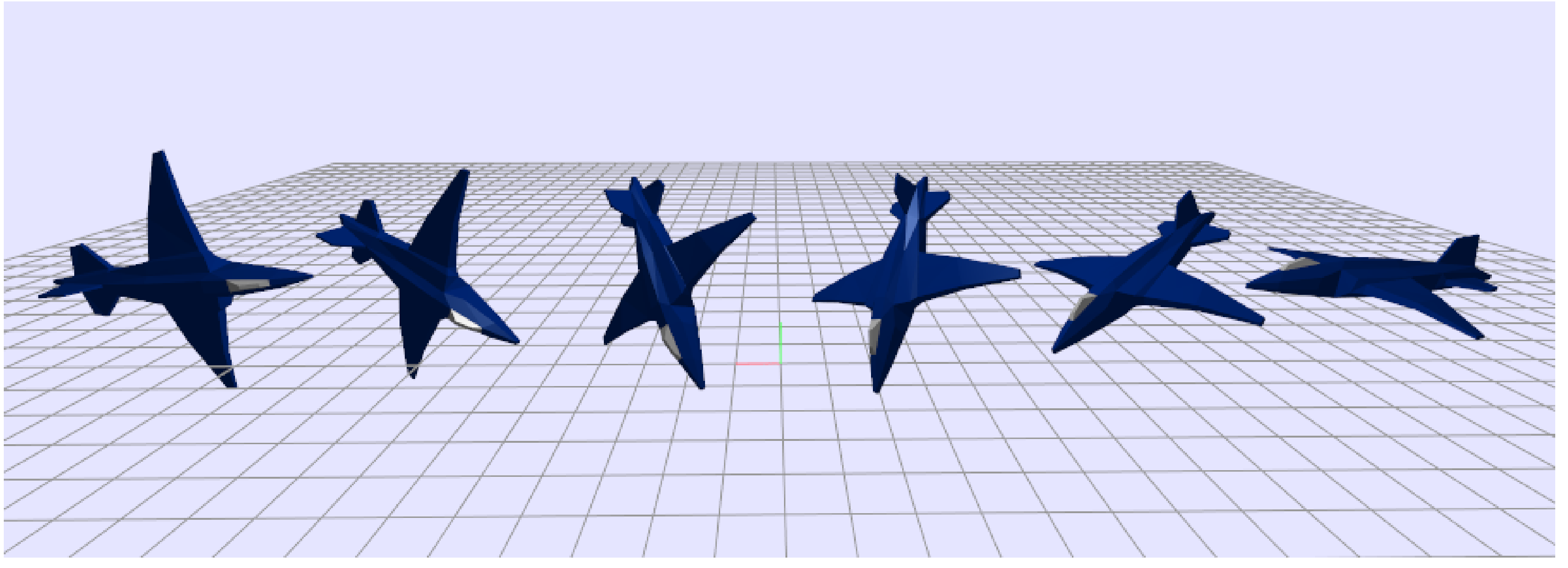


…don’t worry about it.

SLERP

Spherical **L**inear Int**erp**olation

- smoothly blends two rotations



SLERP

Spherical **L**inear Int**erp**olation

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Why not interpolate rotations?

$$R(t) = (1 - t)R_1 + tR_2$$

SLERP

Spherical **L**inear Int**erp**olation

- smoothly blends two rotations

Turns out easy to do with quaternions

$$(q_2\bar{q}_1)^t q_1$$

- don't worry about the mechanics

Representation 4: Quaternions

Pros:

- just four numbers
- easy to compose
- easy to interpolate

Cons:

- unintuitive

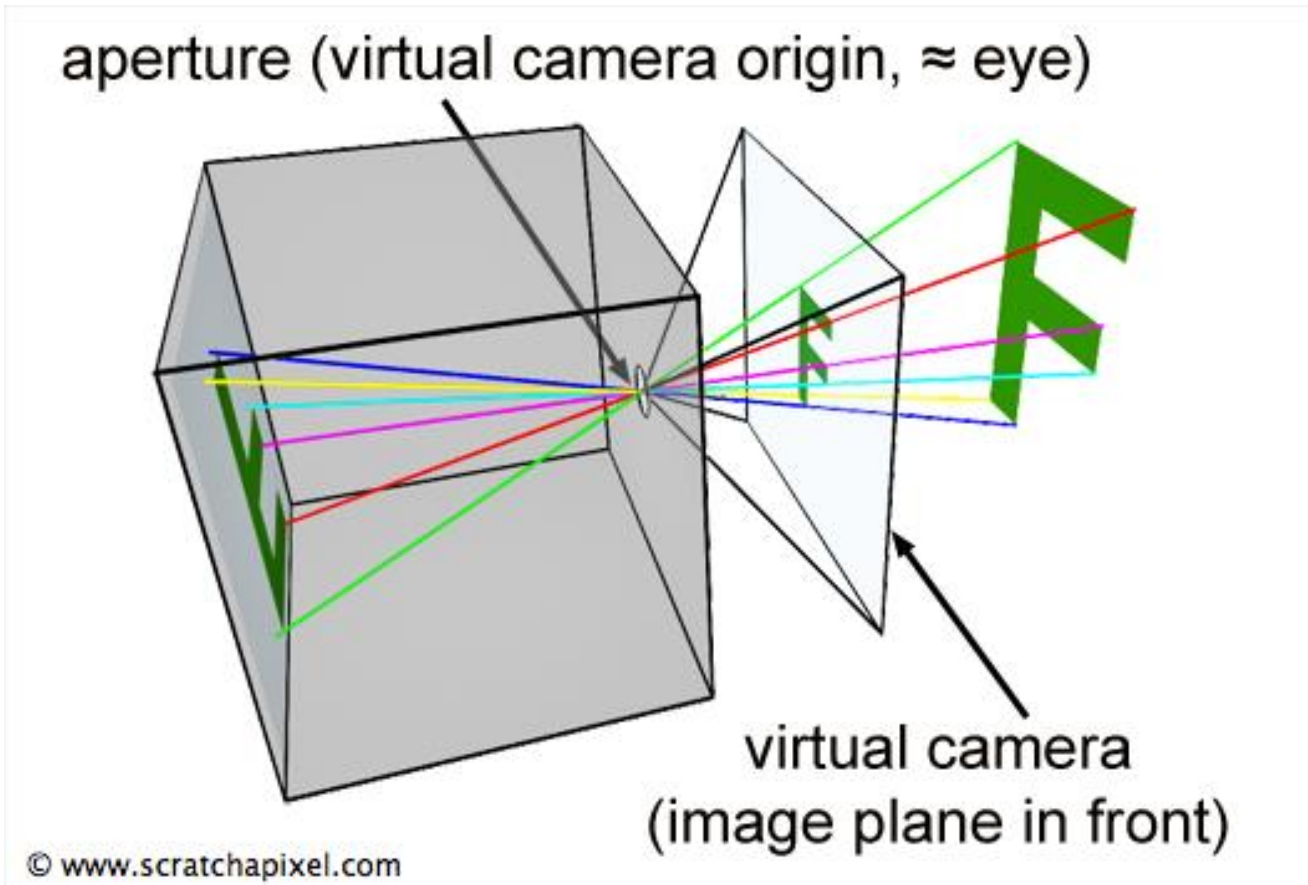
Rotations in Practice

Shaders work with **matrices** at end of day

High-performance intermediate computation is done with **quaternions**

Axis-angle most intuitive for camera controls and physics

Recall: Pinhole Camera



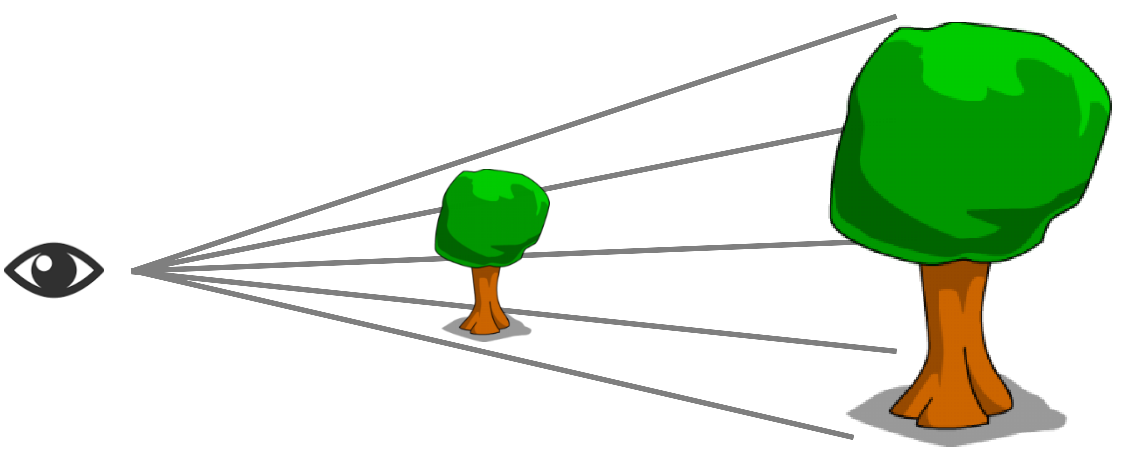
Pinhole Camera: Consequences

We see only **projection** of the world

Pinhole Camera: Consequences

We see only **projection** of the world

- many-to-one mapping
- distance vs scale ambiguity



Distance vs Scale Ambiguity

How we work around it:

- “deep learning”
- depth cues

Distance vs Scale Ambiguity

How we work around it:

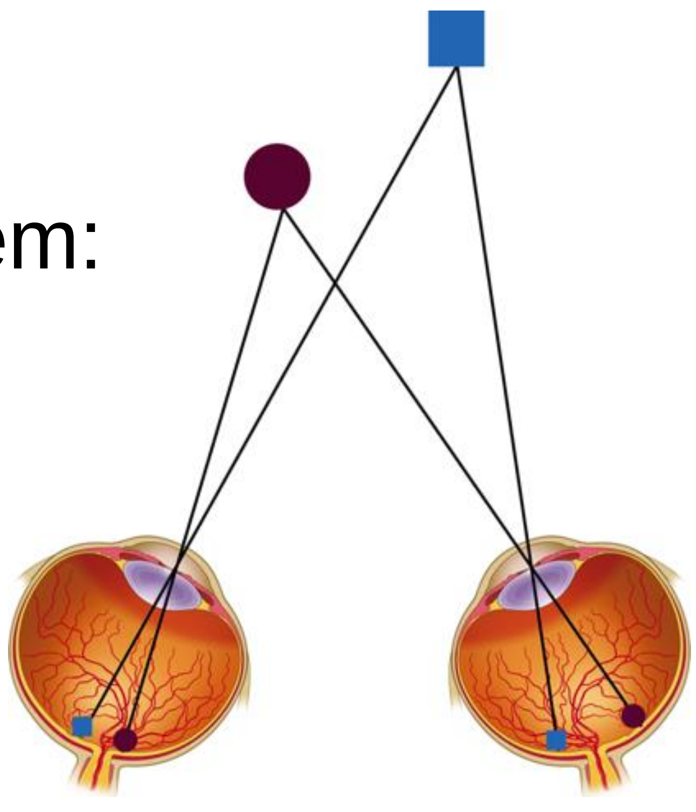
- “deep learning”
- depth cues

we assume:
similar objects have
similar shapes & sizes

Distance vs Scale Ambiguity

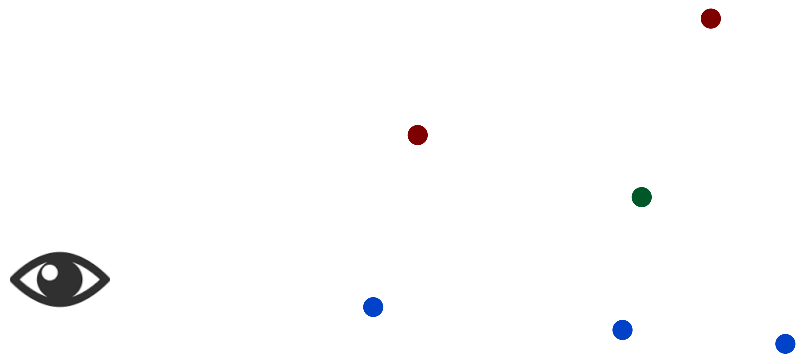
How we work around it:

- “deep learning”
- depth cues
- throw hardware at problem: binocular vision



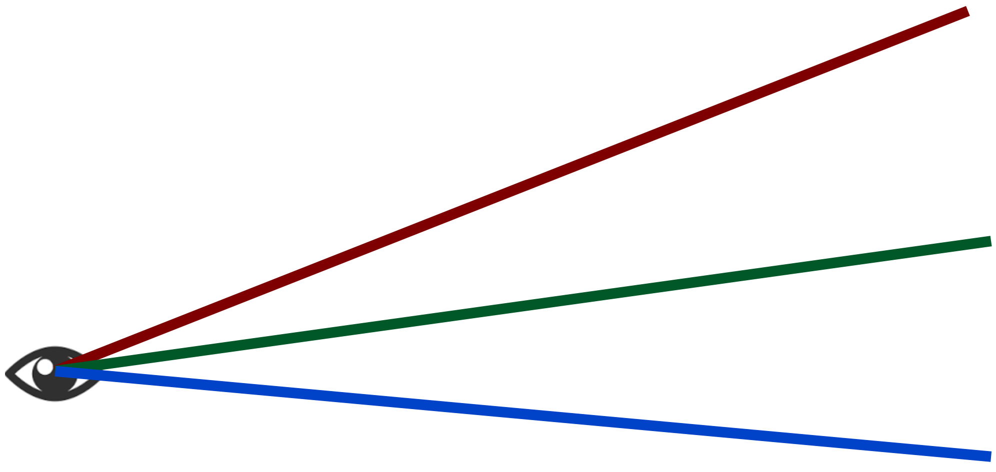
Monocular Vision Many-to-One

Points in space wrong abstraction



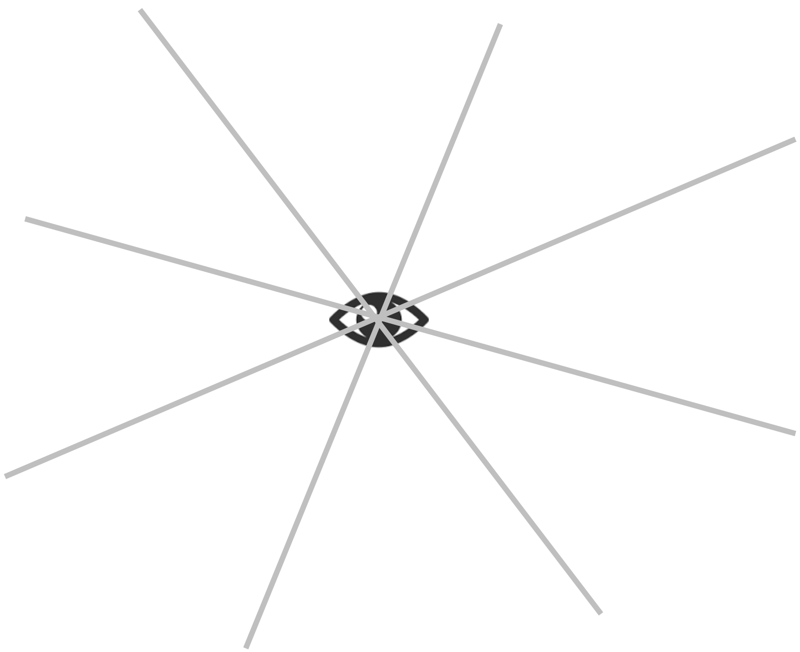
1D Projective Space

Space of **lines through origin**



1D Projective Space Reps.

1. All lines through the origin



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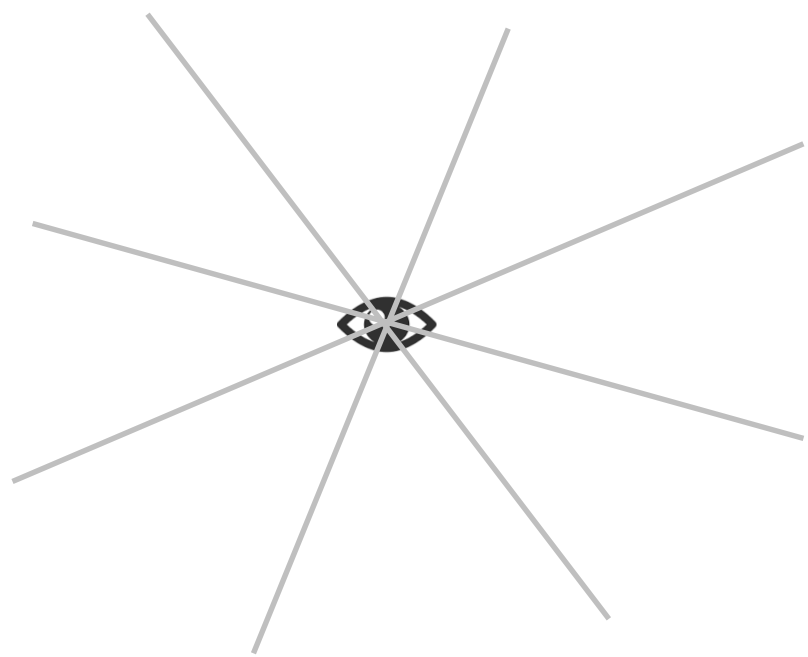
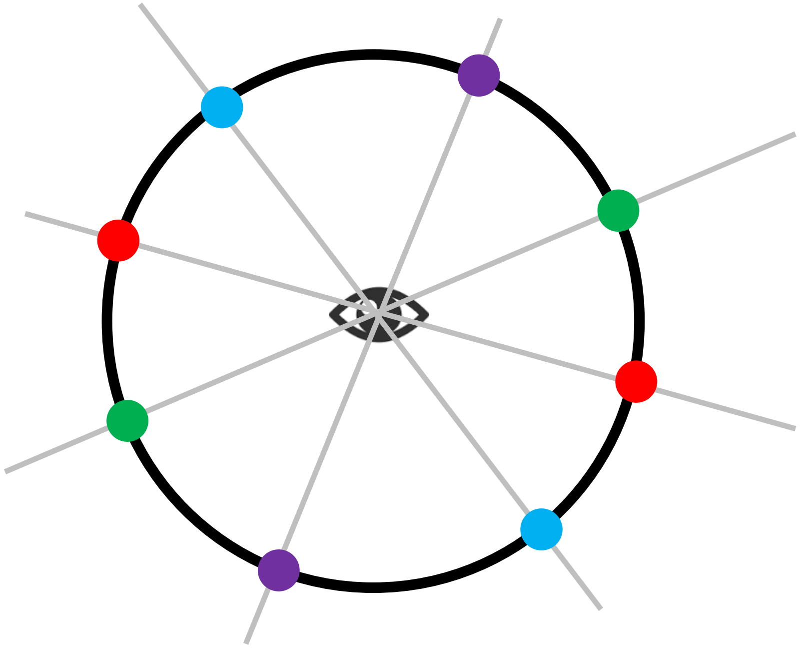


Note:
no notion of sign

1D Projective Space Reps.

1. All lines through the origin
2. Circle with **antipodal points glued**



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1D Projective Space Reps.

1. All lines through the origin
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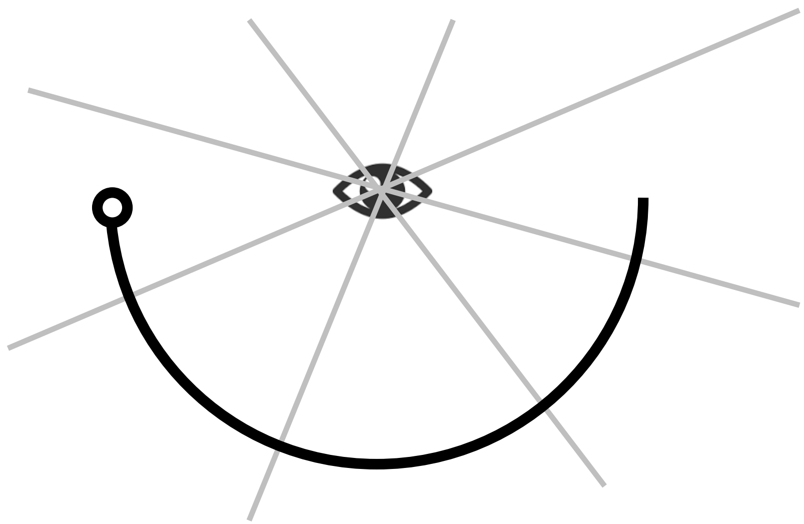
3. Interval $[0, \pi]$ with **boundary glued**



1D Projective Space Reps.

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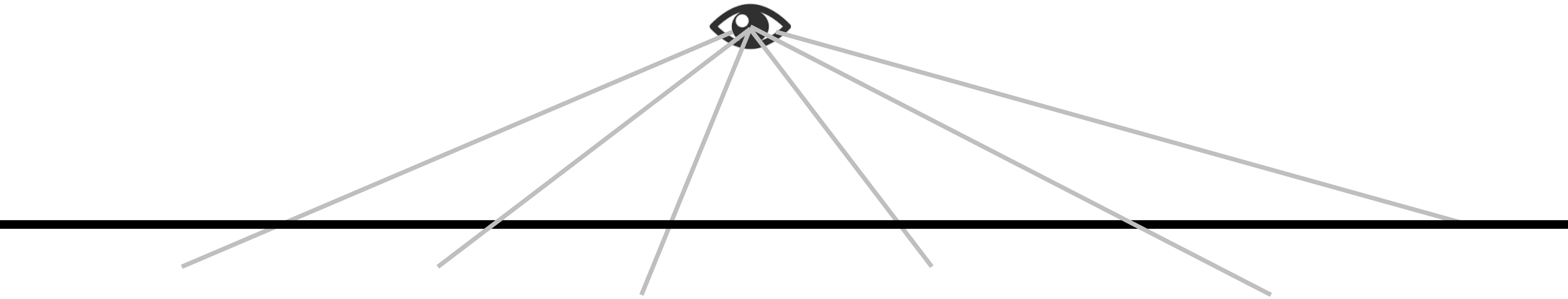
4. The real line…



1D Projective Space Reps.

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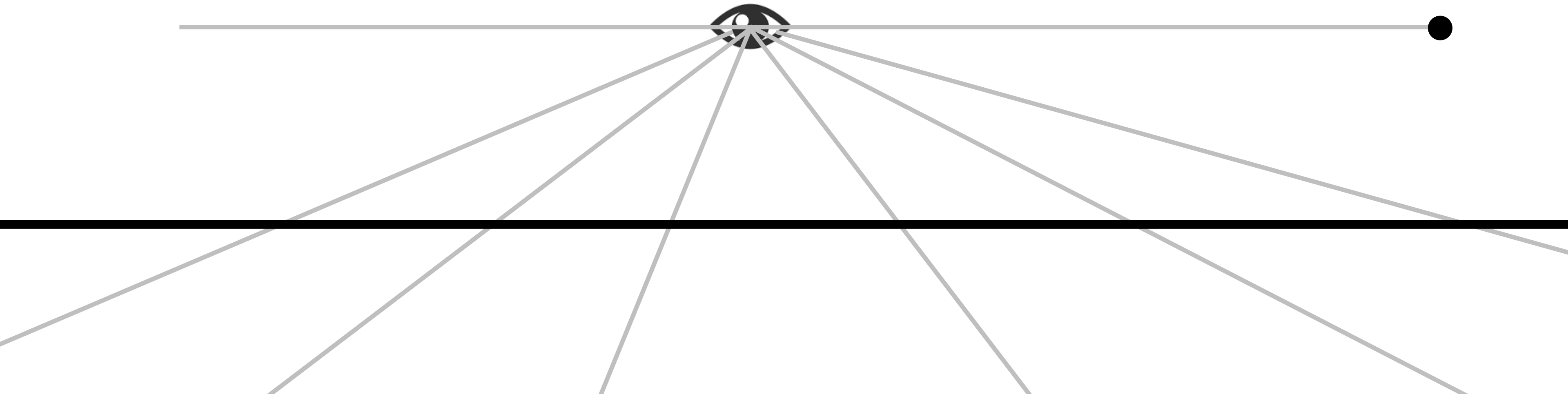

1D Projective Space Reps.

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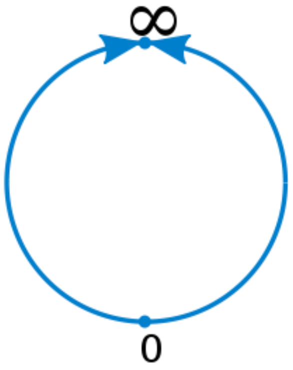
    - “ends” of line meet at infinity



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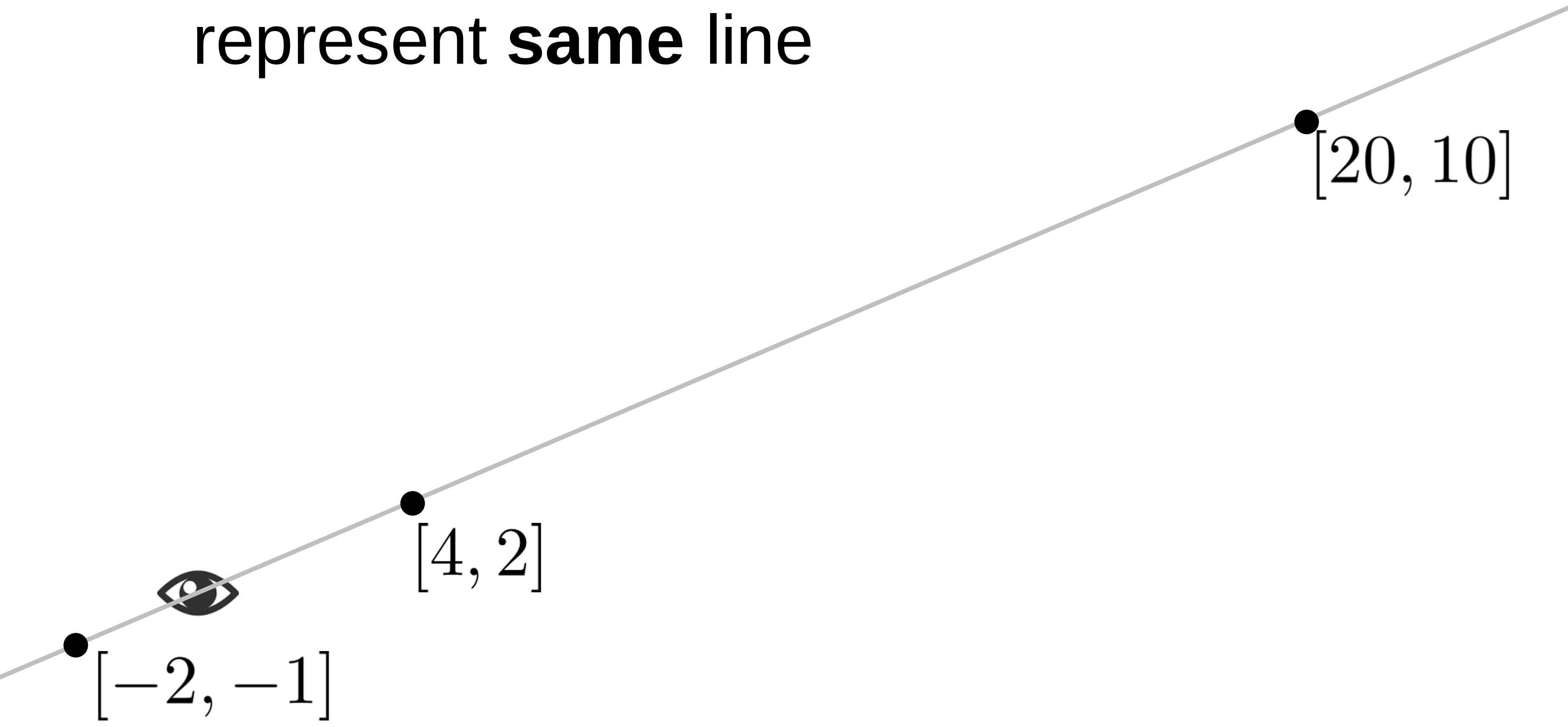
5. Homogeneous coordinates $[x, w]$

Homogeneous Coordinates

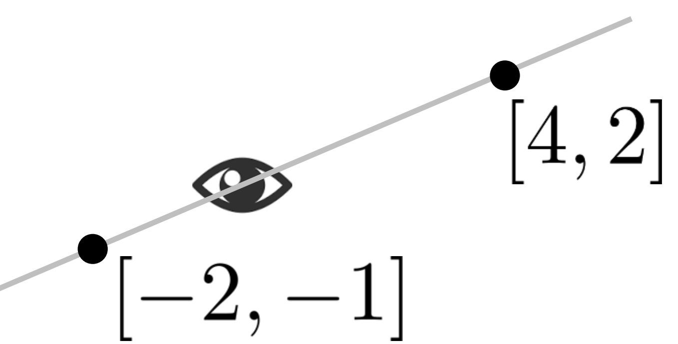
All homogeneous coordinates $[\alpha x, \alpha w]$ represent **same** line



Homogeneous Coordinates

All homogeneous coordinates $[\alpha x, \alpha w]$ represent **same** line

- say $[x, w] \sim [\alpha x, \alpha w]$ ("are **equivalent**")



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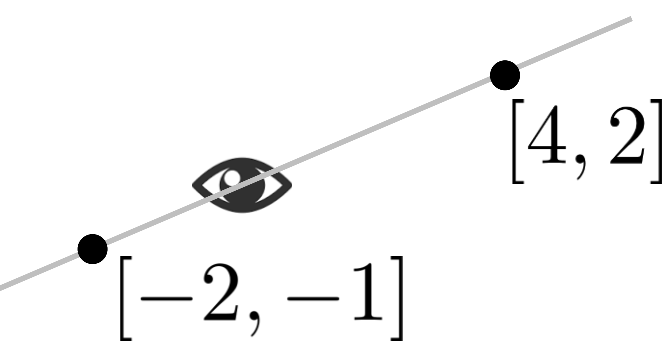
All points in 1D projective space equiv to:



1. $[x, 1]$
2. $[1, 0]$

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All homogeneous coordinates $[\alpha x, \alpha w]$ represent **same** line

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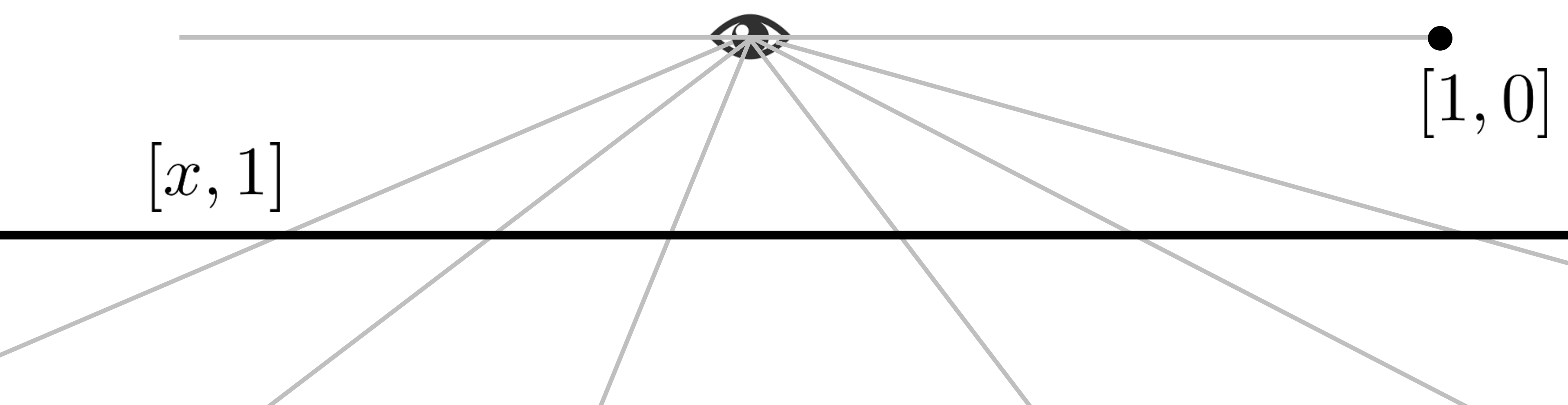
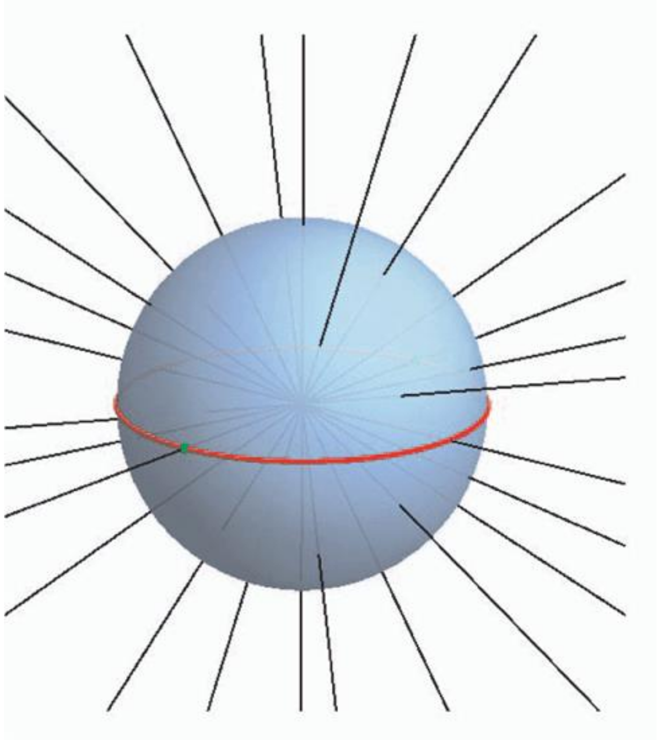
All points in 1D projective space equiv to:



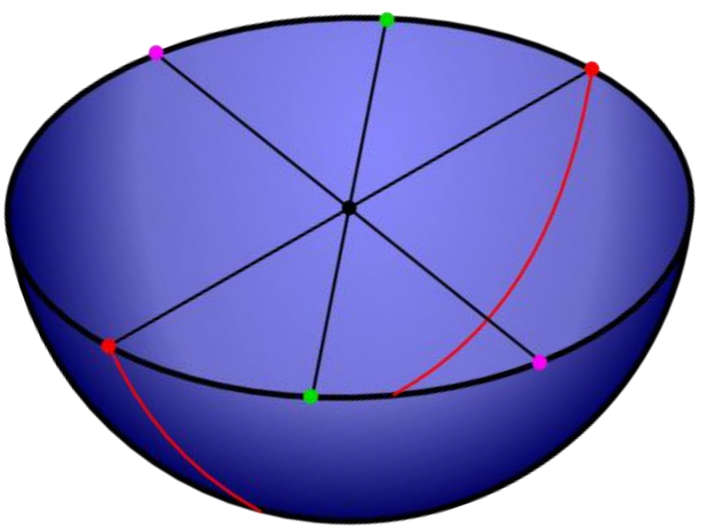
2D Projective Plane

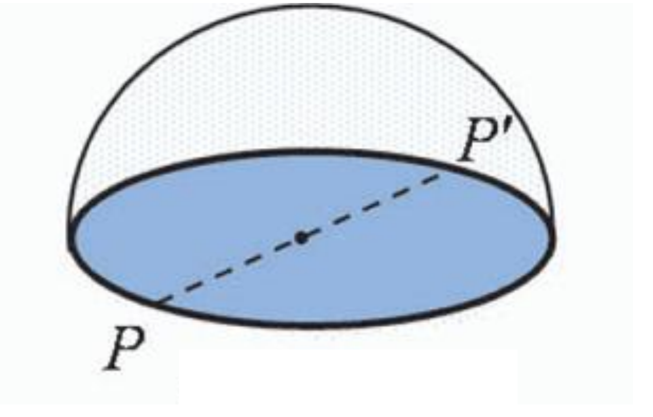
1. All lines through the origin
2. Sphere with **antipodal points glued**



2D Projective Plane

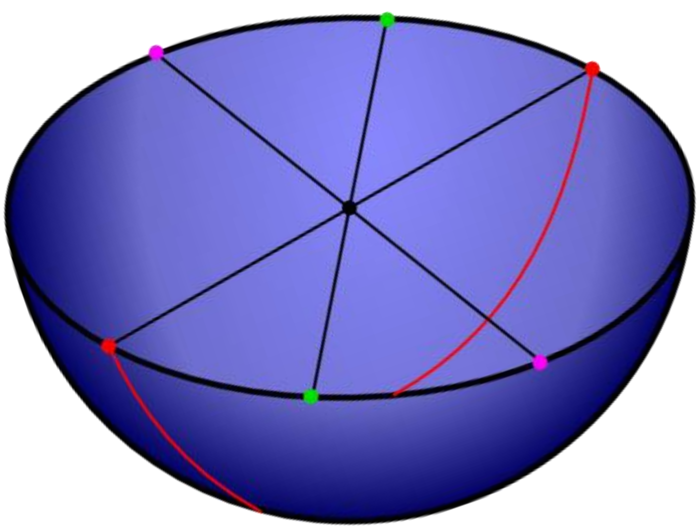
1. All lines through the origin
2. Sphere with **antipodal points glued**
3. Radius π disk with **boundary glued**

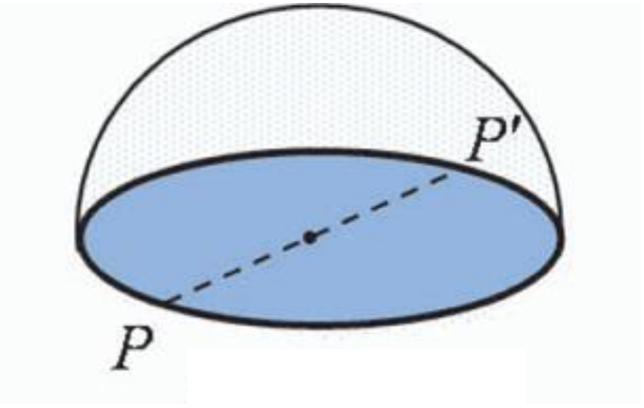




2D Projective Plane

1. All lines through the origin
2. Sphere with **antipodal points glued**
3. Radius π disk with **boundary glued**



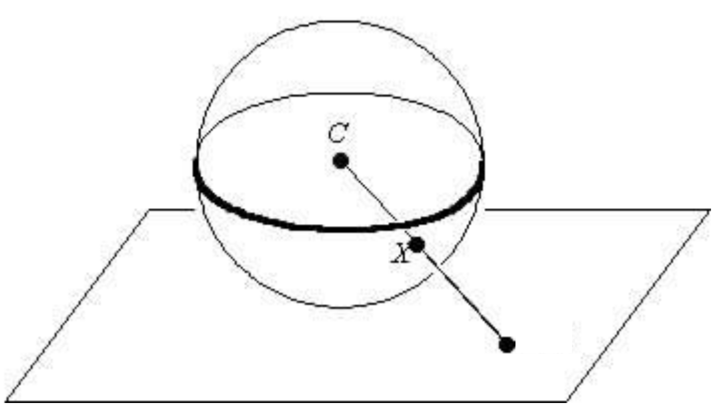


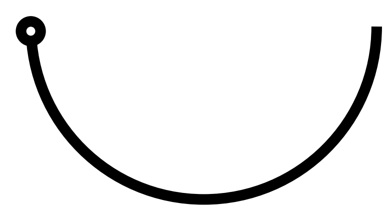
what does this look like?

Boy’s Surface

2D Projective Plane

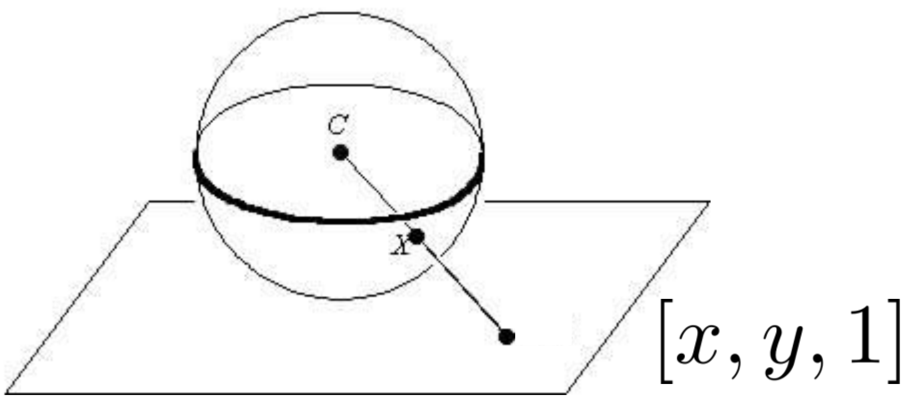
1. All lines through the origin
2. Sphere with **antipodal points glued**
3. Radius π disk with **boundary glued**
4. Plane plus “**line at infinity**”

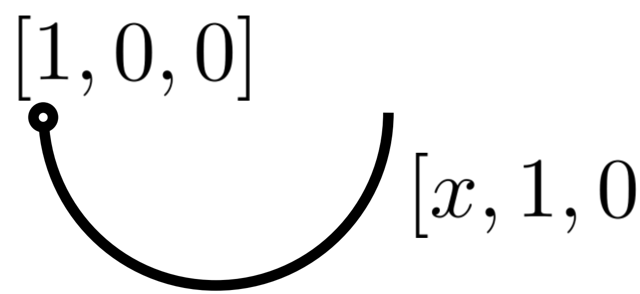




2D Projective Plane

1. All lines through the origin
2. Sphere with **antipodal points glued**
3. Radius π disk with **boundary glued**
4. Plane plus "**line at infinity**"
5. Homogeneous coordinates $[x, y, w]$





on projective plane, all lines intersect at one point

2D Projective Plane

1. All lines through the origin
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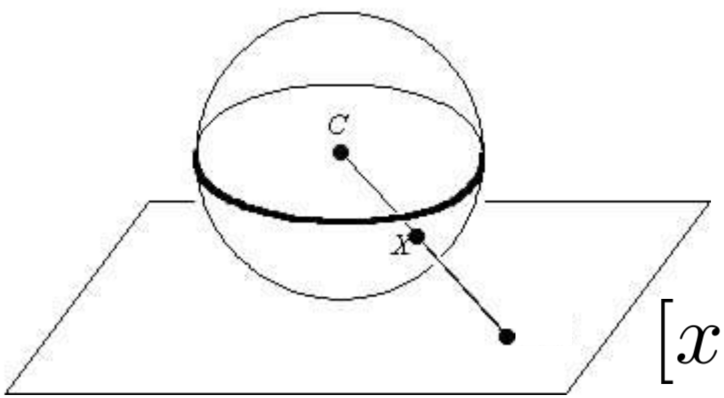
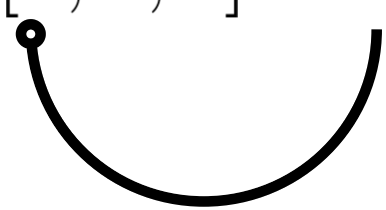


$[x, y, 1]$

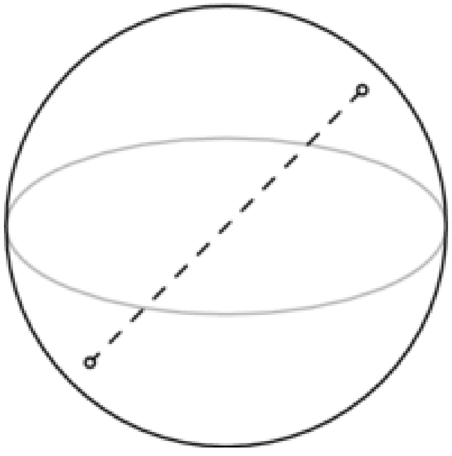
$[1, 0, 0]$



$[x, 1, 0]$

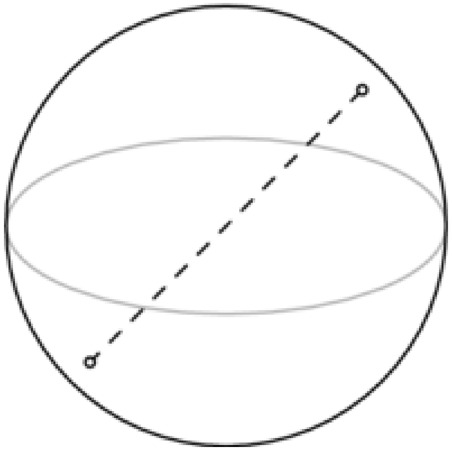
3D Projective Space

1. All lines through the origin
2. Hypersphere with **antipodal pts glued**
3. Radius π ball with **boundary glued**



3D Projective Space

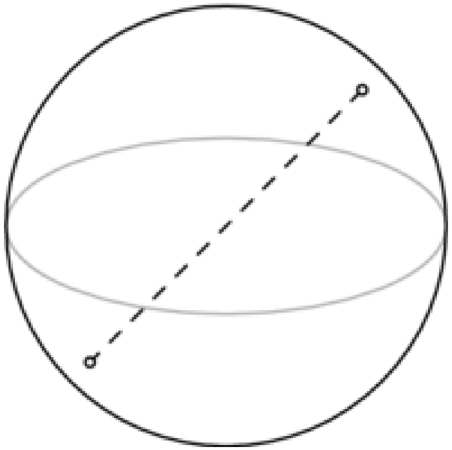
1. All lines through the origin
2. Hypersphere with **antipodal pts glued**
3. Radius π ball with **boundary glued**



remind you of anything?

3D Projective Space

1. All lines through the origin
2. Hypersphere with **antipodal pts glued**
3. Radius π ball with **boundary glued**
4. Rotations in 3D (axis-angle)!



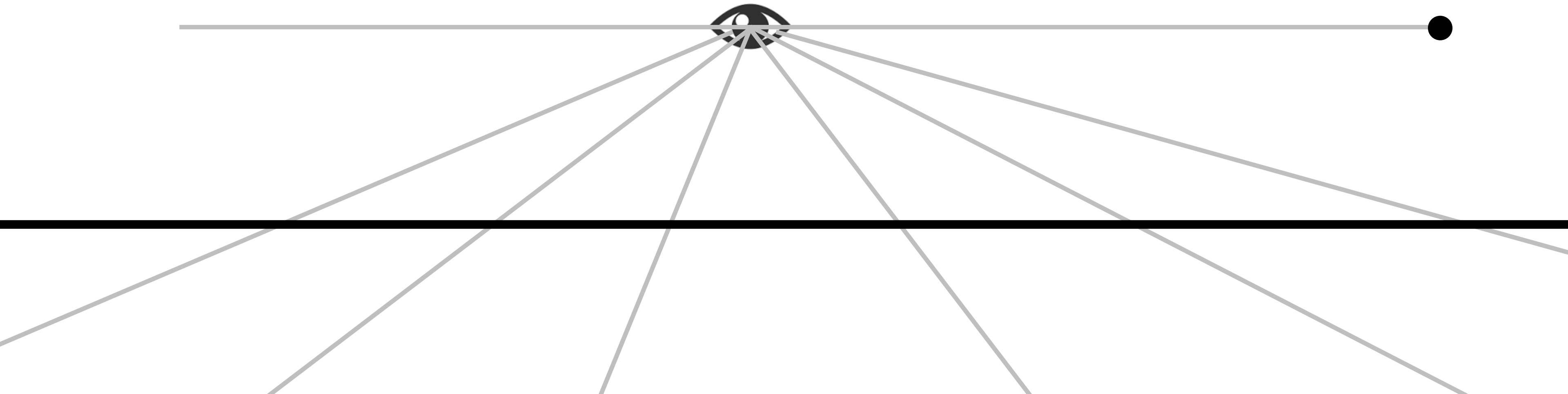
remind you of anything?

3D Projective Space

1. All lines through the origin
2. Hypersphere with **antipodal pts glued**
3. Radius π ball with **boundary glued**
4. Rotations in 3D (axis-angle)!
5. Homogeneous coordinates $[x, y, z, w]$

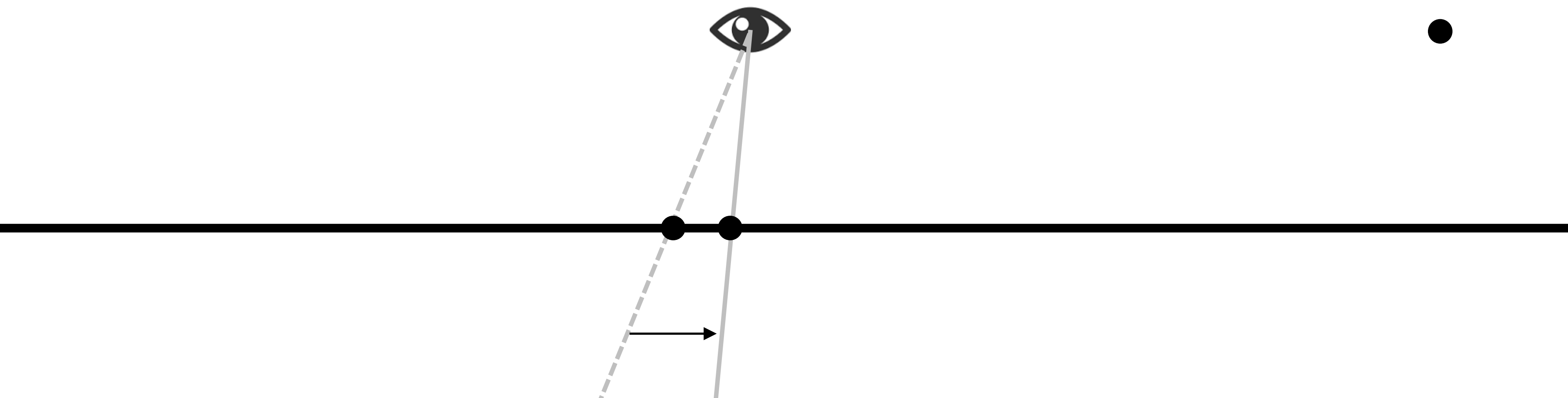
A Final Note

What does shear do in projective space?



A Final Note

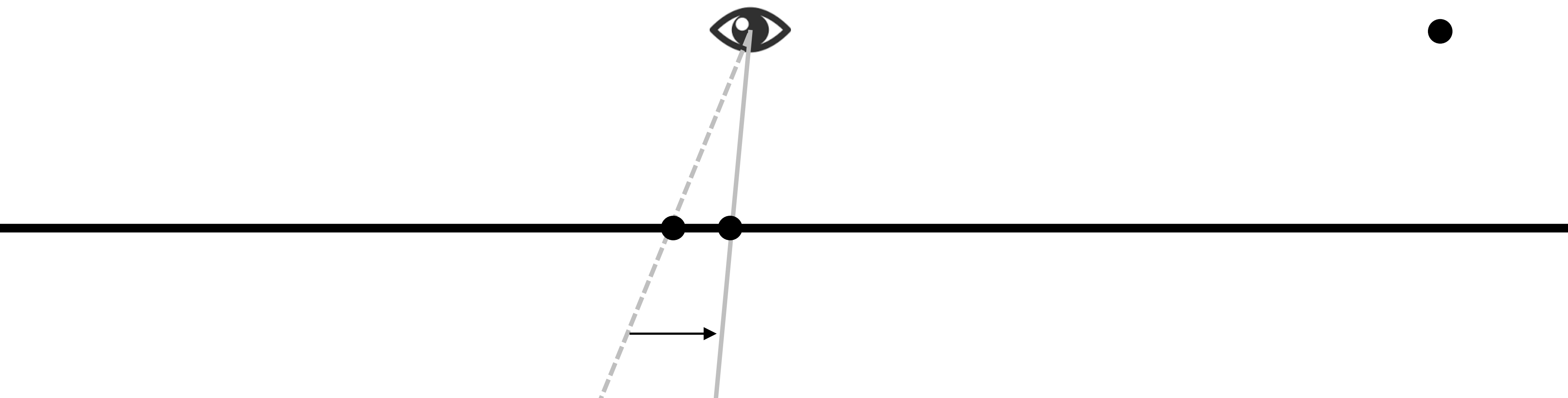
What does shear do in projective space?



A Final Note

What does shear do in projective space?

- points on line **translate**



A Final Note

What does shear do in projective space?

- points on line **translate**
- point at infinity untouched