SAT MOD ODEs

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http://dreal.cs.cmu.edu

Decision Problems over Real Numbers

Given an arbitrary first-order φ over

$$\langle \mathbb{R}, \geq, \mathcal{F} \rangle$$

decide the truth value of φ.

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With a rich enough $\mathcal{F}$, we would be able to:

- solve many control-engineering problems
- verify and synthesize safety-critical embedded systems

High-speed Parking

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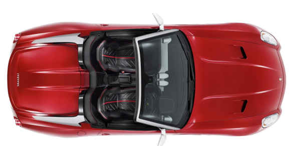

Speed up

High-speed Parking

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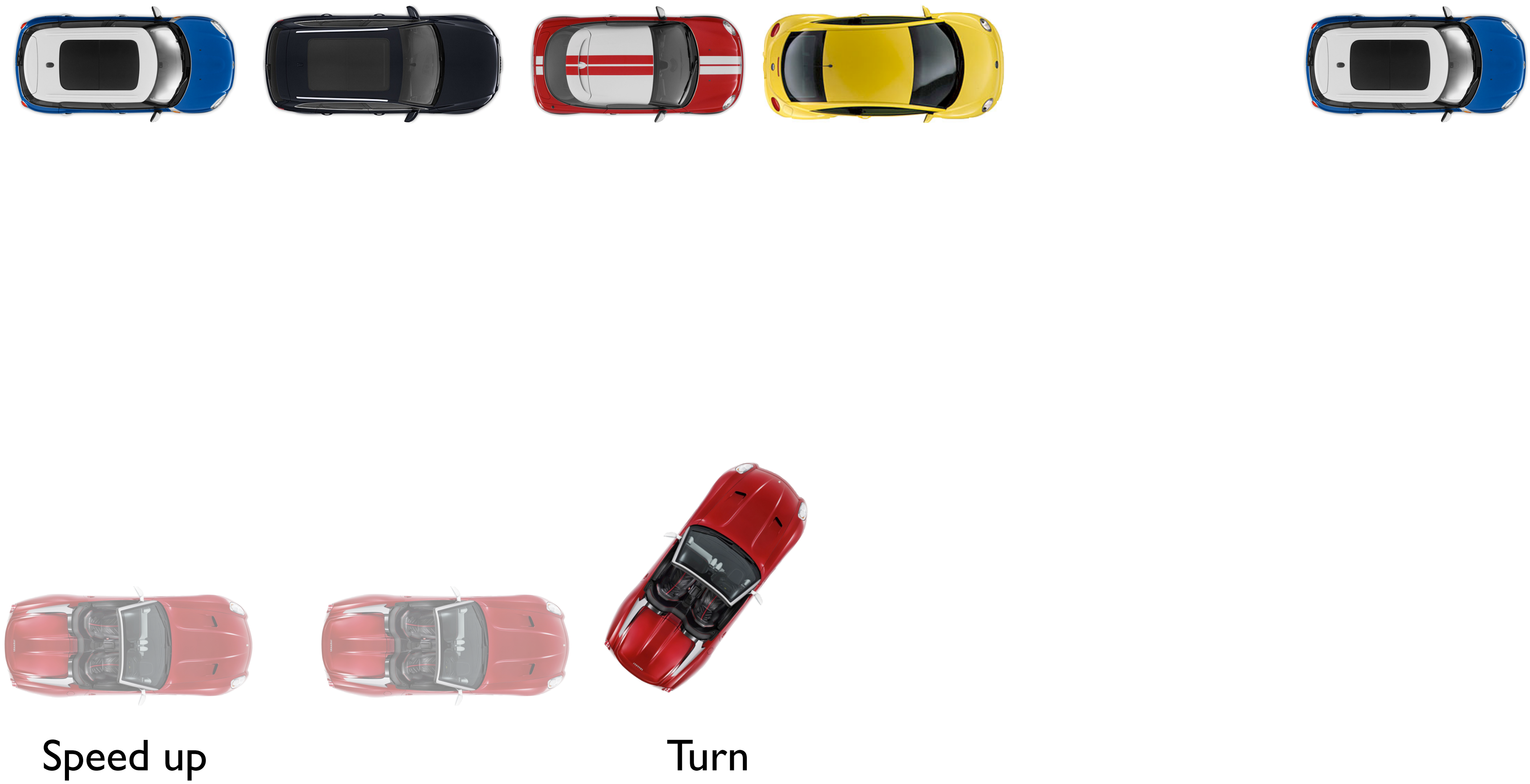

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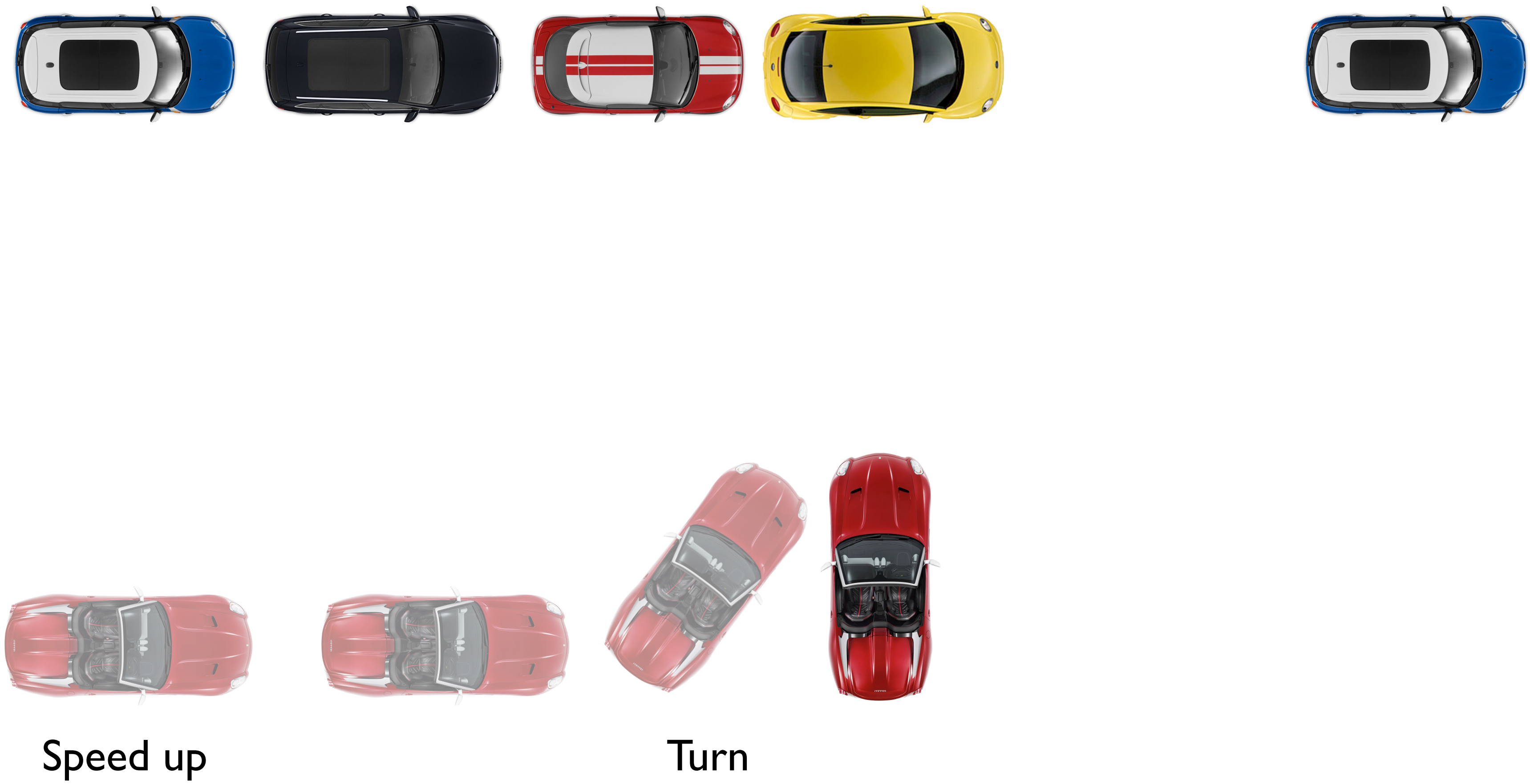

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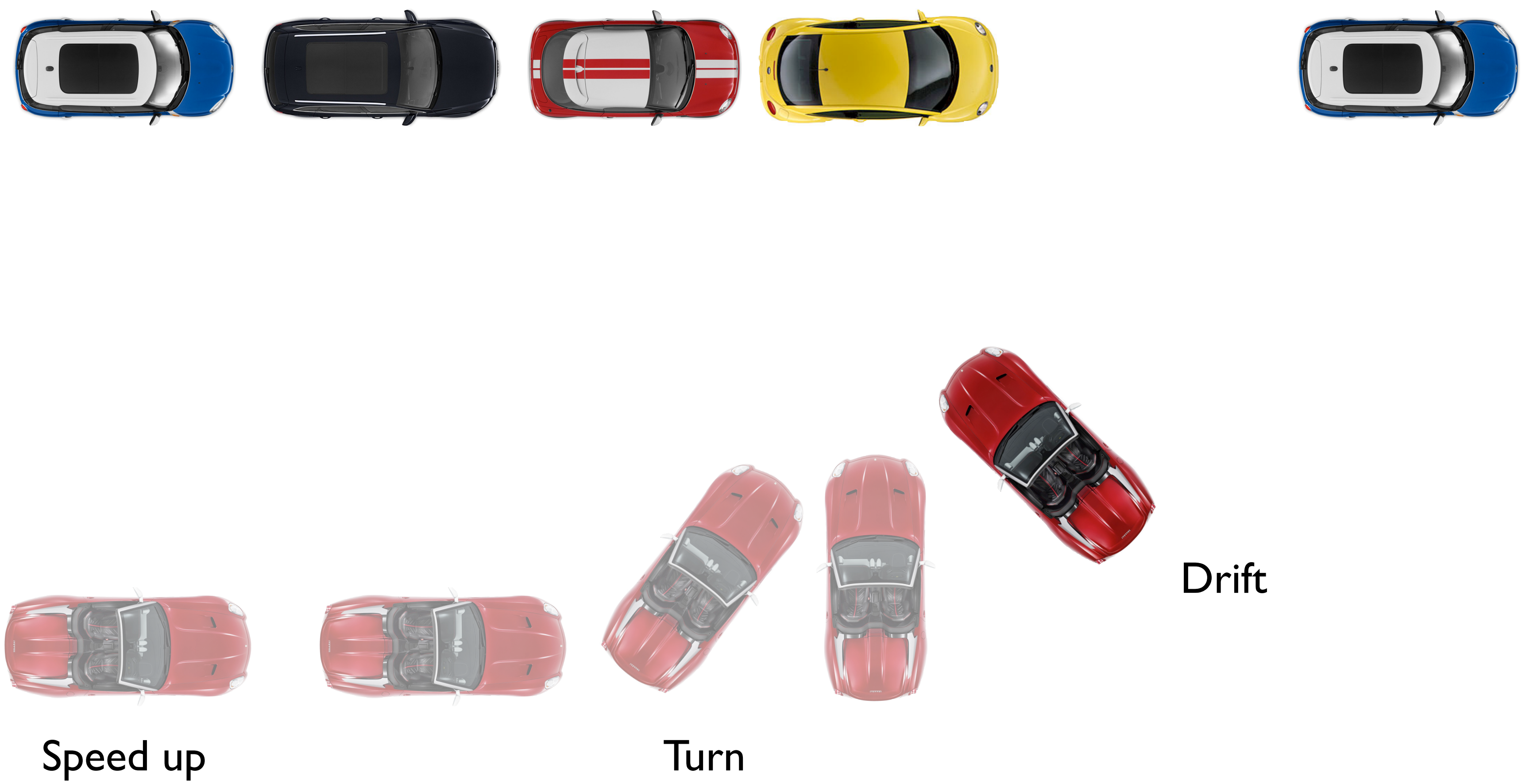

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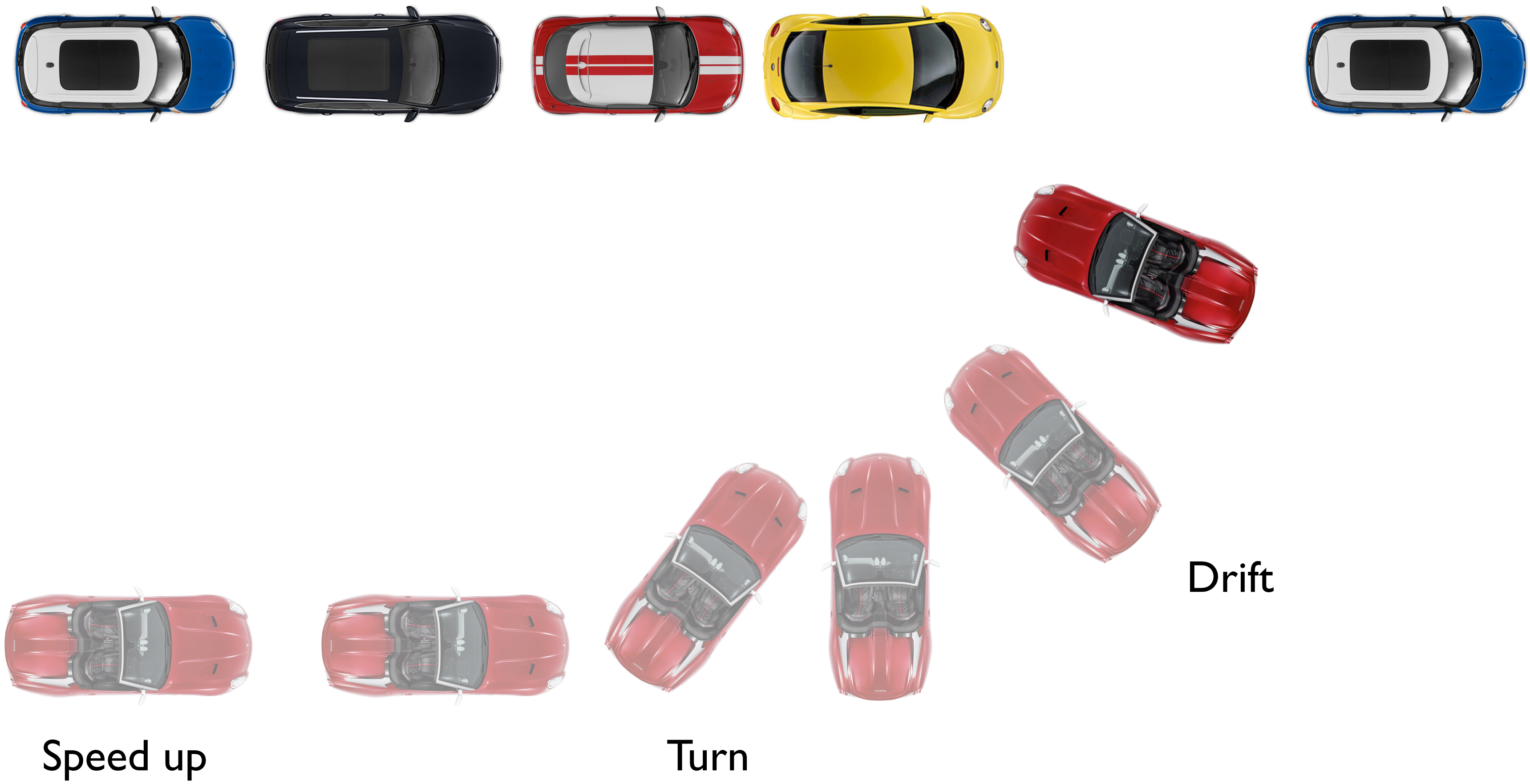

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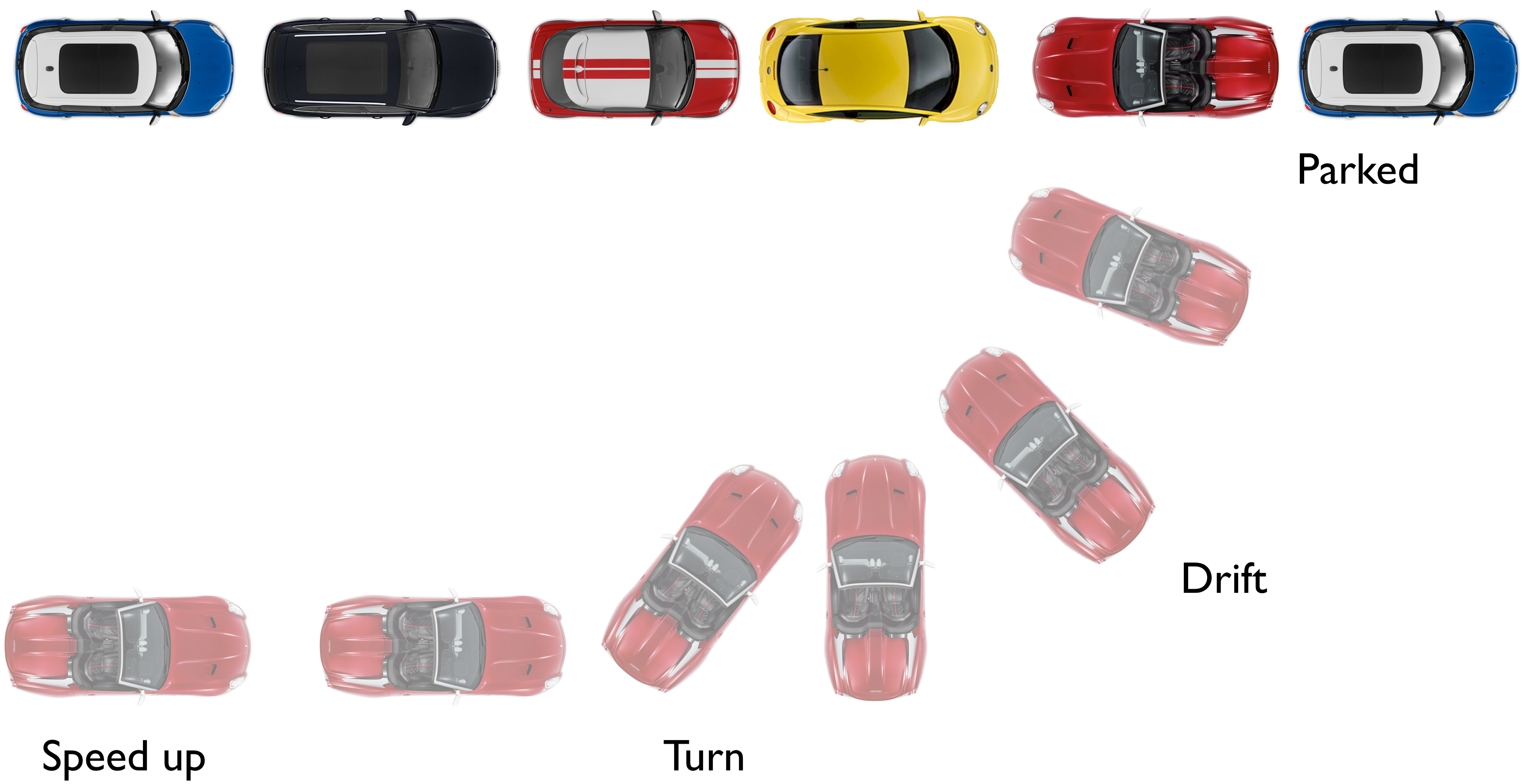
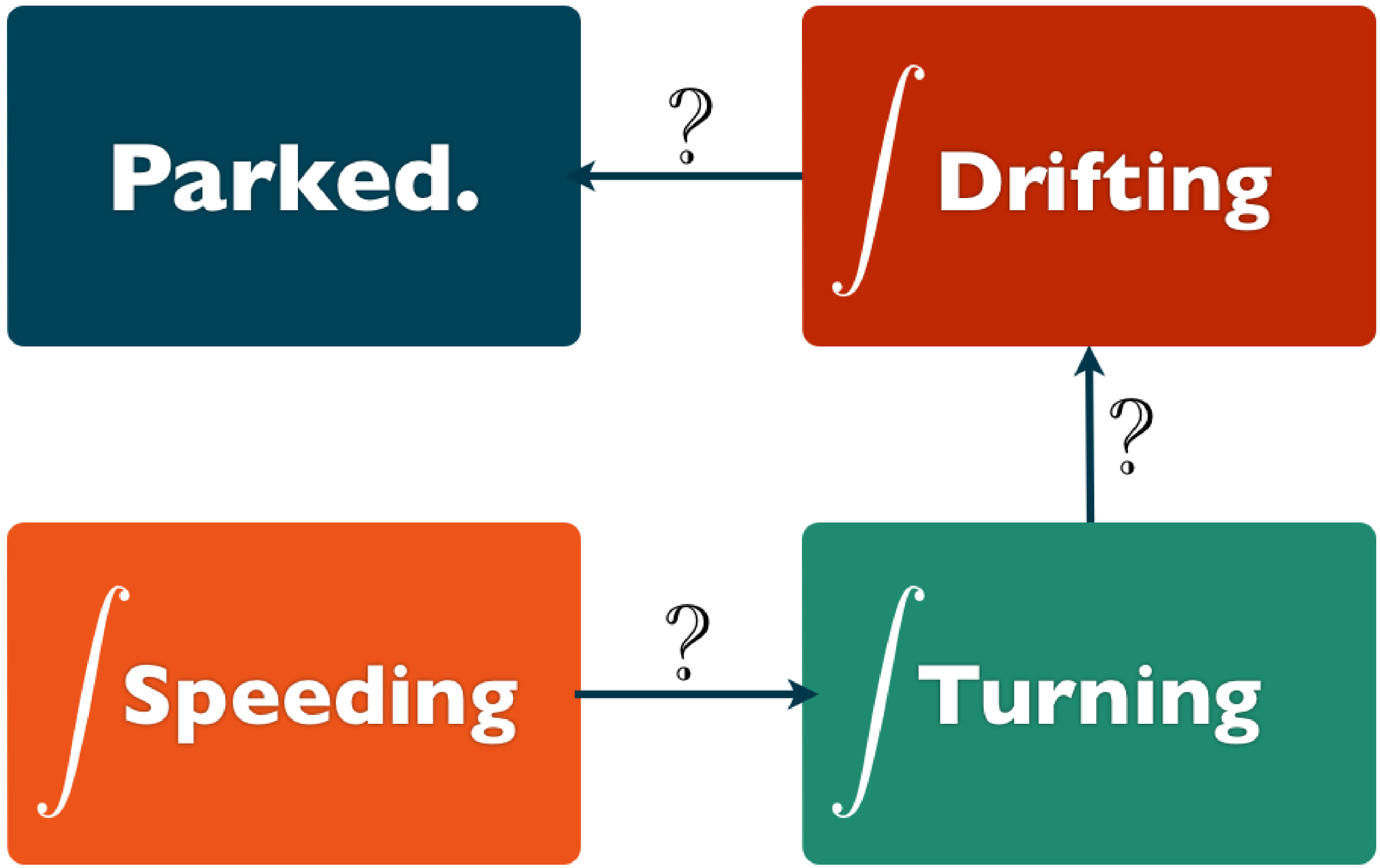


High-speed Parking



Logic Encoding

We can do this if we can solve the following SMT formula in real-time:

$$\mathsf{speedup}(\vec{x}_0) \wedge \left(\vec{x}_1 = \vec{x}_0 + \int_0^{t_1} \mathsf{speeding}(s)\mathrm{d}s\right) \bigwedge$$

$$\mathsf{steer}(\vec{x}_1, \vec{x}_2) \wedge \left(\vec{x}_3 = \vec{x}_2 + \int_0^{t_2} \mathsf{turning}(s)\mathrm{d}s\right) \bigwedge$$

$$\mathsf{brake}(\vec{x}_3, \vec{x}_4) \wedge \left(\vec{x}_5 = \vec{x}_4 + \int_0^{t_3} \mathsf{drifting}(s)\mathrm{d}s\right) \bigwedge \mathsf{parked}(\vec{x}_5)$$

Isn't this problem too hard?

Difficulty

Suppose $\mathcal{F}$ is $\{+, \times\}$.

$$\mathbb{R} \overset{?}{\models} \exists a \forall b \exists c\ (ax^2 + bx + c > 0)$$

- Decidable [Tarski 1948].
- Double-exponential lower-bound. Extensive research on practical solvers.

Difficulty

Suppose $\mathcal{F}$ further contains **sine**:

$$\mathbb{R} \overset{?}{\models} \exists x, y, z\ (\sin^2(\pi x) + \sin^2(\pi y) + \sin^2(\pi z) = 0 \bigwedge x^3 + y^3 = z^3)$$

- Σ_1 case already undecidable.
- Partial algorithms are of extremely high complexity.
- Engineers would rather be left alone.

The key is to change the decision problem.

The Delta-Decision Problem (one version)

Given φ and $\delta \in \mathbb{Q}^+$, return one of the following:

- φ is false.
- A weakening of the original formula, $\varphi^{-\delta}$, is true.

We now define what $\varphi^{-\delta}$ is.

δ-Variants

Any bounded $\mathcal{L}_{\mathcal{F}}$ -sentence φ can be written in the form

$$Q_1^{[u_1,v_1]}x_n \cdots Q_n^{[u_n,v_n]}x_n \bigwedge(\bigvee t(\vec{x}) > 0 \vee \bigvee t(\vec{x}) \geq 0)$$

Definition (δ-weakening)

Let $\delta \in \mathbb{Q}^+ \cup \{0\}$. The **δ-weakening** $\varphi^{-\delta}$ of φ is

$$Q_1^{[u_1,v_1]}x_1 \cdots Q_n^{[u_n,v_n]}x_n \bigwedge(\bigvee t(\vec{x}) > -\delta \vee \bigvee t(\vec{x}) \geq -\delta)$$

δ-Decisions

Let $\delta \in \mathbb{Q}^+$ be arbitrary.

Definition (δ-decisions)

Decide, for any given bounded φ, whether

- φ is false, or
- $\boldsymbol{\varphi^{-\delta}}$ is true.

When the two cases overlap, either answer can be returned.

δ-Decidability

Let $\mathcal{F}$ be an arbitrary collection of Type 2 computable functions.

Theorem [Gao et al. LICS'12]

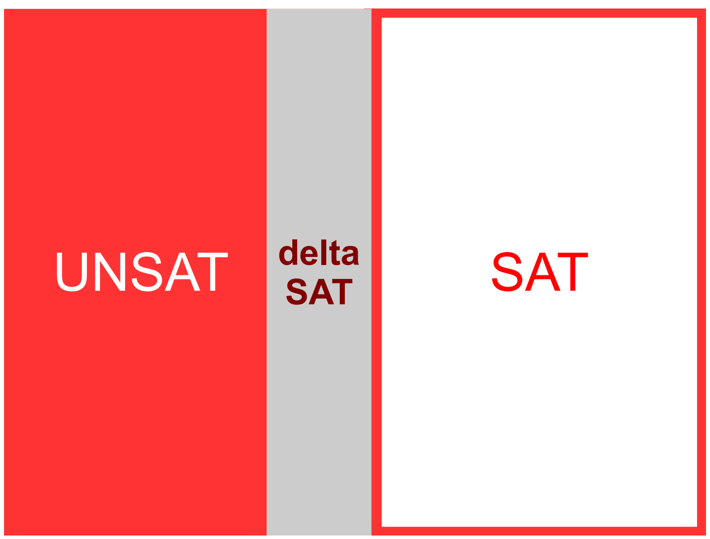
The δ-decision problem over $\mathbb{R}_{\mathcal{F}}$ is decidable.

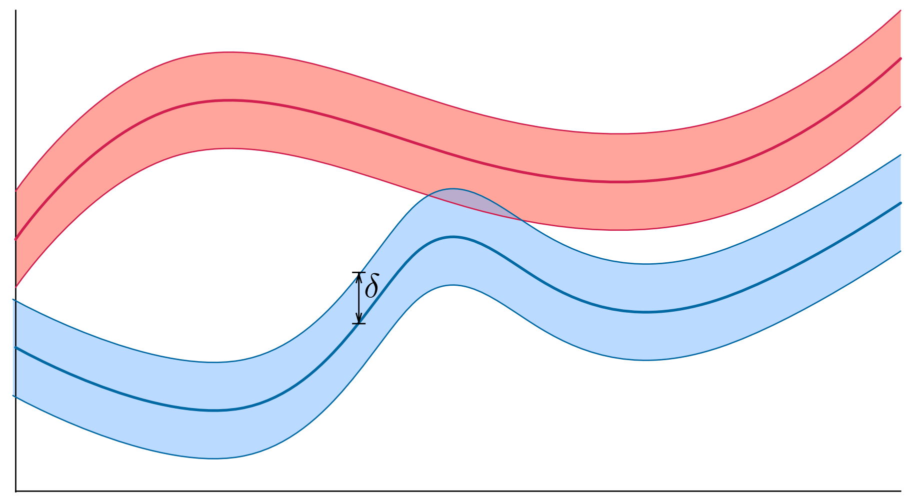
Type 2 computable functions:

- Polynomials
- exp, sine, …
- L-continuous ODEs
- PDEs, …

δ-Decisions

There is a grey area that a δ-complete algorithm can be wrong about.





δ is good

A system $\mathcal{S}$ is **safe** if some formula φ is false.

- $\exists x_0 \exists t \exists x_t(\mathsf{Reach}(x_0, t, x_t) \wedge \mathsf{Unsafe}(x_t))$

Now the interpretation of δ-decisions is:

- False: $\mathcal{S}$ is **safe** (within bounds, for BMC).
- δ-True: $\mathcal{S}$ is **unsafe**, or **some δ-perturbation would make it unsafe**. You shouldn't rely on it anyway.

Complexity

Theorem

- $\mathcal{F} = \{+, \times, \exp, \sin, \dots\}$: $\mathbf{\Sigma_k^P}$**-complete**.
- $\mathcal{F} = \{\text{ODEs with PTIME deriv.}\}$: **PSPACE -complete**.

These are extremely low compared to the original ones.

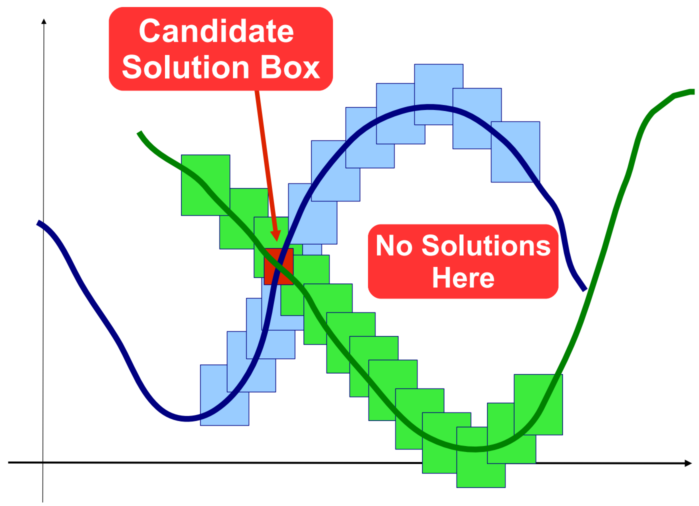
In theory, it may be possible to solve some. In practice?

Formal Analysis of Numerical Algorithms

- We say an algorithm is **δ-complete** if it solves δ-decision problem.
- Many numerically-driven procedures satisfy δ-completeness after **formal analysis** [Gao et al, IJCAR'12].

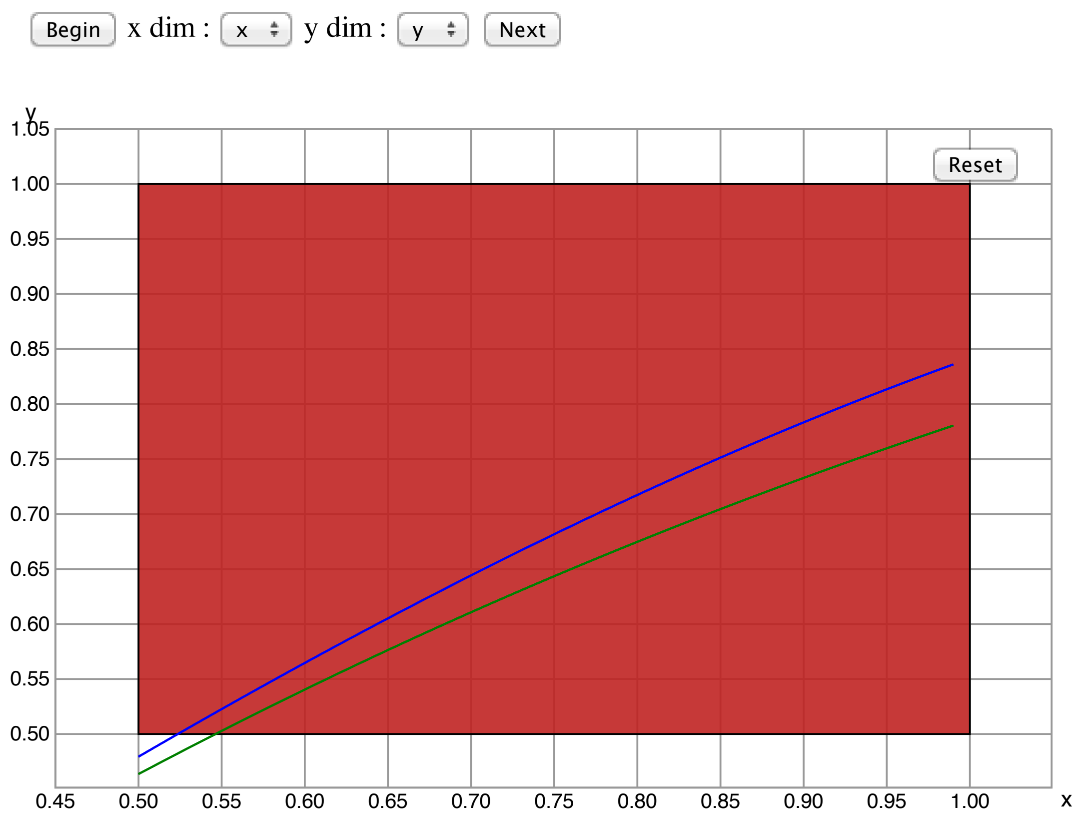
Interval Constraint Propagation

- Contract big initial interval boxes to small ones that cover solutions.
- If some constraints are satisfiable, then the interval relaxations always have overlapping boxes.



Interval Constraint Propagation

$$\exists x, y \in [0.5, 1.0] : y = \sin(x) \land y = \operatorname{atan}(x)$$



δ-Completeness of ICP

We gave conditions for a pruning operator to be **well-defined**, formalizing practical implementation strategies used in ICP.

Theorem [Gao et al. IJCAR'12]

DPLL(ICP) is δ-complete **iff** its pruning operators are well-defined.

We now go into the details of ODE solving.

Handling Differential Equations

An ODE system

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = \vec{f}\,(\vec{x}, t)$$

when put in Picard–Lindelöf form:

$$\vec{x}_t = \vec{x}_0 + \int_0^t f(\vec{x}, s)\mathrm{d}s$$

is seen as a constraint between $\vec{x}_0, \vec{x}_t$, and t.

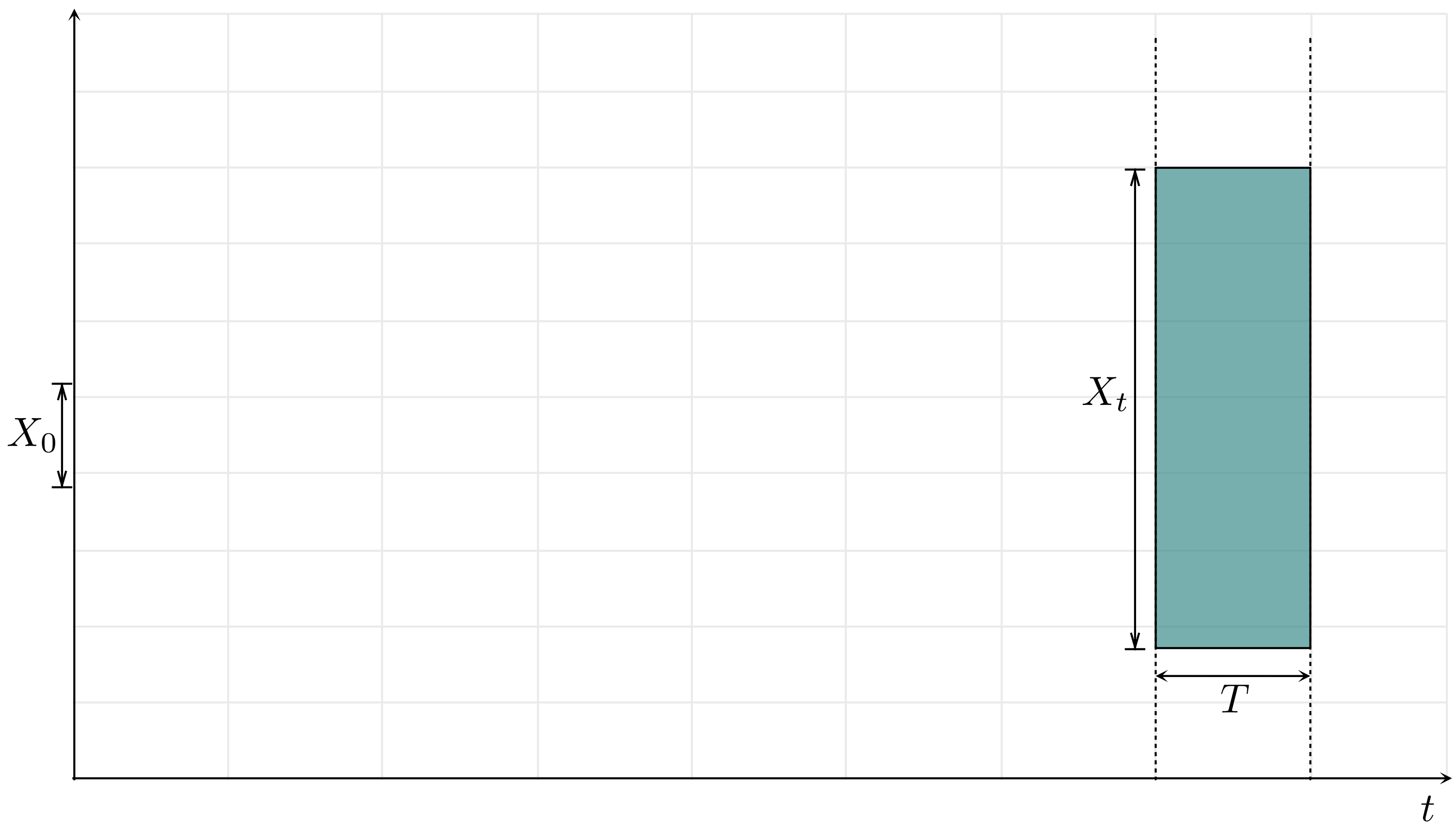
ODE Pruning

Starting with big intervals for

$$\vec{x}_t, \vec{x}_0, t$$

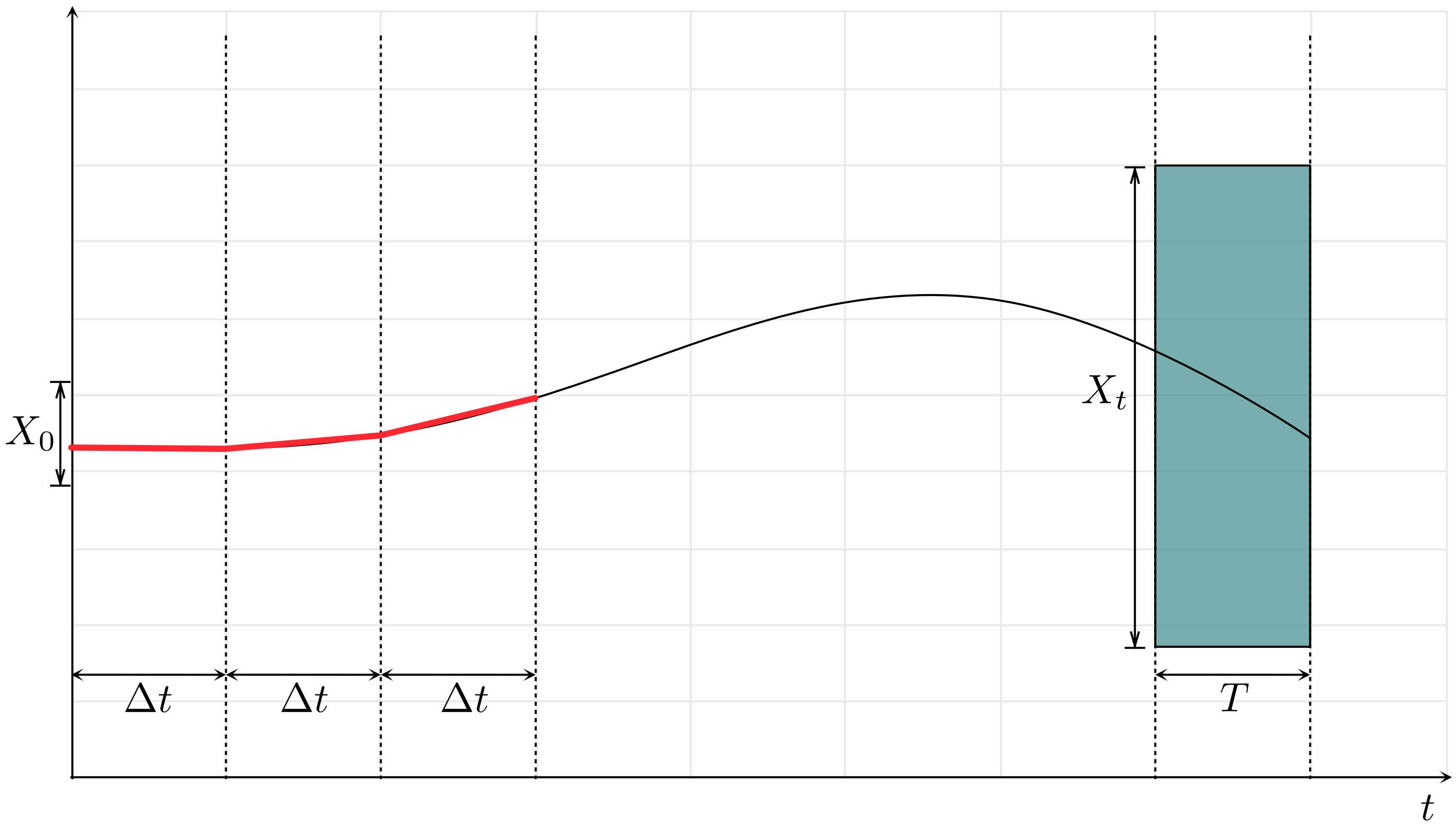
use the ODE constraints to find smaller intervals for them.

Forward Pruning (on X_t)



$\Rightarrow$

Forward Pruning (on X_t)



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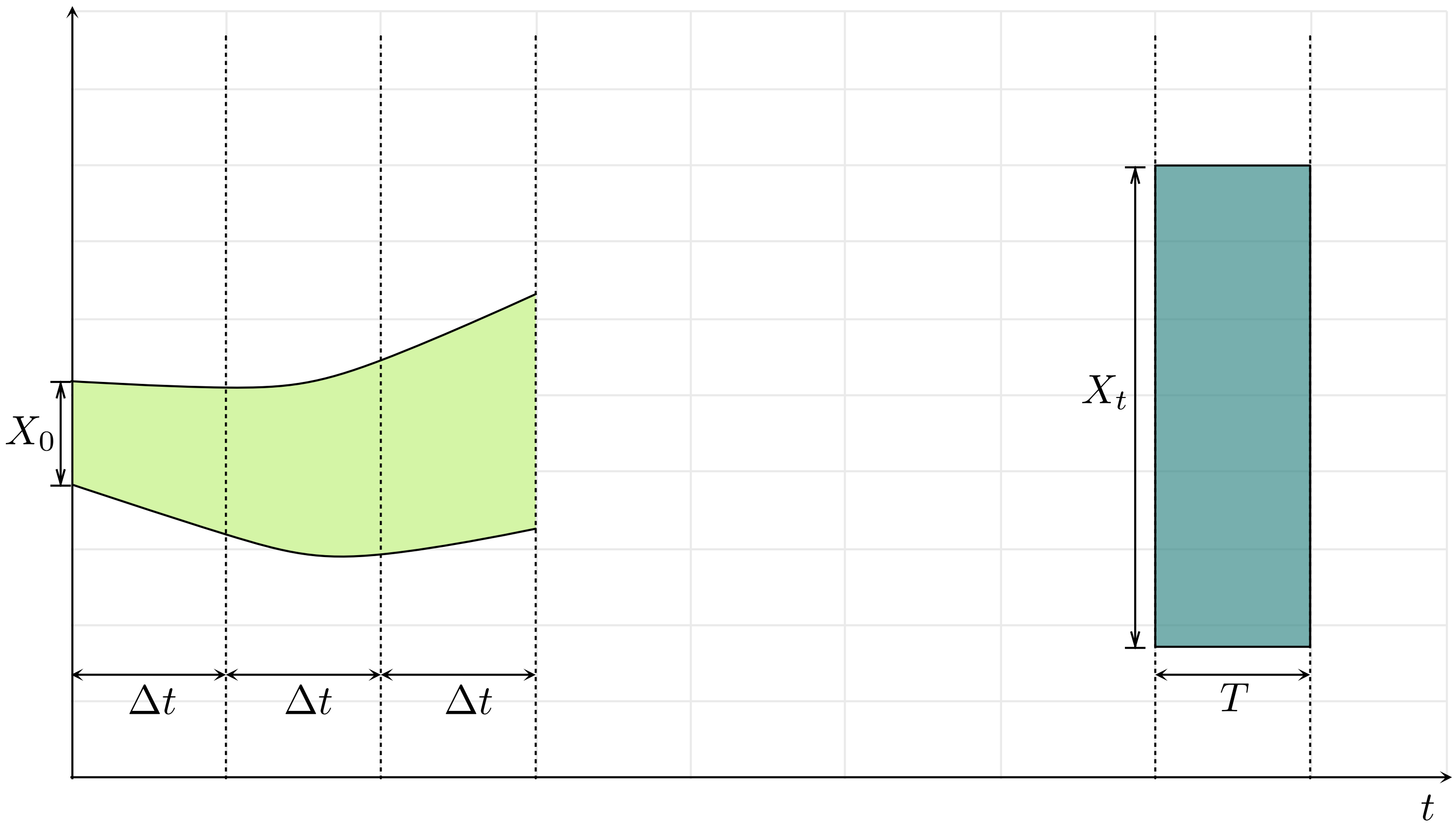
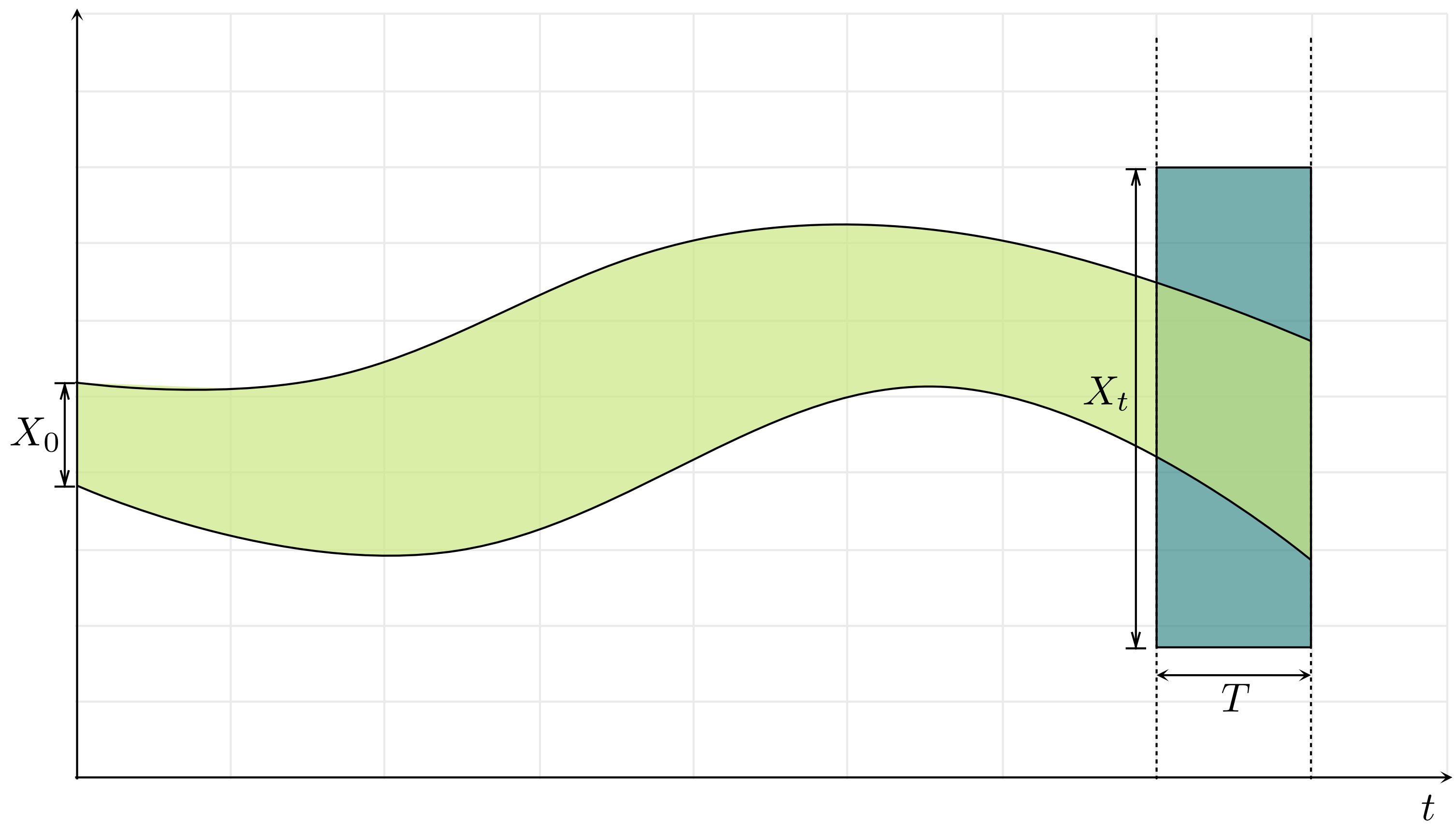
Forward Pruning (on X_t)



$\Rightarrow$

Forward Pruning (on X_t)



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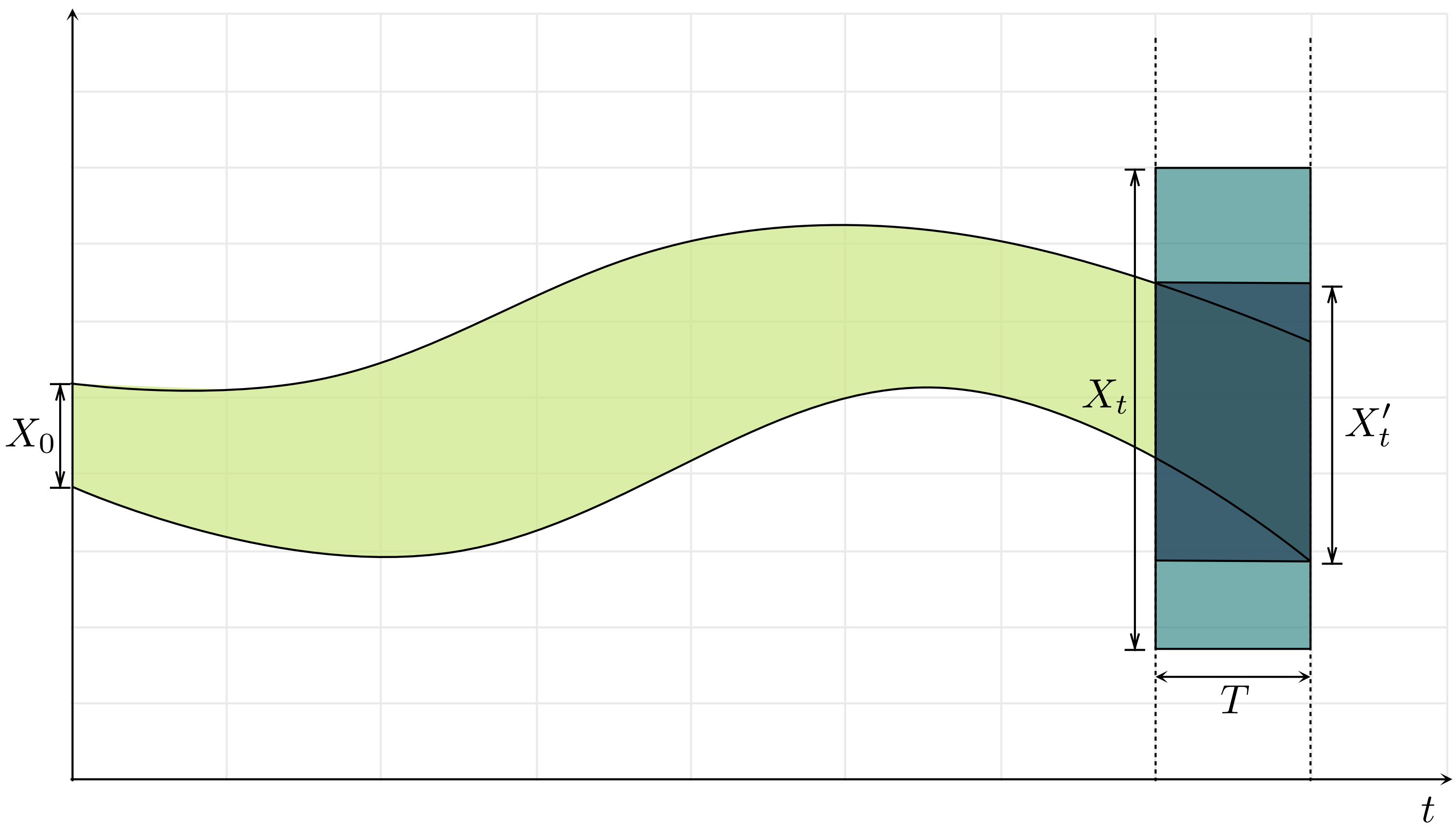
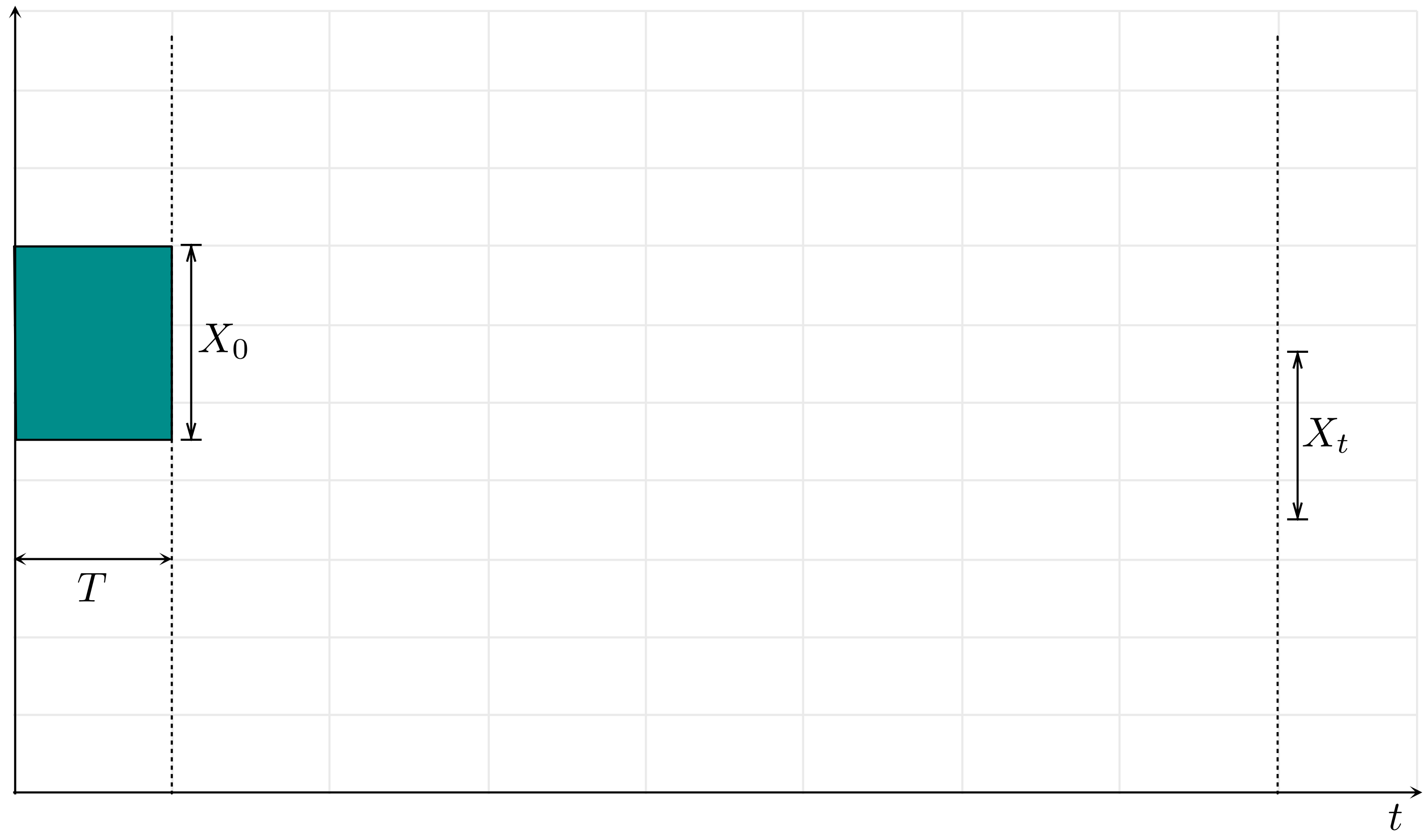
Forward Pruning (on X_t)

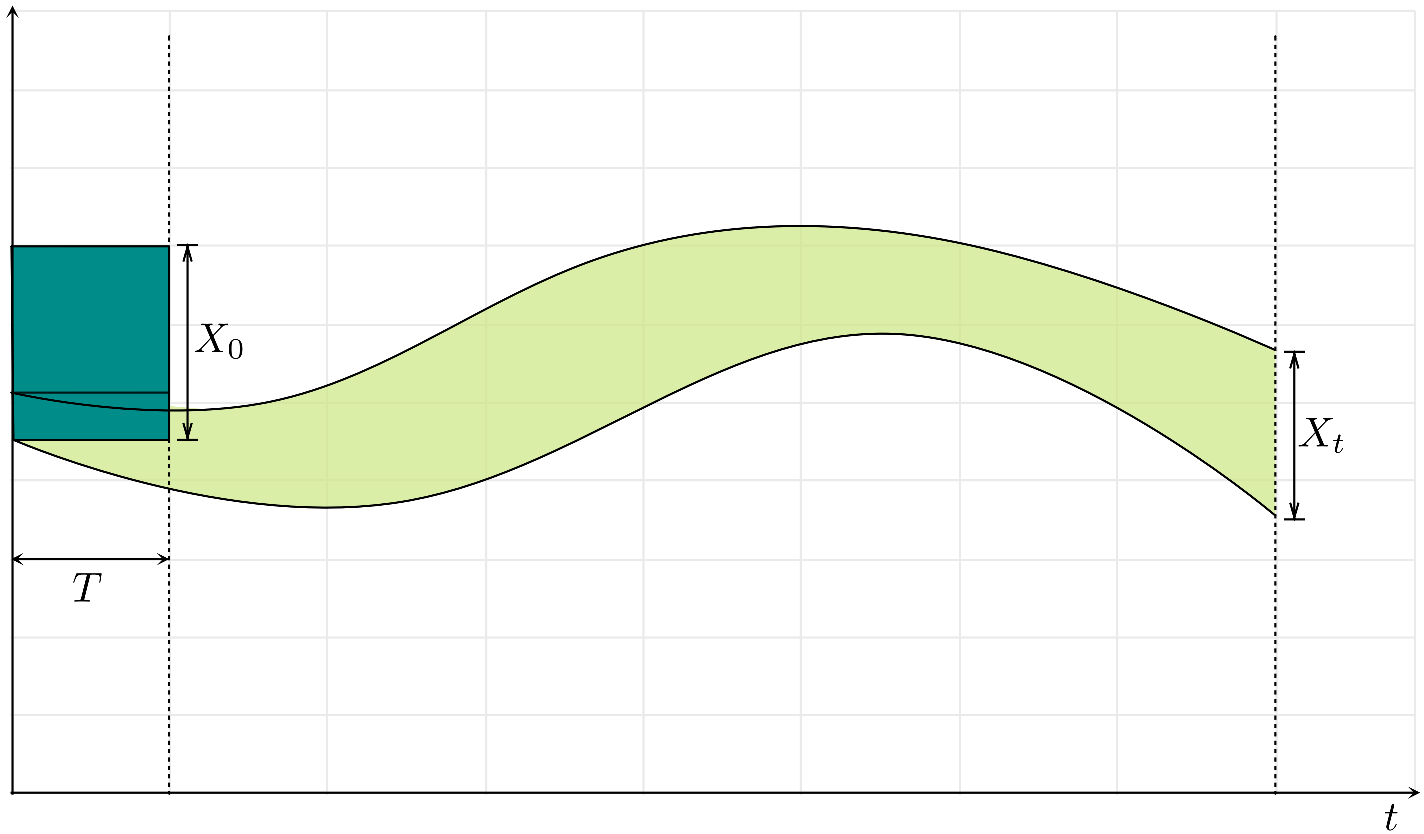


$\Rightarrow$

Backward Pruning (on X_0)

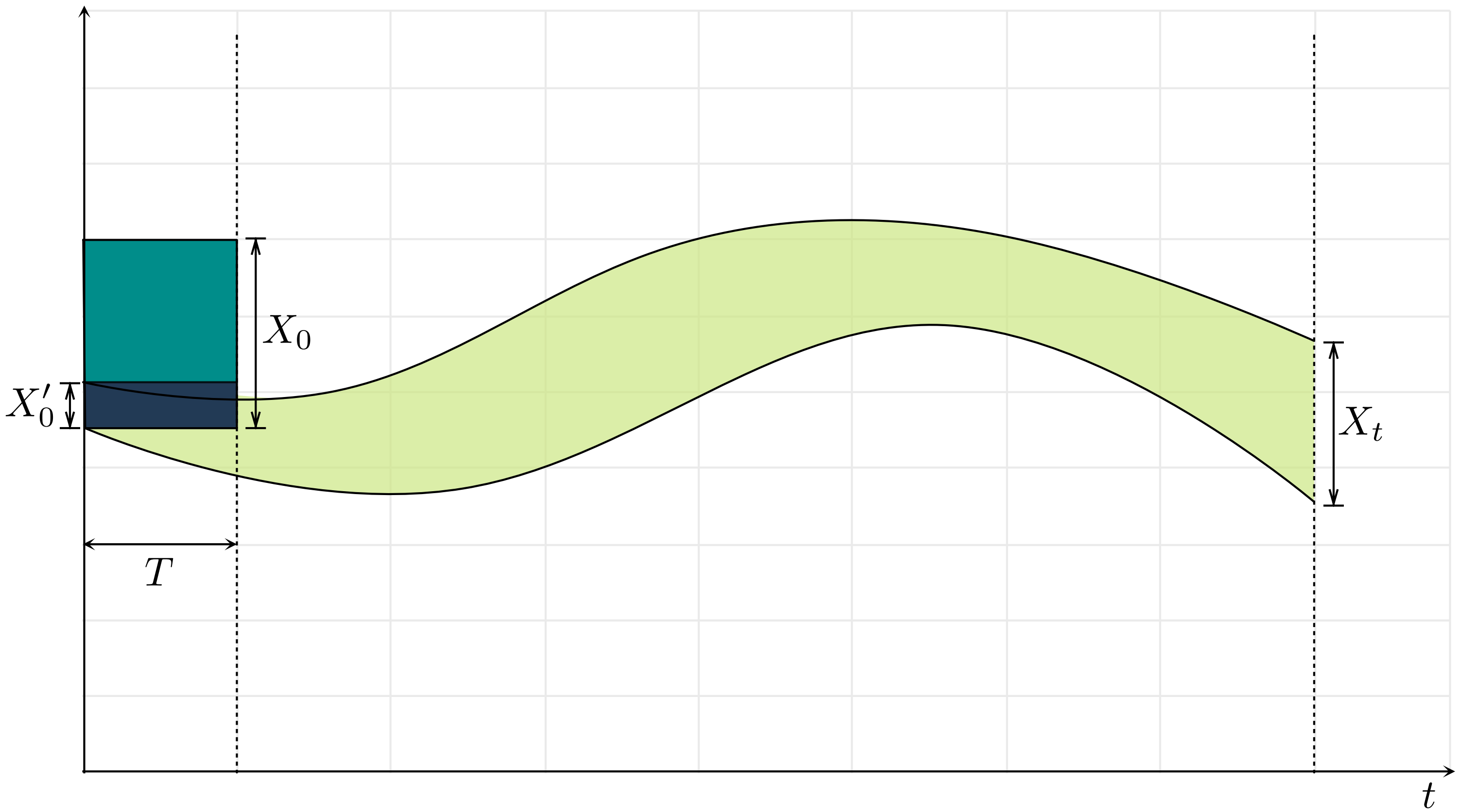


Backward Pruning (on X_0)



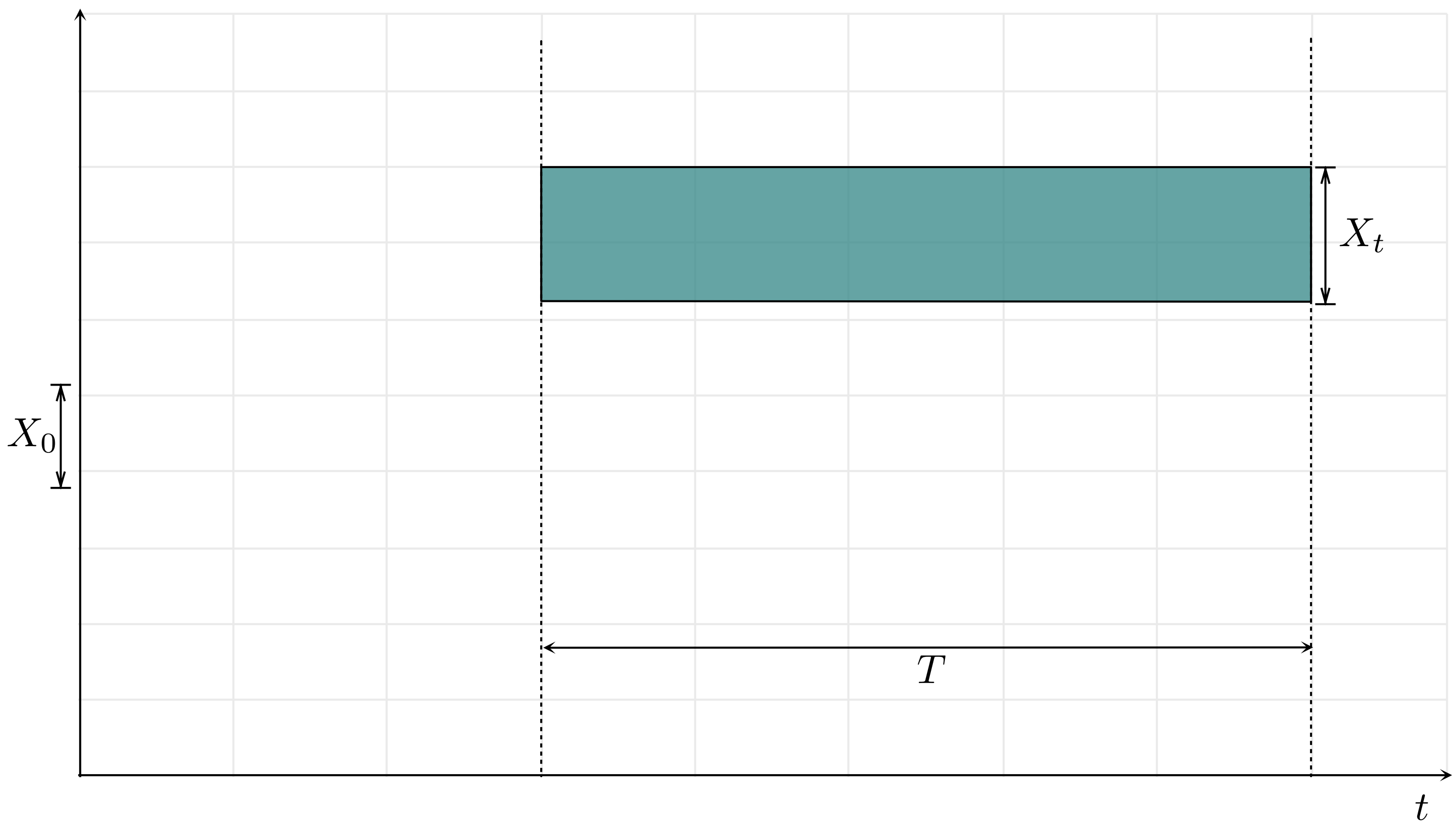
$\Leftarrow$

Backward Pruning (on X_0)

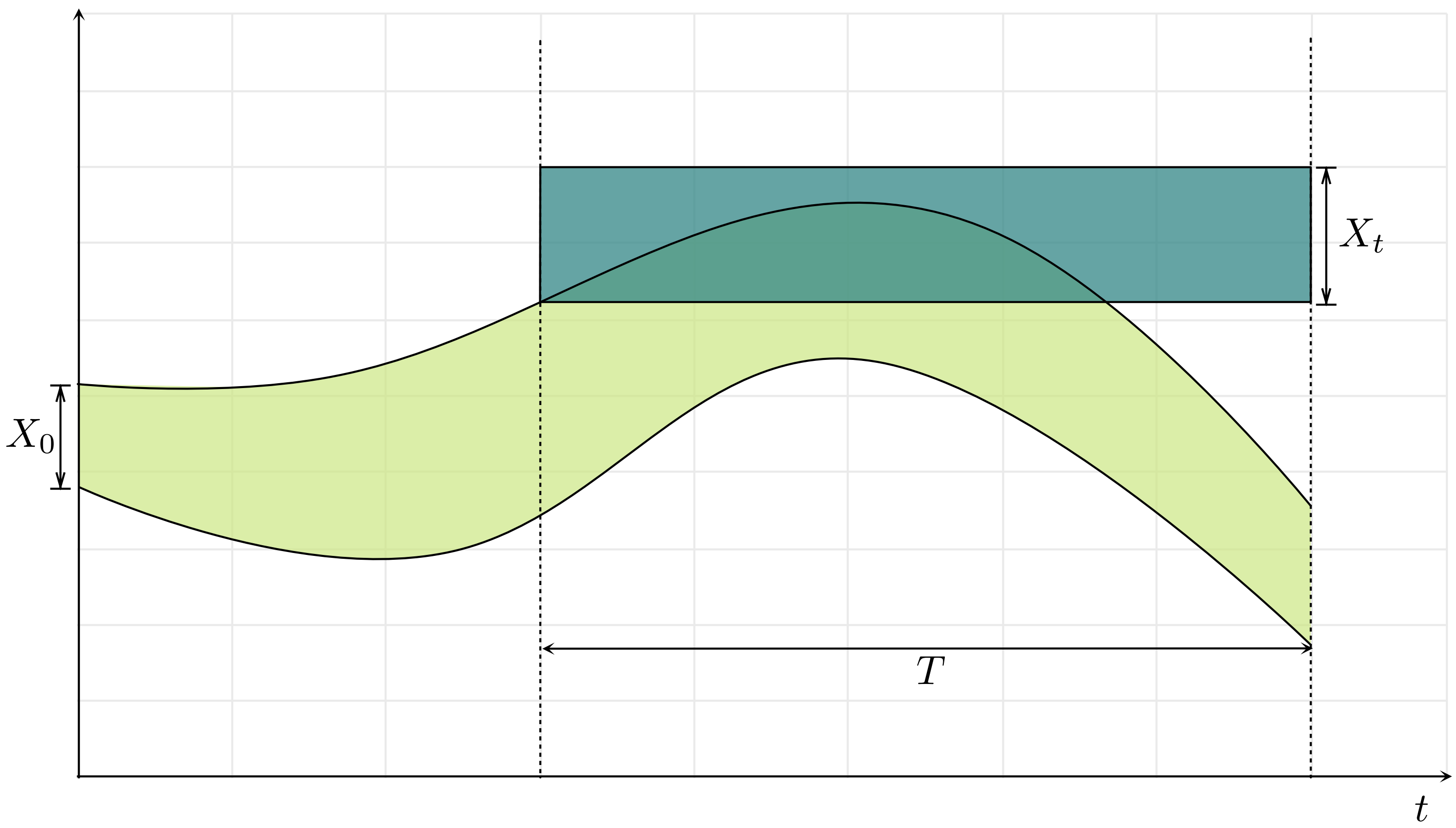


$\Leftarrow$

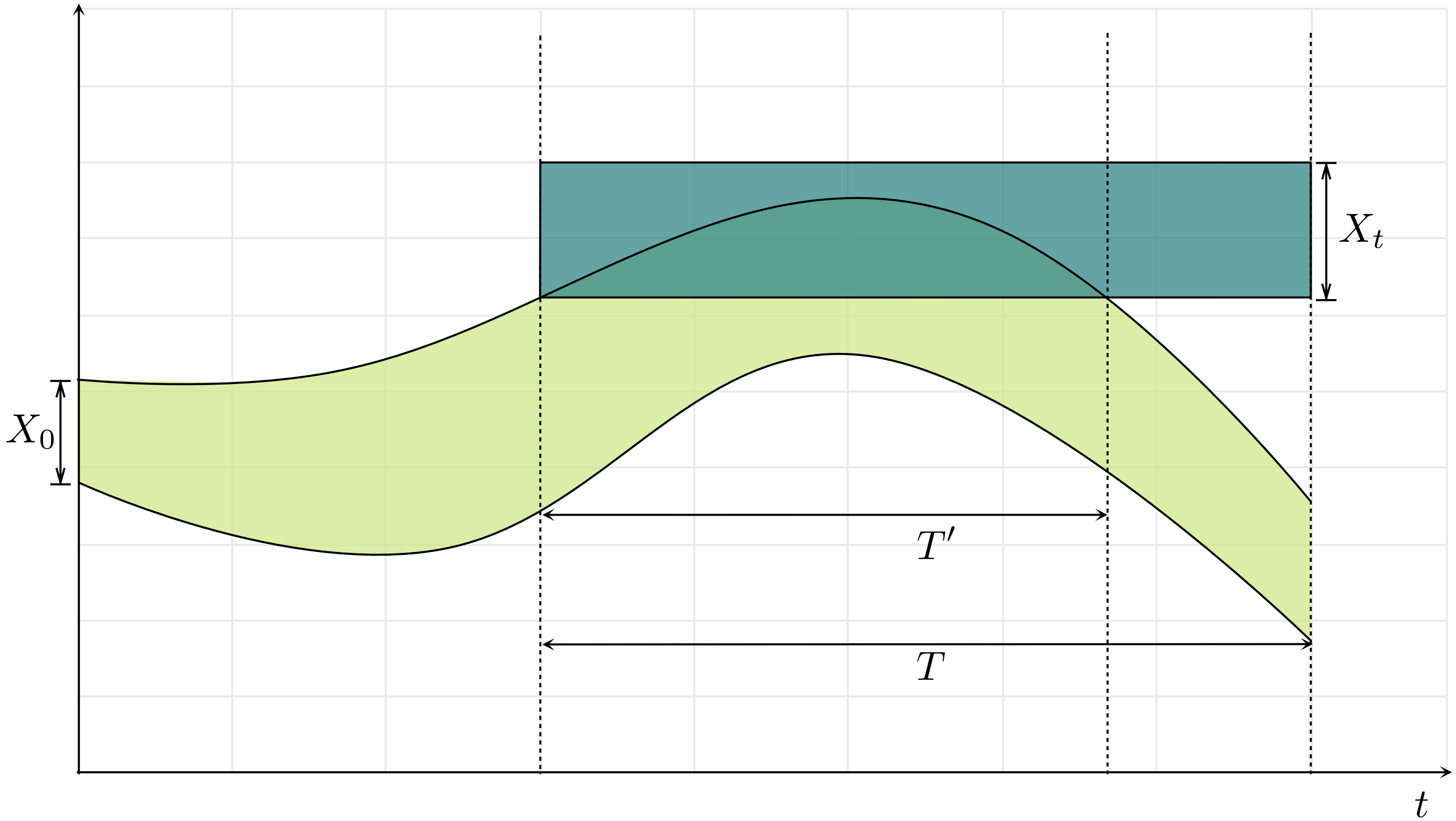
Time Pruning (on T)



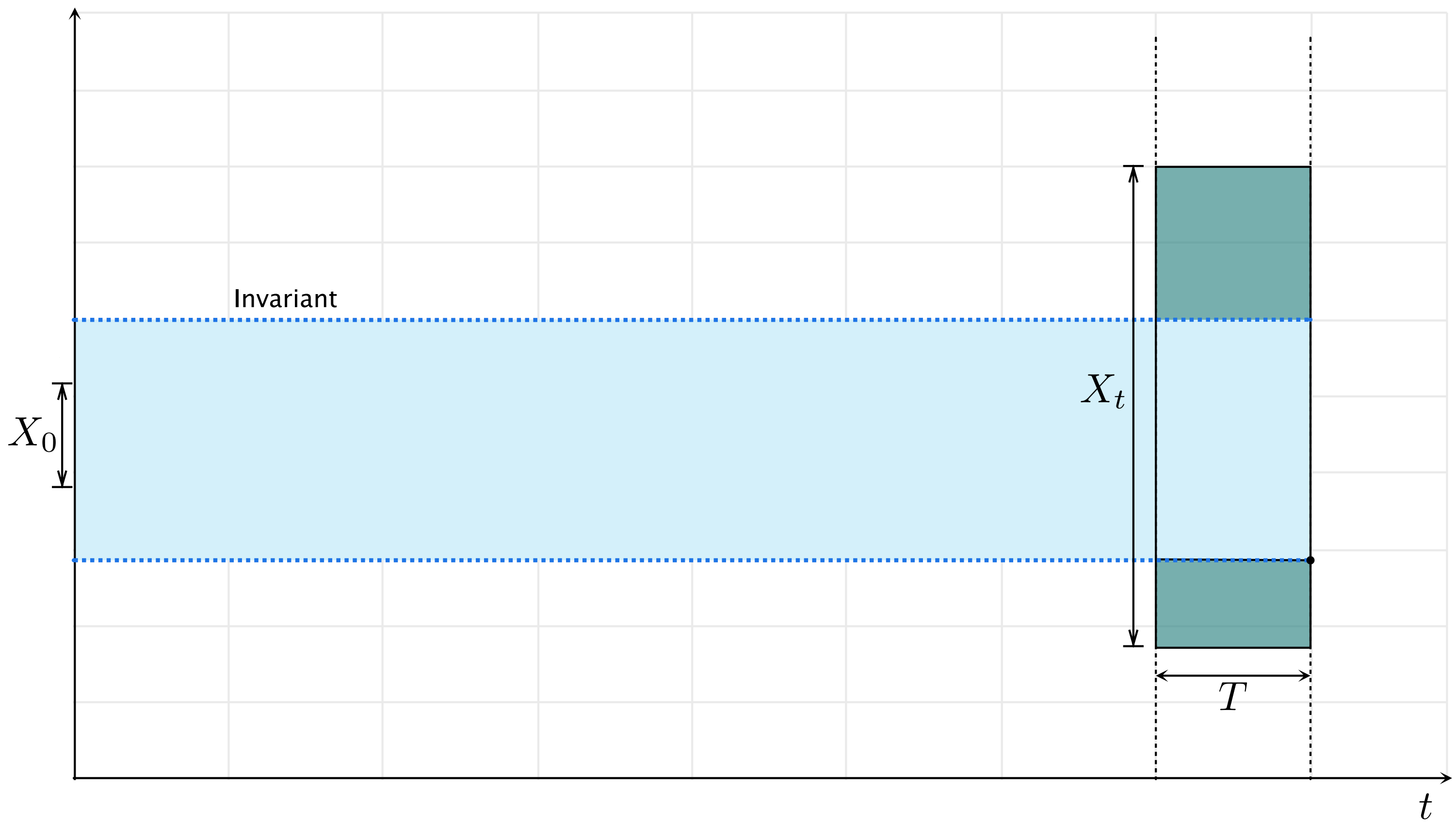
Time Pruning (on $\boldsymbol{T}$)



Time Pruning (on $\boldsymbol{T}$)



Pruning with Invariant



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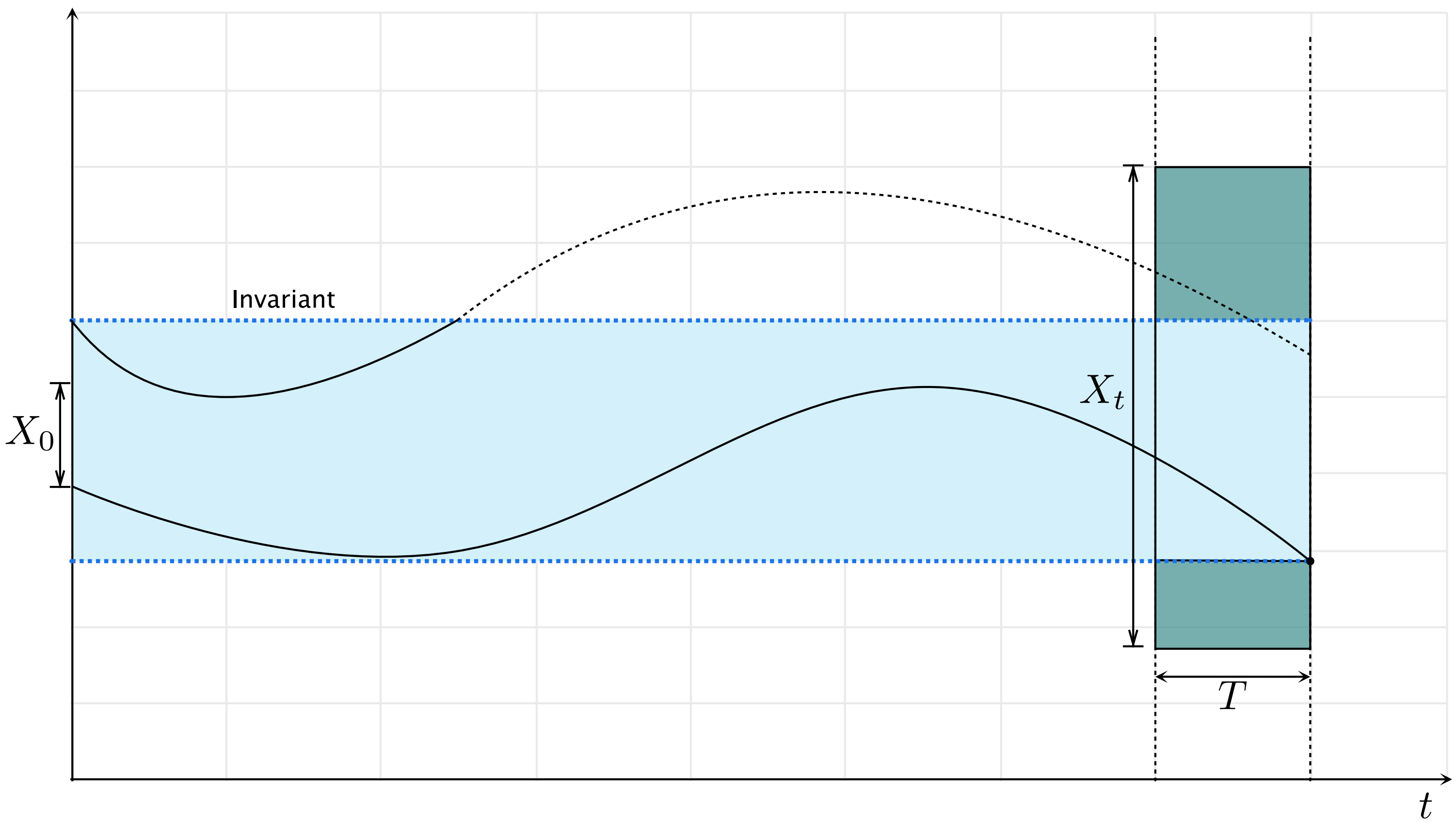
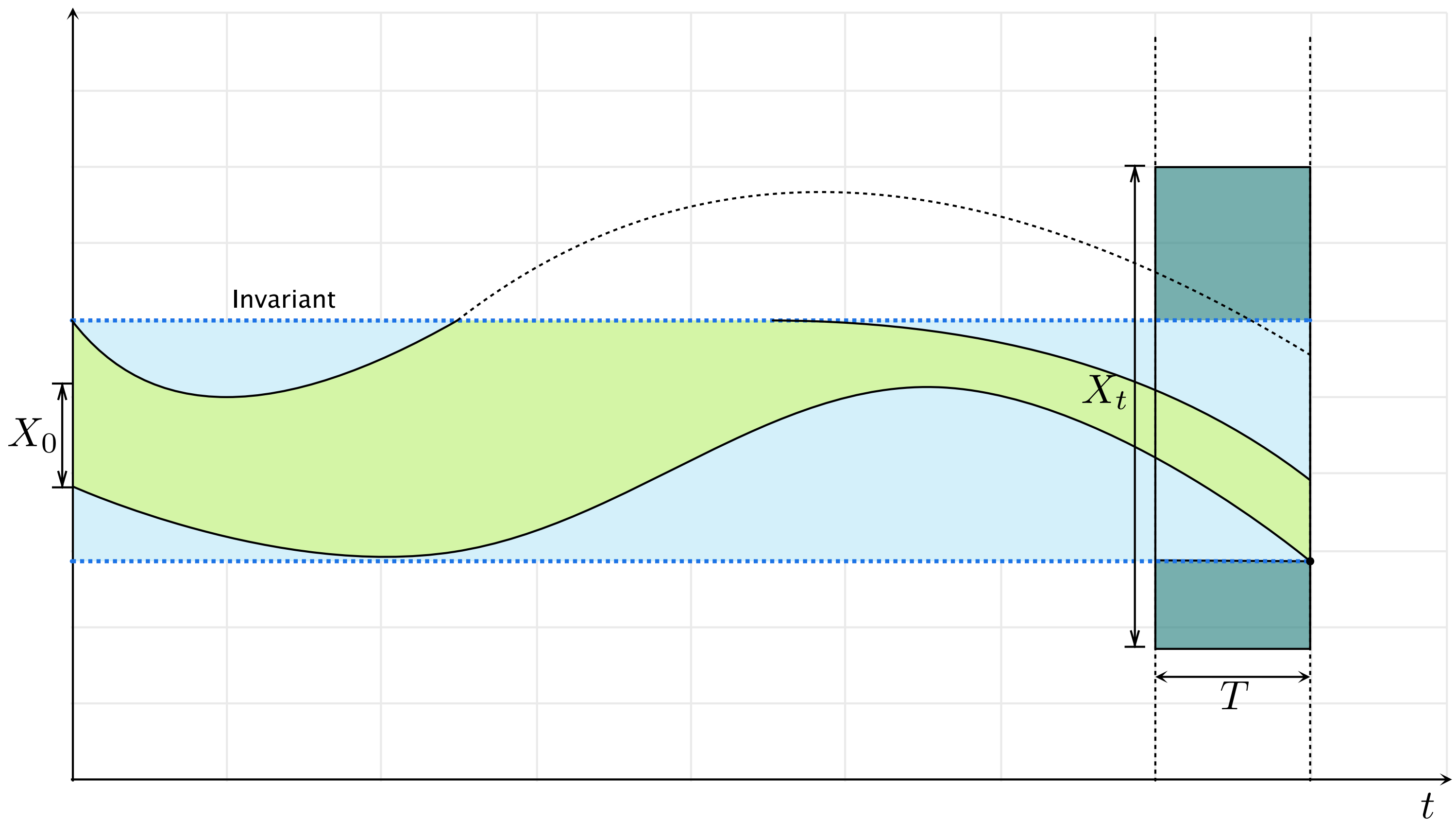
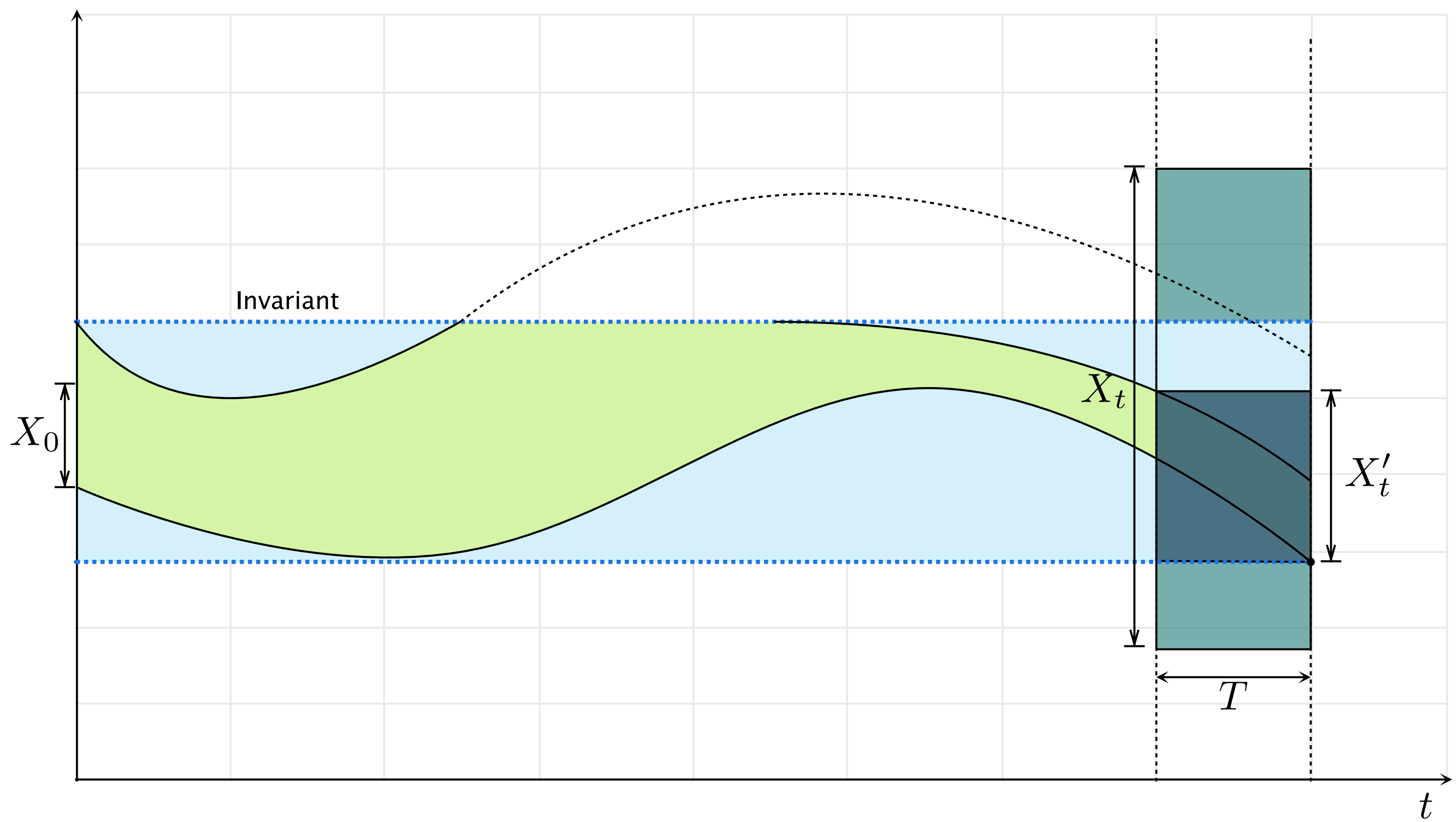


Pruning with Invariant

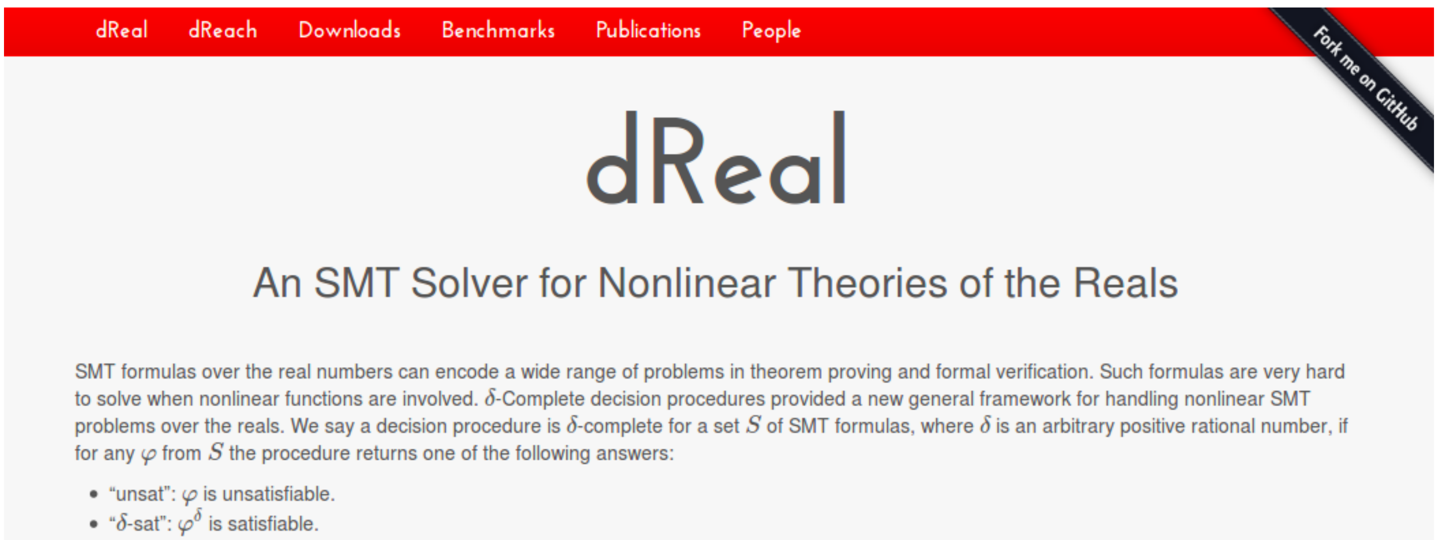


Pruning with Invariant



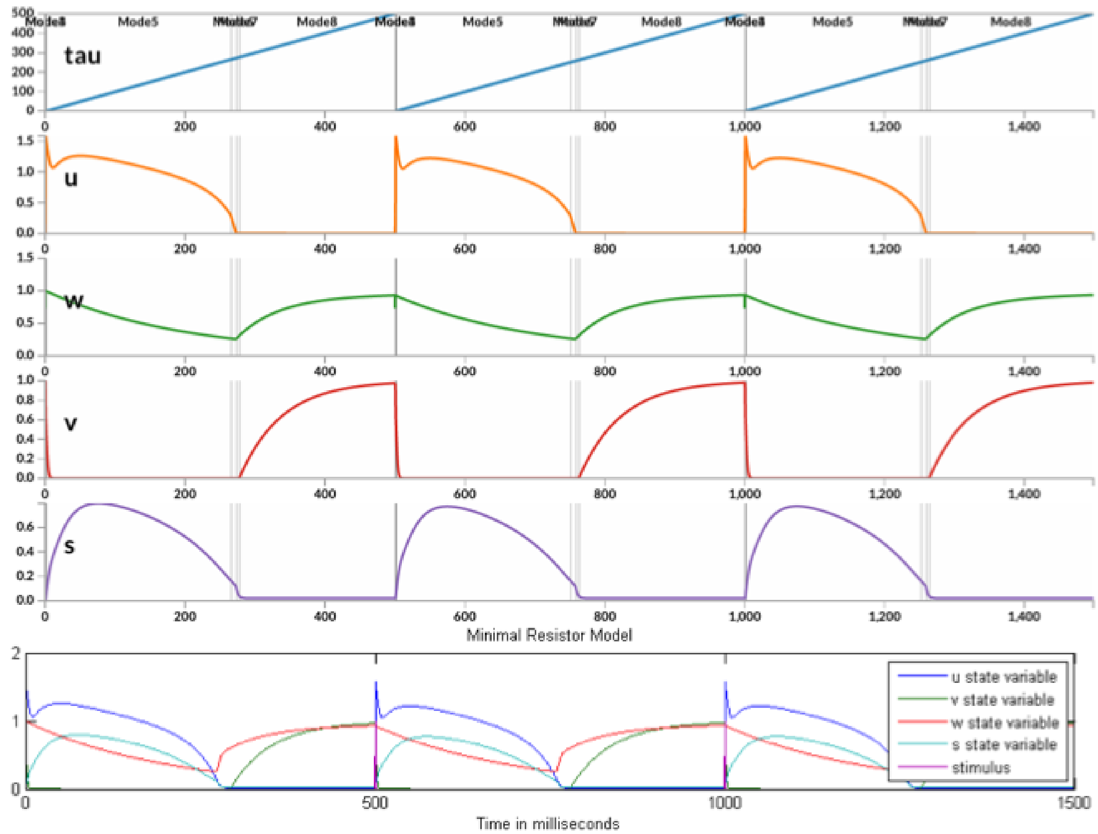
Tool

- Open-source at **http://dreal.cs.cmu.edu**
- Nonlinear ODEs, and of course, polynomials, transcendental functions, etc.
- Formulas with hundreds of nonlinear ODEs have been solved.



Experiments

$$
\begin{aligned}
\frac{du}{dt} &= e + (u - \theta_v)(u_u - u)vg_{fi} \\
&\quad + wsg_{si} - g_{so}(u) \\
\frac{ds}{dt} &= \frac{g_{s2}}{(1 + e^{-2k(u-us)})} - g_{s2}s \\
\frac{dv}{dt} &= -g_v^+ \cdot v \\
\frac{dw}{dt} &= -g_w^+ \cdot w
\end{aligned}
$$

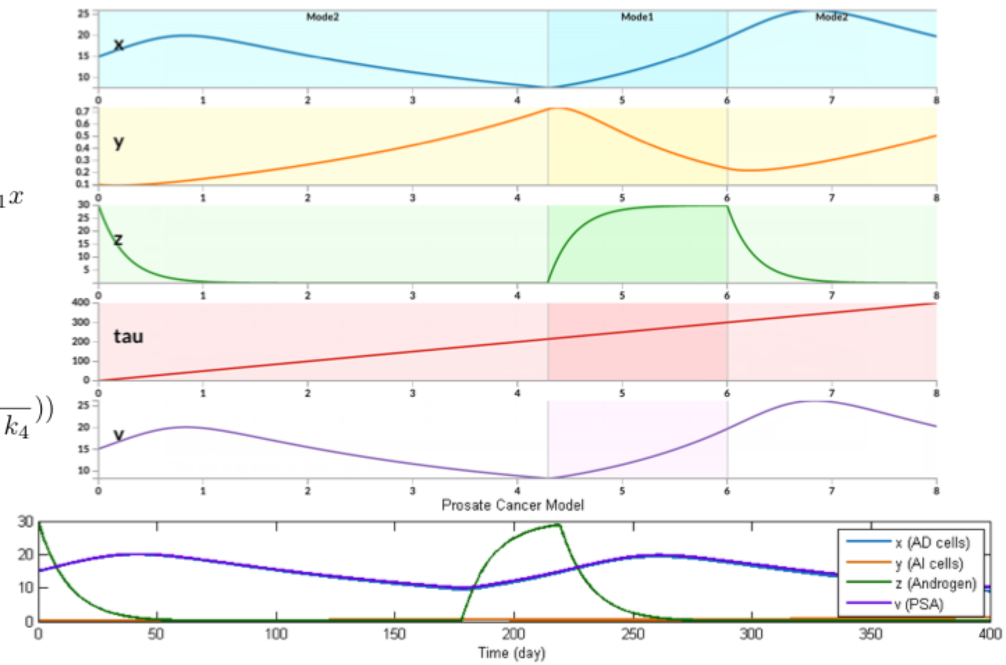


Experiments

```
(set-logic QF_NRA_ODE)
(declare-fun w_0_1_t () Real)
(declare-fun w_0_1_0 () Real)
(declare-fun w_1_2_t () Real)
(declare-fun w_1_2_0 () Real)
(declare-fun w_2_3_t () Real)
(declare-fun w_2_3_0 () Real)
(declare-fun w_3_4_t () Real)
(declare-fun w_3_4_0 () Real)
(declare-fun w_4_5_t () Real)
(declare-fun w_4_5_0 () Real)
(declare-fun w_5_6_t () Real)
(declare-fun w_5_6_0 () Real)
(declare-fun w_6_7_t () Real)
(declare-fun w_6_7_0 () Real)
(declare-fun w_7_8_t () Real)
(declare-fun w_7_8_0 () Real)
(declare-fun w_8_1_t () Real)
(declare-fun w_8_1_0 () Real)
(declare-fun w_9_2_t () Real)
(declare-fun w_9_2_0 () Real)
(declare-fun w_10_3_t () Real)
(declare-fun w_10_3_0 () Real)
```

Experiments

$$
\begin{aligned}
\frac{dx}{dt} &= (\alpha_x(k_1 + (1-k_1)\frac{z}{z+k_2} \\
&\quad -\beta_x((1-k_3)\frac{z}{z+k_4} + k_3)) - m_1(1-\frac{z}{z_0}))x + c_1 x \\
\frac{dy}{dt} &= m_1(1-\frac{z}{z_0})x + (\alpha_y(1-d\frac{z}{z_0}) - \beta_y)y + c_2 y \\
\frac{dz}{dt} &= \frac{-z}{\tau} + c_3 z \\
\frac{dv}{dt} &= (\alpha_x(k_1 + (1-k_1)\frac{z}{z+k_2} - \beta_x(k_3 + (1-k_3)\frac{z}{z+k_4})) \\
&\quad -m_1(1-\frac{z}{z_0}))x + c_1 x + m_1(1-\frac{z}{z_0})x \\
&\quad +(\alpha_y(1-d\frac{z}{z_0}) - \beta_y)y + c_2 y
\end{aligned}
$$



Experiments

P	#M	#D	#O	#V	delta	R	Time(s)	Trace
AF	4	3	20	44	0.001	S	43.10	90K
AF	8	7	40	88	0.001	S	698.86	20M
AF	8	23	120	246	0.001	S	4528.13	59M
AF	8	31	160	352	0.001	S	8485.99	78M
AF	8	47	240	528	0.001	S	15740.41	117M
AF	8	55	280	616	0.001	S	19989.59	137M
CT	2	2	15	36	0.005	S	345.84	3.1M
CT	2	2	15	36	0.002	S	362.84	3.1M
EO	3	2	18	42	0.01	S	52.93	998K
EO	3	2	18	42	0.001	S	57.67	847K
EO	3	11	72	168	0.01	U	7.75	–
BB	2	10	22	66	0.01	S	0.25	123K
BB	2	20	42	126	0.01	S	0.57	171K
BB	2	20	42	126	0.001	S	2.21	168K
BB	2	40	82	246	0.01	U	0.27	—-
BB	2	40	82	246	0.001	U	0.26	—-
D1	3	2	9	24	0.1	S	30.84	72K
DU	3	2	6	16	0.1	U	0.04	–

TABLE I: Experimental results. #M = Number of modes in the hybrid system, #D = Unrolling depth, #O = Number of ODEs in the unrolled formula, #V = Number of variables in the unrolled formula, R = Bounded Model Checking Result (delta-SAT/UNSAT) Time = CPU time (s), Trace = Size of the ODE trajectory, AF = Atrial Filbrillation Model, CT = Cancer Treatment Model, EO = Electronic Oscillator Model, BB = Bouncing Ball with Drag Model, D1,DU = Decay Model.

Conclusion

Conclusion

This is not much harder than SAT solving.