An elementary proof of Hall’s marriage theorem

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1 Introduction

Hall’s marriage theorem [1] is applied in many combinatorial problems. Given
is a bipartite graph \( B \) with non-empty node sets \( X \) and \( Y \). A matching is a set
of edges that have no common incident nodes. A \((X, Y)\) matching is a matching
in which every node of \( X \) is incident on some edge in the matching.

Hall condition (HC): Subset \( S \) of \( X \) meets HC if the number of neighbors of
\( S \) is greater than or equal to the size of \( S \).

Theorem 1 [Hall] There is a \((X, Y)\) matching if and only if every subset of \( X \)
meets HC.

Proof: The proof in one direction, that if there is a \((X, Y)\) matching every
subset of \( X \) meets HC, is straightforward. I prove the converse of the statement
by induction on the size of set \( X \). If \( X \) is empty, there is a trivial matching.
For the general case assume, using induction, that there is a matching over all
nodes of \( X \) except one node \( r \).

Henceforth \( u \overset{n}{\rightarrow} v \) and \( u \overset{m}{\rightarrow} v \) denote, respectively, that \((u, v)\) is a non-matching
edge and \((u, v)\) a matching edge. An alternating path is a simple path of alternating
matching and non-matching edges. Let \( Z \) be the subset of nodes of \( X \)
that are connected to \( r \) by an alternating path.

Every node of \( Z \) except \( r \) is connected to a unique node in \( Z' \) by a matching
edge, from the induction hypothesis; so, there are \(|Z| - 1\) nodes in \( Z' \) that are
so connected. Since \( Z \) meets HC, \(|Z'| \geq |Z|\). Therefore, there is a node \( v \) in \( Z' \)
that is not connected to any node in \( Z \) by a matching edge. I show that \( v \) is
not incident on any matching edge.

Let \( v \) be a neighbor of \( u \) in \( Z \), so \( u \overset{n}{\rightarrow} v \). Since \( u \in Z \), there is an alternating
path between \( r \) and \( u \); extend the path to include edge \( u \overset{n}{\rightarrow} v \), as shown below
in \( P \). I color the matching edges blue and non-matching edges red for emphasis.

\[ P : r = x_0 \overset{n}{\rightarrow} y_0 \overset{m}{\rightarrow} x_1 \cdots x_i \overset{n}{\rightarrow} y_i \overset{m}{\rightarrow} x_{i+1} \cdots x_t \overset{n}{\rightarrow} y_t \overset{m}{\rightarrow} x_{t+1} = u \overset{n}{\rightarrow} v. \]
If $v$ is incident on a matching edge, say $v \leftarrow m \rightarrow w$, then $w \not\in Z$ because $v$ is not connected to any node in $Z$ by a matching edge. However, we can extend $P$ as follows which shows that $r$ is connected to $w$ by an alternating path, so $w \in Z$, a contradiction.

$P' : r = x_0 \leftarrow n \rightarrow y_0 \leftarrow m \rightarrow x_1 \leftarrow n \rightarrow y_1 \leftarrow m \rightarrow x_i \leftarrow n \rightarrow y_i \leftarrow m \rightarrow x_{i+1} = u \leftarrow n \leftarrow v \leftarrow m \rightarrow w.$

So, we conclude that $v$ is not incident on any matching edge.

Flip the edge labels in $P$ from $n$ to $m$ and $m$ to $n$ to obtain a matching that includes all previously matched nodes of $X$ and now includes $r$, so all nodes of $X$ are in the matching.

$r = x_0 \leftarrow m \rightarrow y_0 \leftarrow n \rightarrow x_1 \leftarrow m \rightarrow y_1 \leftarrow n \rightarrow x_i \leftarrow m \rightarrow y_i \leftarrow n \rightarrow x_{i+1} = u \leftarrow m \leftarrow v.$

This completes the inductive proof.

References