An elementary proof of Hall's marriage theorem

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November 4, 2021

1 Introduction

I prove Hall's marriage theorem [1] for a bipartite graph over non-empty node sets X and Y. A *matching* is a set of edges that have no common incident nodes. A (X,Y) matching is a matching in which every node of X is incident on some edge in the matching.

Hall condition (HC): Subset S of X *meets* HC if the number of neighbors of S is greater than or equal to the size of S.

Theorem 1 [Hall] There is a (X,Y) matching if and only if every subset of X meets HC.

Proof: The proof in one direction, that if there is a (X,Y) matching every subset of X meets HC, is straightforward. I prove the converse of the statement by induction on the size of set X. If X is empty, there is a trivial matching. For the general case assume, using induction, that there is a matching over all nodes of X except one node r.

Henceforth $u \overset{n}{—} v$ and $u \overset{m}{—} v$ denote, respectively, that (u,v) is a non-matching edge and (u,v) a matching edge. An *alternating path* is a simple path of alternating matching and non-matching edges. Let Z be the subset of nodes of X that are connected to r by an alternating path.

Every node of Z except r is connected to a unique node in Z' by a matching edge, from the induction hypothesis; so, there are $|Z|-1$ nodes in Z' that are so connected. Since Z meets HC, $|Z'| \geq |Z|$. Therefore, there is a node v in Z' that is not connected to any node in Z by a matching edge; let v be a neighbor of u in Z, so $u \overset{n}{—} v$. Let P be the alternating path connecting r and v,

$$P: \; r = x_0 \overset{n}{—} y_0 \overset{m}{—} x_1 \cdots x_i \overset{n}{—} y_i \overset{m}{—} x_{i+1} \cdots x_t \overset{n}{—} y_t \overset{m}{—} x_{t+1} = u \overset{n}{—} v.$$

If v is incident on a matching edge, say $v \overset{m}{—} w$, then r is connected to w by extending P with the edge $v \overset{m}{—} w$, so $w \in Z$. Then v is connected to w in Z by a matching edge; a contradiction. So, v is not incident on any matching edge.

Flip the edge labels on P from n to m and m to n to obtain a matching in which all nodes of X, including r, are in the matching:

$$r = x_0 \overset{m}{—} y_0 \overset{n}{—} x_1 \cdots x_i \overset{m}{—} y_i \overset{n}{—} x_{i+1} \cdots x_t \overset{m}{—} y_t \overset{n}{—} x_{t+1} = u \overset{m}{—} v.$$

References

[1] Philip Hall. On representatives of subsets. *J. London Math. Soc.*, 10(1):2630, 1935.