Rules for Division by 3 and 11

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1 Division by 3

Show that for any natural number n, $n \bmod 3 = r.n \bmod 3$ where $r.n$ is the sum of the decimal digits in n.

Define a natural number as follows:

nat $\rightarrow$ *digit* | *(nat,digit)*

i.e., a natural number is either a single digit or a pair consisting of a natural number and a digit. Then $r.n$ can be defined:

$r.(d{:}\ digit) = d$
$r.(m, d) = r.m + d$

The proof is by structural induction on n.

- $n{:}\ digit :: n \bmod 3 = r.n \bmod 3$, because $n = r.n$
- $n = (m, d)$::

$$
\begin{array}{ll}
 & n \bmod 3 \\
= & \{n = 10m + d\} \\
 & (10m + d) \bmod 3 \\
= & \{(a + b) \bmod 3 = (a \bmod 3 + b \bmod 3) \bmod 3\} \\
 & (10m \bmod 3 + d \bmod 3) \bmod 3 \\
= & \{10m \bmod 3 = m \bmod 3\} \\
 & (m \bmod 3 + d \bmod 3) \bmod 3 \\
= & \{\text{induction hypothesis: } m \bmod 3 = r.m \bmod 3\} \\
 & (r.m \bmod 3 + d \bmod 3) \bmod 3 \\
= & \{(a + b) \bmod 3 = (a \bmod 3 + b \bmod 3) \bmod 3\} \\
 & (r.m + d) \bmod 3 \\
= & \{r.n = r.m + d\} \\
 & r.n \bmod 3
\end{array}
$$

2 Division by 11

The rule for division by 11 is to start from the lowest digit of the number, and add and subtract the digits alternately. Formally, for any n define $s.n$:

$s.(d{:}\ digit) = d$
$s.(m, d) = -s.m + d$

Show that for any natural number n, $n \bmod 11 = s.n \bmod 11$ where $s.n$.
The proof is by structural induction on n.

- n: *digit* :: $n \bmod 11 = s.n \bmod 11$, because $n = s.n$
- $n = (m, d)$::

$$
\begin{array}{ll}
 & n \bmod 11 \\
= & \{n = 10m + d\} \\
 & (10m + d) \bmod 11 \\
= & \{(a + b) \bmod 11 = (a \bmod 11 + b \bmod 11) \bmod 11\} \\
 & (10m \bmod 11 + d \bmod 11) \bmod 11 \\
= & \{10m \bmod 11 = -m \bmod 11\} \\
 & (-m \bmod 11 + d \bmod 11) \bmod 11 \\
= & \{\text{induction hypothesis: } m \bmod 11 = s.m \bmod 11\} \\
 & (-s.m \bmod 11 + d \bmod 11) \bmod 11 \\
= & \{(a + b) \bmod 11 = (a \bmod 11 + b \bmod 11) \bmod 11\} \\
 & (-s.m + d) \bmod 11 \\
= & \{s.n = -s.m + d\} \\
 & s.n \bmod 11
\end{array}
$$