

## Parities of Binomial Coefficients

Jayadev Misra

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In this note, variables  $k, n$  are integers and  $\binom{n}{k}$  is a binomial coefficient. We write  $\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle$  for the lowest bit, i.e., parity, of  $\binom{n}{k}$ . We derive a result about parity bits.

**Theorem:** For any natural number  $t$ , and integers  $k$  and  $n$ ,

$$\left\langle \begin{smallmatrix} n+2^t \\ k \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \oplus \left\langle \begin{smallmatrix} n \\ k-2^t \end{smallmatrix} \right\rangle$$

Proof: Proof is by induction on  $t$ .

**Case  $t = 0$ :** We have to show

$$\left\langle \begin{smallmatrix} n+1 \\ k \end{smallmatrix} \right\rangle = \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \oplus \left\langle \begin{smallmatrix} n \\ k-1 \end{smallmatrix} \right\rangle$$

We exploit the following identity over binomial coefficients.

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

The lowest bit of the lhs is 0 iff the terms in the rhs have identical lowest bits. Hence the result.

**Case  $t + 1, t \geq 0$ :**

$$\begin{aligned} & \left\langle \begin{smallmatrix} n+2^{t+1} \\ k \end{smallmatrix} \right\rangle \\ = & \text{{rewriting}} \\ & \left\langle \begin{smallmatrix} (n+2^t)+2^t \\ k \end{smallmatrix} \right\rangle \\ = & \text{{induction}} \\ & \left\langle \begin{smallmatrix} n+2^t \\ k \end{smallmatrix} \right\rangle \oplus \left\langle \begin{smallmatrix} n+2^t \\ k-2^t \end{smallmatrix} \right\rangle \\ = & \text{{induction applied to both terms}} \\ & \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \oplus \left\langle \begin{smallmatrix} n \\ k-2^t \end{smallmatrix} \right\rangle \oplus \left\langle \begin{smallmatrix} n \\ k-2^t \end{smallmatrix} \right\rangle \oplus \left\langle \begin{smallmatrix} n \\ k-2^{t+1} \end{smallmatrix} \right\rangle \\ = & \text{{property of } \oplus} \\ & \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle \oplus \left\langle \begin{smallmatrix} n \\ k-2^{t+1} \end{smallmatrix} \right\rangle \end{aligned}$$