A Theorem Relating leads-to and unless

Notes on UNITY: 04-88

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12/20/88

The following theorem has recently been discovered by Ambuj Singh [3]; a very similar result had been observed by Ernie Cohen and J. R. Rao [1] about a year back, in proving a corollary of this theorem directly.

In the following p, q, r denote predicates.

Theorem:

$$p \mapsto q \equiv \langle \exists\, r :: p \Rightarrow r \wedge r \mapsto q \wedge r \textit{ unless } q \rangle$$

Proof: Clearly if the right side holds then $p \mapsto q$, from $p \Rightarrow r$ and $r \mapsto q$. We show that given $p \mapsto q$, there is a predicate r satisfying $p \Rightarrow r$, $r \mapsto q$ and r *unless* q. The proof is constructive: We show how to compute r from a proof of $p \mapsto q$.

A proof of $p \mapsto q$ can be one of three types:

1) p *ensures* q or,
2) $p \mapsto s$, $s \mapsto q$, for some predicte s or,
3) $\langle \forall\, i :: p.i \mapsto q \rangle$, where i ranges over some given set and $p = \langle \vee\, i :: p.i \rangle$

1) p *ensures* q : Let $r = p$.
Then $p \Rightarrow r$, trivially
$r \mapsto q$, since $p \mapsto q$ from p *ensures* q
r *unless* q , from p *ensures* q we have p *unless* q

2) $p \mapsto s$, $s \mapsto q$:

Inductively, we may assume from $p \mapsto s$ that there is a predicate rp satisfying

$$p \Rightarrow rp \text{ , } rp \mapsto s \text{ , } rp \textit{ unless } s$$

Similarly, from $s \mapsto q$, we have a predicate rs satisfying

$$s \Rightarrow rs \text{ , } rs \mapsto q \text{ , } rs \textit{ unless } q$$

Let $r = rp \vee rs$. We show that r has the desired properties.

*This work was partially supported by ONR Contracts N00014-87-K-0510 and N00014-86-0763, by a grant from the John Simon Guggenheim Foundation, and by a University of Texas URI-FRA.

Proof of $p \Rightarrow r$:

$p \Rightarrow rp$	, from the induction hypothesis
$rp \Rightarrow r$	, $r = rp \vee rs$
$p \Rightarrow r$	, from the above two

Proof of $r \mapsto q$:

$rp \mapsto s$	, from the induction hypothesis
$s \mapsto q$	, given
$rp \mapsto q$	, from the transitivity of $\mapsto$ and the above two
$rs \mapsto q$	, from the induction hypothesis
$rp \vee rs \mapsto q$	, disjunction on the above two
$r \mapsto q$	, $r = rp \vee rs$

Proof of r *unless* q :

rp *unless* s	, from the induction hypothesis
rp *unless* rs	, weakening the rhs of the above using $s \Rightarrow rs$
rs *unless* q	, from the induction hypothesis
$rp \vee rs$ *unless* $(\neg rp \wedge q) \vee (rs \wedge q)$	
	, disjunction on the above two
r *unless* q	, rewriting lhs and weakening the rhs of the above

3) $\langle \forall\, i :: p.i \mapsto q \rangle$

Inductively, assume that for every i there is an $r.i$ such that

$$p.i \Rightarrow r.i\ ,\ r.i \mapsto q\ ,\ r.i \textit{ unless } q$$

Let $r = \langle \vee\, i :: r.i \rangle$

Proof of $p \Rightarrow r$

$\langle \forall\, i :: p.i \Rightarrow r.i \rangle$	, from the induction hypothesis
$\langle \vee\, i :: p.i \rangle \Rightarrow \langle \vee\, i :: r.i \rangle$	, predicate calculus
$p \Rightarrow r$	, definitions of p and r

Proof of $r \mapsto q$

$\langle \forall\, i :: r.i \mapsto q \rangle$	, from the induction hypothesis
$\langle \vee\, i :: r.i \rangle \mapsto q$	, disjunction on the above
$r \mapsto q$	, definition of r

Proof of r *unless* q

$\langle \forall\, i :: r.i \textit{ unless } q \rangle$	, from the induction hypothesis
$\langle \vee\, i :: r.i \rangle$ *unless* q	, simple disjunction (see Notes on Unity 01-88)
r *unless* q	, definition of r $\bigtriangledown$

Corollary: (originally proved in [1])

$$\frac{p \mapsto q\ ,\ \neg p \wedge \neg q \text{ is stable}}{p \textit{ unless } q}$$

Proof: From $p \mapsto q$ we have a predicate r satisfying $p \Rightarrow r$, $r \mapsto q$ and r *unless* q.

$$
\begin{array}{ll}
\neg p \wedge \neg q \text{ is stable} & \text{, given} \\
r \mapsto q & \text{, given} \\
\neg p \wedge \neg q \wedge r \mapsto \mathit{false} & \text{, PSP on the above two} \\
p \vee q \vee \neg r \text{ is invariant} & \text{, impossibility rule on the above} \\
p \vee q \vee \neg r \text{ is stable} & \text{, from the above} \\
r \ \mathit{unless}\ q & \text{, given} \\
(p \wedge r) \vee (q \wedge r)\ \mathit{unless}\ q & \text{, simple conjunction of the above two} \\
p \vee (q \wedge r)\ \mathit{unless}\ q & \text{, since } p \Rightarrow r \text{ we have } p \wedge r = p \\
\neg(q \wedge r)\ \mathit{unless}\ q \wedge r & \text{, antireflexivity of } \mathit{unless} \\
p \wedge \neg(q \wedge r)\ \mathit{unless}\ q & \text{, simple conjunction of the above two} \\
p \wedge (q \wedge r)\ \mathit{unless}\ q & \text{, } p \wedge q \wedge r \Rightarrow q \\
p\ \mathit{unless}\ q & \text{, simple disjunction of the above two}
\end{array}
\qquad \nabla
$$

The notion of the "weakest predicate that *leads-to* q," $wlt(q)$, has recently been explored by Jutla, Knapp and Rao [2]. We have the following property of $wlt(q)$.

Corollary: $wlt(q)$ *unless* q

Proof: From $wlt(q) \mapsto q$, we have a r such that

$$wlt(q) \Rightarrow r \ , \ r \mapsto q \ , \ r \ \mathit{unless}\ q$$

Since $wlt(q)$ is the weakest predicate that *leads-to* q the only r satisfying

$wlt(q) \Rightarrow r$ and $r \mapsto q$ is $wlt(q)$. Hence
$wlt(q)$ *unless* q , from r *unless* q ∇

References

1. Ernie Cohen and J. R. Rao, personal communication, 1987.

2. Charanjit Jutla, J. R. Rao, Edgar Knapp, "Extensional Semantics for Parallel Programs," *Logic in Computer Science* (submitted).

3. Ambuj Singh, "Some Theorems About *until*," November 13, 1988.