Your Name: ______________________________ Your EID: _______________

Circle Your Discussion Section:
54075: Ian Wehrman, Friday, 9:00 – 10:00a, ENS 109
54080: Ian Wehrman, Friday, 10:00 – 11:00a, GSB 2.122
54085: David Rager, Friday, 10:00 – 11:00a, JES A218A
54088: Behnam Robatmili, Friday, 12:00 – 1:00p, RLM 5.122
54090: Behnam Robatmili, Friday, 1:00 – 2:00p, CBA 4.344
54095: Nathan Wetzler, Friday, 1:00 – 2:00p, JES A209A

Final Exam
CS313K Logic, Sets, and Functions – Spring, 2009

Instructions

Write your name and EID above and circle the unique ID of your discussion section! Write your answers in the space provided. If your proofs fill more than the space provided, you may write on the back of the page but please put "PTO" ("please turn over") at the bottom and put the Question number at the top of each back page you use. If you use extra paper, be sure to put your name and EID and the Question number on each page!

This is a three hour final, Friday, May 15, 2:00–5:00 pm. There are 15 questions worth a total of 300 points. Some questions are worth more than others. Partial credit will be given for proof attempts. Some questions subtract points for incorrect answers; such questions are noted when asked.

N is the set of natural numbers. $\emptyset$ is the empty set; 0 is the natural number zero.

You may refer to the course notes (the red book) during the exam. In your proofs you may use use any theorem in the course notes. Beware: Some Questions in the course notes, e.g., "Question 184 (`end-means-nil`): `(endp x)` $\rightarrow$ `x` = `nil`" are not theorems! So if you cite a Question from the book in one of your proofs, *make sure the formula in the Question is a theorem!*

You may refer to your own notes if they are on paper. No computers are allowed. No talking is allowed. No cellphones. Remove sunglasses, hats, baseball caps, etc.

Question 1 (20 points): Below is a theorem and proof, followed by some numbered justification lines. Fill in the justifications in the proof with the appropriate line numbers. Some justifications may be used more than once; some will not be used at all.

Theorem:
$(A \rightarrow (B \lor C)) \leftrightarrow ((A \land \neg B) \rightarrow C)$
Proof:
$(A \rightarrow (B \lor C)) \leftrightarrow ((A \land \neg B) \rightarrow C)$
$\leftrightarrow \{ \qquad \qquad \}$ ⟵ fill in this blank
$(\neg A \lor (B \lor C)) \leftrightarrow ((A \land \neg B) \rightarrow C)$
$\leftrightarrow \{ \qquad \qquad \}$ ⟵ fill in this blank
$((\neg A \lor B) \lor C) \leftrightarrow ((A \land \neg B) \rightarrow C)$
$\leftrightarrow \{ \qquad \qquad \}$ ⟵ fill in this blank
$((\neg A \lor \neg \neg B) \lor C) \leftrightarrow ((A \land \neg B) \rightarrow C)$
$\leftrightarrow \{ \qquad \qquad \}$ ⟵ fill in this blank
$(\neg(A \land \neg B)) \lor C) \leftrightarrow ((A \land \neg B) \rightarrow C)$
$\leftrightarrow \{ \qquad \qquad \}$ ⟵ fill in this blank
$(A \land \neg B) \rightarrow C) \leftrightarrow ((A \land \neg B) \rightarrow C)$
$\leftrightarrow \{ \qquad \qquad \}$ ⟵ fill in this blank
T

Possible Justifications

1. Associativity
2. Contraposition
3. De Morgan
4. Double Negation
5. Implicative Disjunction
6. Taut $p \leftrightarrow p$
7. Transitivity

Question 2 (20 points): Prove

```
((A → (B ∨ C))
 ∧
 (D → A))
→
(B ∨ (D → C))
```

Hint: You may use the theorem proved in Question 1.

Question 3 (30 points): Suppose `f` is a function that produces an answer satisfying property `p`. Suppose `mapf` takes a list and copies it, applying `f` to each element. Suppose `chkp` takes a list and checks that every element has property `p`. Then obviously, `chkp` succeeds on the output of `mapf`. Prove it. Formally, you are given:

Axiom `p-f`

```
(p (f x))
```

```
(defun mapf (x)
  (if (endp x)
      nil
      (cons (f (car x))
            (mapf (cdr x)))))
```

```
(defun chkp (x)
  (if (endp x)
      t
      (and (p (car x))
           (chkp (cdr x)))))
```

and must prove `(chkp (mapf x))`.

Question 4 (10 points): Let `(neighborp x y)` mean "`x` has `y` as a neighbor," `(safep x)` mean "`x` is safe," and `(partner x)` mean "`x`'s partner." Then formalize this remark:

> If every neighbor of `A` is safe, then `A`'s partner is safe.

You may use ACL2 notation or standard infix notation. You need not try to prove the formula you write.

Question 5 (20 points): Given the terminology in Question 4 and assuming that if every neighbor of `A` is safe, then `A`'s partner is safe, can you conclude that `A`'s partner is a neighbor of `A`? If so, prove it. If not, give definitions of `neighborp`, `safep`, and `partner`, and a value for `A` that falsify the claim.

Question 6 (10 points): Given the terminology of Question 4, formalize the remark that `A` has *exactly* two neighbors. No proof is required.

Question 7 (10 points): Given the terminology of Question 4, formalize the remark that there are at least two neighbors of `A` that have the same partner. No proof is required.

Question 8 (15 points): Suppose `A` is a true list of length 16. Let's denote the length of `A` by $|\texttt{A}|$ and the $\texttt{i}^{th}$ element of `A` by `A[i]`. Suppose `A` has the following property.

$(\forall i : i \in \mathbf{N} \wedge i < |\texttt{A}|) \rightarrow \texttt{A[i]} = \texttt{i}^2$

1. What do you know about `A[12]`?

2. What do you know about `A[16]`?

3. Is this always true: $(\exists \texttt{j} : \texttt{A[j]=j})$?

Question 9 (5 points, but see below): One of the formulas below is equivalent to the *negation* of

$\texttt{(neighborp A B)} \rightarrow (\exists x : (\forall y : \texttt{(neighborp B y)} \rightarrow \texttt{(neighborp A x)}))$.

Circle the number of the appropriate formula. The correct answer is worth 5 points. But an incorrect answer will cost you 5 points! Leaving this question blank (not circling any of the numbers below) will neither add nor subtract points.

1. $\texttt{(neighborp A B)} \rightarrow \neg(\exists x : (\forall y : \texttt{(neighborp B y)} \rightarrow \texttt{(neighborp A x)}))$

2. $\neg\texttt{(neighborp A B)} \vee \neg(\exists x : (\forall y : \texttt{(neighborp B y)} \rightarrow \texttt{(neighborp A x)}))$

3. $\texttt{(neighborp A B)} \wedge (\forall x : (\exists y : \texttt{(neighborp B y)} \wedge \neg\texttt{(neighborp A x)}))$

4. $\texttt{(neighborp A B)} \wedge (\forall x : (\exists y : \texttt{(neighborp B y)} \rightarrow \neg\texttt{(neighborp A x)}))$

5. $\texttt{(neighborp A B)} \vee (\exists x : (\forall y : \texttt{(neighborp B y)} \rightarrow \neg\texttt{(neighborp A x)}))$

Question 10 (30 points): Write down in roster notation each of the following finite sets:

1. $(\{1, 2, 3\} \cap \{0, 2, 4, 6\}) \cup \{1, 3, 5\}$

2. $(\{\{\}\} \cap \{\emptyset\}) \cup (\{1, 3, 5\} \cap \{0, 2, 4\})$

3. $\{x : x \in \mathbf{N} \wedge x^2 \leq 16\}$

4. $\{x^2 : (\exists y : y \in \mathbf{N} \wedge (y < 4) \wedge 2y = x)\}$

5. $\{\texttt{A}, \texttt{B}, \texttt{C}\} \times \{\texttt{t}, \texttt{nil}\}$

6. $\wp(\{\texttt{A}\}) \times \wp(\{\texttt{B}\})$

Question 11 (30 points): Prove $(A \cup (B \cap C)) = ((A \cup B) \cap (A \cup C))$.

Question 12 (30 points): Prove $(A \setminus B) \subseteq A$.

Question 13 (30 points, but see below):

Let $R = \{< \mathtt{x}, \mathtt{y} >: \mathtt{(evenp\ x)} \rightarrow \mathtt{(evenp\ y)}\}$ where `(evenp x)` means "`x` is an even natural number."

Check the properties R has (on the natural numbers). A correct answer (whether "yes" or "no") is worth 2 points, but an incorrect answer will cost 2 points! A missing answer will not add or subtract points. Be careful! I recommend that you write down the required formulas on scratch paper and prove or disprove them; but you need not show your work, just check the correct answers.

		yes	*no*
1.	$R \subseteq \mathbf{N} \times \{\mathtt{t}, \mathtt{nil}\}$		
2.	$< 1, 2 > \in R$		
3.	R is a relation		
4.	R is a function		
5.	R is reflexive		
6.	R is irreflexive		
7.	R is symmetric		
8.	R is asymmetric		
9.	R is antisymmetric		
10.	R is transitive		
11.	R is total		
12.	R is connected		
13.	R is an equivalence relation		
14.	R is a partial order		
15.	R is a total order		

Question 14 (20 points, but see below):

Let $A_1 = \{\texttt{t}, \texttt{yes}, \texttt{1}\}$, $A_2 = \{\texttt{nil}, \texttt{no}, \texttt{-1}\}$, and $A_3 = \{\texttt{undef}, \texttt{0}\}$.

Let $A = A_1 \cup A_2 \cup A_3$.

Let $R = (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_3 \times A_3)$. Note that R is an equivalence relation on A. Define `can` (for "canonical") as follows:

```
(defun can (x) (cond ((mem x '(t yes 1)) 1)
                     ((mem x '(nil no -1)) -1)
                     (t 0)))
```

Each correct answer is worth 2 points; each incorrect answer costs 2 points. Omitting an answer neither adds nor subtracts points.

1. How many elements are in A?

2. How many elements are in $A \times A$?

3. Show an element in $A \times A$ that is not in R.

4. How many elements are in R?

5. Show the partitions induced on A by R.

6. Is $\texttt{undef} \in [1]_R$?

7. Is $\texttt{nil} \in [-1]_R$?

8. What is `(can 23)`?

9. Show an element of $\{u : \texttt{(consp u)} \wedge \texttt{(can (car u))} = \texttt{(can (cdr u))}\}$ that is not an element of R.

10. Is it a theorem that: $R = (\{< x, y >: \texttt{(can x)} = \texttt{(can y)}\} \cap (A \times A))$?

Question 15 (20 points, but see below):

Let $f = \{< \mathtt{x}, \mathtt{y} >: (\mathtt{x} \in \mathbf{N}) \wedge (\mathtt{y} = 2\mathtt{x})\}$.

Each correct answer is worth 2 points; each incorrect answer costs 2 points. Omitting an answer neither adds nor subtracts points.

1. What is $dom(f)$?

2. What is $f(123)$?

3. What is $ran(f)$? Write the answer in set notation without using "f".

4. Is it a theorem that $(f : \mathbf{N} \rightarrow \mathbf{N})$?

5. Is f 1:1?

6. Is f onto $\mathbf{N}$?

7. What is $(f \circ f)(3)$?

8. Define a function g such that $(f \circ g) \neq (g \circ f)$.

9. Prove, for your g above, $(\exists x : (f \circ g)(x) \neq (g \circ f)(x))$

10. If f^{-1} is a function, what is $f^{-1}(32)$? If f^{-1} is not a function, explain why not.