Problem Set #1

This problem set is due at the start of class on Tuesday, February 21st.

1. Consider an n-person game in which each player has only two actions. This game has 2^n possible outcomes, one for each of the 2^n possible pure strategy profiles. Therefore the game in matrix form is exponentially large. Let T be a tree (i.e., an acyclic, connected, undirected graph) with maximum degree 3 and with n vertices, one corresponding to each player. Assume that the payoff to player i only depends on the strategies of player i and the (at most 3) neighbors of player i in T. Give an algorithm with running time that is polynomial in n to decide whether such a game has a pure Nash equilibrium.

2. Let A be a set of three or more candidates. Assume that n voters, numbered from 1 to n, each submit a ballot that ranks these candidates from best to worst. Each ballot also indicates the number of the corresponding voter. Let C be a social choice function that takes the preference profile specified by the n ballots and determines the winning candidate. Assume that C satisfies the properties MON and PE′ defined in the lecture. In the proof of the Muller-Satterthwaite theorem that was presented in class, we showed how to construct from C a social welfare function W satisfying the properties IIA and PE; this allowed us to apply Arrow's impossibility theorem. Let W' be the social welfare function that is derived from C in the following different manner. For a preference profile $I = I_0$, we define the highest candidate in $W'(I)$ as $C(I_0)$. We then obtain a preference profile I_1 from I_0 by moving $C(I_0)$ to the bottom of every ballot, and we define the second highest candidate in $W'(I)$ as $C(I_1)$. We then obtain a preference profile I_2 from I_1 by by moving $C(I_1)$ to the bottom of every ballot, and we define the third highest candidate in $W'(I)$ as $C(I_2)$, and so on, until all of the candidates have been ranked in $W'(I)$.

   (a) Prove that W' is guaranteed to be a valid social welfare function.

   (b) Prove or disprove: $W = W'$.

3. This question is concerned with rules for voting with single-peaked preferences. Let n denote the number of voters. Fix a multiset $Y = \{y_1, \ldots, y_{n-1}\}$ of $n - 1$ real numbers in $[0, 1]$. Let R denote a rule that produces as output the median of the multiset of $2n - 1$ numbers consisting of the n peaks specified on the ballots and the elements of Y.

   (a) Briefly explain why R is anonymous.

   (b) Prove that R is onto.

   (c) Prove that R is strategyproof.

4. Let I be an instance of the stable marriage problem in which each man x specifies a strict preference order over some subset of the women (x prefers to remain single than to marry any woman not in this subset), and each woman y specifies a strict preference order over some subset of the men. The number of men need not be equal to the number of women. Let M and M' be stable matchings for instance I.

   (a) Prove that if a man x is matched in M, then x is matched in M'. (By a symmetric argument, the same claim holds for the women.)

   (b) Let X denote the set of all men matched by M, and let Y denote the set of all women matched by M. By part (a), the set of men matched by M' is equal to X, and the set of women matched by M' is equal to Y. For any man x who is matched in M and M', let $f(x)$ denote x's preferred mate under either M or M', and let $g(x)$ denote x's least preferred mate under either M or M'. (If x has the same mate y in M and M', then $f(x) = g(x) = y$.) Let M_0 denote the set of all man-woman pairs (x, y) such that $f(x) = y$, and let M_1 denote the set of all man-woman pairs (x, y) such that $g(x) = y$. Prove that M_0 and M_1 are each perfect matchings of the set of men X with the set of women Y.

   (c) In part (b) we have chosen to define the matchings M_0 and M_1 in terms of the preferences of the men. Give an equivalent definition of the matchings M_0 and M_1 in terms of the preferences of the women. You are not required to prove equivalence, since the proof details are similar to those associated with part (b).

   (d) Let M_0 and M_1 be the matchings defined in part (b). Prove that M_0 is stable. (A symmetric argument can be used to show that M_1 is stable.)