Problem Set #3

This problem set is due at the start of class on Thursday, March 29th.

Fix an instance of the assignment game with m buyers and n sellers, and where all of the $\alpha_{i,j}$ values are integers. As a technical convenience, we assume that the set of buyers includes n buyers i such that $\alpha_{i,j} = 0$ for each item j. (These buyers may be viewed as dummy buyers; we will use them to ensure that there is a stable assignment in which every item is assigned to a buyer.)

Let p^* denote the minimum (i.e., buyer-optimal) stable price vector for this instance. In the lecture we saw how to use the (incremental) Hungarian algorithm to compute p^*. In this problem set, we analyze another method for computing p^*. For any price vector p, and any buyer i, we define $\mathit{gap}(p, i)$ as the maximum, over all items j, of $\alpha_{i,j} - p_j$. Given any price vector p, let $\mathit{yes}(p)$ denote the set of all buyers i such that $\mathit{gap}(p, i) > 0$, let $\mathit{maybe}(p)$ denote the set of all buyers i such that $\mathit{gap}(p, i) = 0$, and let $\mathit{no}(p)$ denote the set of all buyers i such that $\mathit{gap}(p, i) < 0$.

For any buyer i, we define $\mathit{demand}(p, i)$ as the set of all items j such that $\alpha_{i,j} - p_j = \max\{0, \mathit{gap}(p, i)\}$. For any set of items S, we define $\mathit{confined}(p, S)$ as the set of all buyers i in $\mathit{yes}(p)$ such that $\mathit{demand}(p, i)$ is contained in S. [NOTE ADDED 3/16/12: In the original version of the problem set, I had erroneously written "belongs to S" at the end of the previous sentence, rather than "is contained in S".] We define $\mathit{overdemanded}(p)$ as the collection of all sets of items S such that $|\mathit{confined}(p, S)| > |S|$. We define a subset of $\mathit{overdemanded}(p)$, denoted $\mathit{minimal}(p)$, as follows: A set S in $\mathit{overdemanded}(p)$ belongs to $\mathit{minimal}(p)$ if no proper subset of S belongs to $\mathit{overdemanded}(p)$.

For any set of items S, we define $\mathit{interested}(p, S)$ as the set of all buyers i such that $\mathit{demand}(p, i) \cap S$ is nonempty.

1. Let p be a price vector such that $p \le p^*$ (i.e, for any item j, $p_j \le p^*_j$), let S be a set of items in $\mathit{minimal}(p)$, and let p' denote the price vector that is obtained from p by incrementing the prices of all items in S (i.e., for each item j in S, $p'_j = p_j + 1$, and for each item j that does not belong to S, $p'_j = p_j$). Prove that $p' \le p^*$.

2. Consider the following nondeterministic algorithm $\mathcal{A}$ for computing a price vector p. Start by initializing p to the all-zeros vector. Then, while $\mathit{overdemanded}(p)$ is nonempty, nondeterministically choose a set S from $\mathit{minimal}(p)$ and update p by incrementing each p_j such that j belongs to S. It is easy to argue that this algorithm terminates. In the following parts, let p denote the final price vector produced by some execution of algorithm $\mathcal{A}$.

   (a) Use the result of question 1 to argue that p is at most p^*.

   (b) Prove that for any set of items S, we have $|\mathit{interested}(p, S)| \ge |S|$. Hint: Use induction on the number of iterations performed by $\mathcal{A}$, and bear in mind the existence of the "dummy" buyers.

(c) Prove that there is an assignment x' such that every buyer i in $yes(p)$ is assigned to an item in $demand(p, i)$. Hint: It is known (Hall, 1935) that if $G = (U, V, E)$ is a bipartite graph such that every subset U' of U has a neighborhood of size at least $|U'|$ in V, then G admits a matching M such that every vertex in U is matched in M. (The "neighborhood" of a subset U' of U is the set of all vertices in V that are adjacent to at least one vertex in U'.)

(d) Prove that there is an assignment x'' such that every item j is assigned to some buyer i such that j belongs to $demand(p, i)$. Hint: Make use of the result of part (b) and the hint of part (c).

(e) Prove that there is an assignment x such that every buyer i in $yes(p)$ is assigned to an item in $demand(p, i)$, and every item j is assigned to some buyer i such that j belongs to $demand(p, i)$. Hint: It is known (Mendelson and Dulmage, 1958) that if bipartite graph $G = (U, V, E)$ admits a matching M' such that every vertex in a subset U' of U is matched in M', and a second matching M'' such that every vertex in a subset V' of V is matched in M'', then G admits a matching M such that every vertex in $U' \cup V'$ is matched in M.

(f) Let x be an assignment satisfying the conditions of the previous part. Prove that there is a stable outcome u, v, x such that $v = p$.

(g) Prove that $p = p^*$.