Strategyproof Pareto-Stable Mechanisms for Two-Sided Matching with Indifferences*

Nevzat Onur Domaniç Chi-Kit Lam C. Gregory Plaxton

March 2017

Abstract

We study variants of the stable marriage and college admissions models in which the agents are allowed to express weak preferences over the set of agents on the other side of the market and the option of remaining unmatched. For the problems that we address, previous authors have presented polynomial-time algorithms for computing a "Pareto-stable" matching. In the case of college admissions, these algorithms require the preferences of the colleges over groups of students to satisfy a technical condition related to responsiveness. We design new polynomial-time Pareto-stable algorithms for stable marriage and college admissions that correspond to strategyproof mechanisms. For stable marriage, it is known that no Pareto-stable mechanism is strategyproof for all of the agents; our algorithm provides a mechanism that is strategyproof for the agents on one side of the market. For college admissions, it is known that no Pareto-stable mechanism can be strategyproof for the colleges; our algorithm provides a mechanism that is strategyproof for the students.

*Department of Computer Science, University of Texas at Austin, 2317 Speedway, Stop D9500, Austin, Texas 78712–1757. Email: {onur, geocklam, plaxton}@cs.utexas.edu. This research was supported by NSF Grant CCF–1217980. An earlier version of this work appears as a UTCS technical report [3].

1 Introduction

Gale and Shapley [7] introduced the stable marriage model and its generalization to the college admissions model. Their work spawned a vast literature on two-sided matching; see Manlove [11] for a recent survey. The present paper is primarily concerned with variants of the stable marriage and college admissions models where the agents have weak preferences, i.e., where indifferences are allowed.

In the most basic stable marriage model, we are given an equal number of men and women, where each man (resp., woman) has complete, strict preferences over the set of women (resp., men); we refer to this model as SMCS. For SMCS, an outcome is a matching that pairs up all of the men and women into disjoint man-woman pairs. A man-woman pair (p, q) is said to form a *blocking pair* for a matching M if p prefers q to his partner in M and q prefers p to her partner in M. A matching is *stable* if it does not have a blocking pair. It is straightforward to prove that any stable matching is also Pareto-optimal. Gale and Shapley presented the deferred acceptance (DA) algorithm for the SMCS problem and proved that the man-proposing version of the DA algorithm produces the unique man-optimal (and woman-pessimal) stable matching. Roth [12] showed that the associated mechanism, which we refer to as the *man-proposing DA mechanism*, is *strategyproof* for the men, i.e., it is a weakly dominant strategy for each man to declare his true preferences. Unfortunately, the man-proposing DA mechanism is not strategyproof for the women. In fact, Roth [12] showed that no stable mechanism for SMCS is strategyproof for all of the agents.

The SMCW model is the generalization of the SMCS model in which each man (resp., woman) has weak preferences over the set of women (resp., men). When indifferences are allowed, we need to refine our notion of a blocking pair. A man-woman pair (p, q) is said to form a *strongly blocking pair* for a matching M if p prefers q to his partner in M and q prefers p to her partner in M. A matching is *weakly stable* if it is individually rational and does not have a strongly blocking pair. Two other natural notions of stability, namely strong stability and super-stability, have been investigated in the literature (see Manlove [11, Chapter 3] for a survey of these results). We focus on weak stability because every SMCW instance admits a weakly stable matching (this follows from the existence of stable matchings for SMCS, coupled with arbitrary tie-breaking), but not every SMCW instance admits a strongly stable or super-stable matching. It is straightforward to exhibit SMCW instances (with as few as two men and two women) for which some weakly stable matching is not Pareto-optimal. Sotomayor [16] argues that *Pareto-stability* (i.e., Pareto-optimality plus weak stability) is an appropriate solution concept for SMCW and certain other matching models with weak preferences, and proves that every SMCW instance admits a Pareto-stable matching.

Erdil and Ergin [4] and Chen and Ghosh [2] present polynomial-time algorithms for computing a Pareto-stable matching of a given SMCW instance; in fact, these algorithms are applicable to certain more general models to be discussed shortly. Given the existence of a stable mechanism for SMCS that is strategyproof for the men (or, symmetrically, for the women), it is natural to ask whether there is a Pareto-stable mechanism for SMCW that is strategyproof for the men. We cannot hope to find a Pareto-stable mechanism for SMCW that is strategyproof for all agents, since that would imply a stable mechanism for SMCS that is strategyproof for all agents. A similar statement holds for the SMIW model, the generalization of the SMCW model in which the agents are allowed to express incomplete preferences. See Section 4 for a formal definition of the SMIW model and the associated notions of weak stability and Pareto-stability. Throughout the remainder of the paper, when we say that a mechanism for a stable marriage model is strategyproof, we mean that it is strategyproof for the agents on one side of the market; moreover, unless otherwise specified, it is to be understood that the mechanism is strategyproof for the men. The Pareto-stable algorithms of Erdil and Ergin, and of Chen and Ghosh, are based on a two-phase approach where the first phase runs the Gale-Shapley DA algorithm after breaking all ties arbitrarily. In Appendix A we show that this approach does not provide a strategyproof mechanism.

This paper provides the first Pareto-stable mechanism for SMIW (and also SMCW) that is shown to be strategyproof. We present a nondeterministic algorithm for SMIW that generalizes Gale and Shapley's DA algorithm as follows: in each iteration, a nondeterministically chosen unmatched man "proposes" simultaneously to all of the women in his next-highest tier of preference (i.e., the highest tier to which he has not already proposed); the women respond to this proposal by solving a certain maximum-weight matching problem to determine which man becomes unmatched (i.e., the man making the proposal or one of the tentatively matched men). Our generalization of the DA mechanism admits a polynomial-time implementation.

The college admissions model with weak preferences, which we denote CAW, is a further generalization of the SMIW model. In the CAW model, students and colleges are being matched rather than men and women, and each college has a positive integer capacity representing the number of students that it can accommodate. See Section 5 for a formal definition of the CAW model and the associated notions of weak stability and Pareto-stability.

A key difference between CAW and SMIW is that in addition to expressing preferences over individual students, the colleges have preferences over *groups* of students. This characteristic is shared by the CAS model, which is the restriction of the CAW model to strict preferences. It is known that no stable mechanism for CAS is strategyproof for the colleges [13]; the proof makes use of the fact that the colleges do not (in general) have unit demand. It follows that no Pareto-stable mechanism for CAW is strategyproof for the colleges. Throughout the remainder of the paper, when we say that a mechanism for a college admissions model is strategyproof, we mean that it is strategyproof for the students.

Gale and Shapley's DA algorithm generalizes easily to the CAS model. Roth [13] has shown that the student-proposing DA algorithm provides a strategyproof stable mechanism for CAS when the preferences of the colleges are *responsive*. When the colleges have responsive preferences, the student-proposing DA mechanism is also known to be student-optimal for CAS [13].

Erdil and Ergin [4] consider the special case of the CAW model where the following restrictions hold for all students x and colleges y: x is not indifferent between being assigned to y and being left unassigned; y is not indifferent between having one of its slots assigned to x and having that slot left unfilled. We remark that this special case of CAW corresponds to the HRT problem discussed in Manlove [11, Chapter 3].[1] For this special case, Erdil and Ergin present a polynomial-time algorithm for computing a Pareto-stable matching when the preferences of the colleges satisfy a technical restriction related to responsiveness. We consider the same class of preferences, which

[1] In the model of Erdil and Ergin, which is stated using worker-firm terminology rather than student-college terminology, a "no indifference to unemployment/vacancy" assumption makes the aforementioned restrictions explicit. In the HRT model of Manlove, which is stated using resident-hospital terminology rather than student-college terminology, it is assumed that a set of acceptable resident-hospital pairs is given, and that each agent specifies weak preferences over the set of agents with whom they form an acceptable pair. We consider the approach of Erdil-Ergin — where the starting point is the preferences of the individual agents, and the "acceptability" of a given pair of agents may be deduced from those preferences — to be more natural, but the resulting models are equivalent.

we refer to as *minimally responsive*; see Section 5 for a formal definition. The algorithm of Erdil and Ergin does not provide a strategyproof mechanism. Chen and Ghosh [2] build on the results of Erdil and Ergin by considering the many-to-many generalization of HRT in which the agents on both sides of the market have capacities (and the agent preferences are minimally responsive). For this generalization, Chen and Ghosh provide a *strongly* polynomial-time algorithm. No strategyproof mechanism (even for the agents on one side of the market) is possible in the many-to-many setting, since it is a generalization of CAS. We provide the first Pareto-stable mechanism for CAW that is shown to be strategyproof. As in the work of Erdil-Ergin and Chen-Ghosh, we assume that the preferences of the colleges are minimally responsive. We can also handle the class of college preferences "induced by additive utility" that is defined in Section 5.2.

In the many-to-many matching setting addressed by Chen and Ghosh [2], a pair of agents (on opposite sides of the market) can be matched with arbitrary multiplicity, as long as the capacity constraints are respected. Chen [1] presents a polynomial-time algorithm for the variation of many-to-many matching in which a pair of agents can only be matched with multiplicity one. Kamiyama [8] addresses the same problem using a different algorithmic approach. (The algorithms of Chen and Kamiyama are strongly polynomial, since we can assume without loss of generality that the capacity of any agent is at most the number of agents on the other side of the market.) Since this variation of the many-to-many setting also generalizes CAS, it does not admit a strategyproof mechanism, even for the agents on one side of the market.

Erdil and Ergin [4, 5] and Kesten [9] consider a second natural solution concept in addition to Pareto-stability. In the context of SMIW (or its special case SMCW), this second solution concept seeks a weakly stable matching M that is "man optimal" in the following sense: for all weakly stable matchings M', either all of the men are indifferent between M and M', or at least one man prefers M to M'. Erdil and Ergin [4] present a polynomial-time algorithm to compute such a man optimal weakly stable matching for SMIW; in fact, their algorithm is presented for the generalization of SMIW to CAW. Erdil and Ergin [5] and Kesten [9] prove that no strategyproof man optimal weakly stable mechanism exists for SMCW. Prior to our work, it was unclear whether such an impossibility result might hold for strategyproof Pareto-stable mechanisms for SMCW (or its generalizations to SMIW and CAW).

The assignment game of Shapley and Shubik [15] can be viewed as an auction with multiple distinct items where each bidder is seeking to acquire at most one item. This class of *unit-demand auctions* has been heavily studied in the literature (see, e.g., Roth and Sotomayor [14, Chapter 8]). In Section 2, we define the notion of a "unit-demand auction with priorities" (UAP) and establish a number of useful properties of UAPs; these are straightforward generalizations of corresponding properties of unit-demand auctions. Section 3 builds on the UAP notion to define the notion of an "iterated UAP" (IUAP), and establishes a number of important properties of IUAPs; these results are nontrivial to prove and provide the technical foundation for our main results. Section 4 presents our first main result, a polynomial-time algorithm for SMIW that provides a strategyproof Pareto-stable mechanism. Section 5 presents our second main result, a polynomial-time algorithm for CAW that provides a strategyproof Pareto-stable mechanism assuming that the preferences of the colleges are minimally responsive.

2 Unit-Demand Auctions with Priorities

In this section, we formally define the notion of a unit-demand auction with priorities (UAP). In Section 2.1, we describe an associated matroid for a given UAP and we use this matroid to define the notion of a "greedy MWM". In Section 2.2, we establish a result related to extending a given UAP by introducing additional bidders. In Section 2.3, we discuss how to efficiently compute a greedy MWM in a UAP. In Section 2.4, we introduce a key definition that is helpful for establishing our strategyproofness results. We start with some useful definitions.

A *(unit-demand) bid β for a set of items V* is a subset of $V \times \mathbb{R}$ such that no two pairs in β share the same first component. (So β may be viewed as a partial function from V to $\mathbb{R}$.)

A *bidder u for a set of items V* is a triple (α, β, z) where α is an integer ID, β is a bid for V, and z is a real priority. For any bidder $u = (\alpha, \beta, z)$, we define $id(u)$ as α, $bid(u)$ as β, $priority(u)$ as z, and $items(u)$ as the union, over all (v, x) in β, of $\{v\}$.

A *unit-demand auction with priorities (UAP)* is a pair $A = (U, V)$ satisfying the following conditions: V is a set of items; U is a set of bidders for V; each bidder in U has a distinct ID.

2.1 An Associated Matroid

A UAP $A = (U, V)$ may be viewed as an edge-weighted bipartite graph, where the set of edges incident on bidder u correspond to $bid(u)$: for each pair (v, x) in $bid(u)$, there is an edge (u, v) of weight x. We refer to a matching (resp., maximum-weight matching (MWM), maximum-cardinality MWM (MCMWM)) in the associated edge-weighted bipartite graph as a matching (resp., MWM, MCMWM) of A. For any edge $e = (u, v)$ in a given UAP, the associated weight is denoted $w(e)$ or $w(u, v)$. For any set of edges E, we define $w(E)$ as $\sum_{e \in E} w(e)$. For any UAP A, we let $w(A)$ denote the weight of an MWM of A.

Lemma 1. Let $A = (U, V)$ be a UAP, and let $\mathcal{I}$ denote the set of all subsets U' of U such that there exists an MWM of A that matches every bidder in U'. Then $(U, \mathcal{I})$ is a matroid.

Proof. The only nontrivial property to show is the exchange property. Assume that U_1 and U_2 belong to $\mathcal{I}$ and that $|U_1| > |U_2|$. Let M_1 be an MWM of A that matches every bidder in U_1, and let M_2 be an MWM of A that matches every bidder in U_2. If M_2 matches some bidder u in $U_1 \setminus U_2$, then $U_2 + u$ belongs to $\mathcal{I}$, as required. Thus, in what follows, we assume that M_2 does not match any bidder in $U_1 \setminus U_2$. The symmetric difference of M_1 and M_2, denoted $M_1 \oplus M_2$, corresponds to a collection of vertex-disjoint paths and cycles. Since M_2 does not match any bidder in $U_1 \setminus U_2$, we deduce that each bidder in $U_1 \setminus U_2$ is an endpoint of one of the paths in this collection. Since $|U_1| > |U_2|$, $|U_1 \setminus U_2| = |U_1| - |U_1 \cap U_2|$, and $|U_2 \setminus U_1| = |U_2| - |U_1 \cap U_2|$, we have $|U_1 \setminus U_2| > |U_2 \setminus U_1|$. It follows that there is at least one path in this collection, call it P, such that one endpoint of P is a bidder u in $U_1 \setminus U_2$ and the other endpoint of P is a vertex y that does not belong to $U_2 \setminus U_1$. Moreover, y does not belong to U_1: if the length of P is odd, then y is an item and hence does not belong to U_1; if the length of P is even, then y is not matched in M_1 and hence does not belong to U_1. Since y does not belong to $U_2 \setminus U_1$ and does not belong to U_1, we conclude that y does not belong to U_2. The edges of P alternate between M_1 and M_2. Let X_1 denote the edges of P that belong to M_1, and let X_2 denote the edges of P that belong to M_2. Since M_1 is an MWM of A and $M_1' = M_1 \oplus P = (M_1 \cup X_2) \setminus X_1$ is a matching of A, we deduce that $w(X_1) \geq w(X_2)$. Since M_2 is an MWM of A and $M_2' = M_2 \oplus P = (M_2 \cup X_1) \setminus X_2$ is a matching of A, we deduce that $w(X_2) \geq w(X_1)$. Hence $w(X_1) = w(X_2)$ and M'_1 and M'_2 are MWMs of A. The MWM M'_2 matches all of the vertices on P except for y. Since y does not belong to U_2, we conclude that M'_2 matches all of the vertices in $U_2 + u$, and so the exchange property holds. □

For any UAP A, we define $\mathit{matroid}(A)$ as the matroid of Lemma 1.

For any UAP $A = (U, V)$ and any independent set U' of $\mathit{matroid}(A)$, we define the *priority of* U' as the sum, over all bidders u in U', of $\mathit{priority}(u)$. For any UAP A, the matroid greedy algorithm can be used to compute a maximum-priority maximal independent set of $\mathit{matroid}(A)$.

For any matching M of a UAP $A = (U, V)$, we define $\mathit{matched}(M)$ as the set of all bidders in U that are matched in M. We say that an MWM M of a UAP A is *greedy* if $\mathit{matched}(M)$ is a maximum-priority maximal independent set of $\mathit{matroid}(A)$. For any UAP A, we define the predicate $\mathit{unique}(A)$ to hold if $\mathit{matched}(M) = \mathit{matched}(M')$ for all greedy MWMs M and M' of A.

For any matching M of a UAP, we define the *priority of* M, denoted $\mathit{priority}(M)$, as the sum, over all bidders u in $\mathit{matched}(M)$, of $\mathit{priority}(u)$. Thus an MWM is greedy if and only if it is a maximum-priority MCMWM.

Lemma 2. All greedy MWMs of a given UAP have the same distribution of priorities.

Proof. This is a standard matroid result that follows easily from the exchange property and the correctness of the matroid greedy algorithm. □

For any UAP A and any real priority z, we define $\mathit{greedy}(A, z)$ as the (uniquely defined, by Lemma 2) number of matched bidders with priority z in any greedy MWM of A.

Lemma 3. Let $A = (U, V)$ be a UAP. Let u be a bidder in U such that (v, x) belongs to $\mathit{bid}(u)$, $\mathit{priority}(u) = z$, and u is not matched in any greedy MWM of A. Let u' be a bidder in U such that (v, x') belongs to $\mathit{bid}(u')$, $\mathit{priority}(u') = z'$, and u' is matched to v in some greedy MWM of A. Then $(x, z) < (x', z')$.[2]

Proof. Let M be a greedy MWM in which u' is matched to v. Thus u is not matched in M. Let M' denote the matching $M - (u', v) + (u, v)$. Since M is an MCMWM of A and $w(M') = w(M) - x' + x$, we conclude that $x \leq x'$. If $x < x'$, the claim of the lemma follows. Assume that $x = x'$. In this case, M' is an MCMWM of A since $w(M') = w(M)$ and $|M'| = |M|$. Since M is a greedy MWM of A and $\mathit{priority}(M') = \mathit{priority}(M) - z' + z$, we conclude that $z \leq z'$. If $z = z'$ then M' is a greedy MWM of A that matches u, a contradiction. Hence $z < z'$, as required. □

2.2 Extending a UAP

Let $A = (U, V)$ be a UAP and let u be a bidder such that $\mathit{id}(u)$ is not equal to the ID of any bidder in U. Then we define $A + u$ as the UAP $(U + u, V)$. For any UAPs $A = (U, V)$ and $A' = (U', V')$, we say that A' *extends* A if $U \subseteq U'$ and $V = V'$.

Lemma 4. Let $A = (U, V)$ be a UAP, let u be a bidder in U that is not matched in any greedy MWM of A, and let $A' = (U', V)$ be a UAP that extends A. Then u is not matched in any greedy MWM of A'.

[2]Throughout this paper, comparisons of pairs are to be performed lexicographically.

Proof. In what follows, we derive a contradiction by proving that u is matched in a greedy MWM M_1 of A'. We need to prove that u is matched in some greedy MWM of A. Let M_0 denote a greedy MWM of A. If u is matched in M_0, we are done, so assume that u is not matched in M_0. Thus $M_0 \oplus M_1$ contains a unique path P with u as an endpoint. The edges of P alternate between M_0 and M_1. Let X_0 denote the edges of P that belong to M_0, and let X_1 denote the edges of P that belong to M_1.

Since u is matched in M_1 and not in M_0, the other endpoint of P is either an item, or it is a bidder that is matched in M_0 and not in M_1. Either way, we deduce that all of the vertices on P belong to A. Thus $M_0' = M_0 \oplus P = (M_0 \cup X_1) \setminus X_0$ is a matching in A. Since M_0 is an MWM of A and M_0' is a matching of A, we deduce that $w(M_0) \geq w(M_0')$ and hence that $w(X_0) \geq w(X_1)$. Since all of the vertices on P belong to A', we conclude that $M_1' = M_1 \oplus P = (M_1 \cup X_0) \setminus X_1$ is a matching in A'. Since M_1 is an MWM of A' and M_1' is a matching of A', we deduce that $w(M_1) \geq w(M_1')$ and hence that $w(X_1) \geq w(X_0)$. Thus $w(X_0) = w(X_1)$, and we conclude that M_0' is an MWM of A.

Since u is matched in M_1 and not in M_0, we deduce that $|X_1| \geq |X_0|$ and hence that $|M_0'| \geq |M_0|$. Since M_0 is a greedy MWM of A, we know that M_0 is an MCMWM of A, and hence that $|M_0| \geq |M_0'|$. Thus $|M_0| = |M_0'|$ and hence $|X_0| = |X_1|$ and M_0' is an MCMWM of A. Since $|X_0| = |X_1|$, the other endpoint of P is a bidder u' that is matched in M_0 and not in M_1. Since M_0 is a greedy MWM of A and M_0' is an MCMWM of A, we deduce that $\mathit{priority}(M_0) \geq \mathit{priority}(M_0')$ and hence that $\mathit{priority}(u') \geq \mathit{priority}(u)$.

Since $|X_0| = |X_1|$ and $w(X_0) = w(X_1)$, we deduce that M_1' is an MCMWM of A'. Since M_1 is a greedy MWM of A' and M_1' is an MCMWM of A', we deduce that $\mathit{priority}(M_1) \geq \mathit{priority}(M_1')$ and hence that $\mathit{priority}(u) \geq \mathit{priority}(u')$. Since we argued above that $\mathit{priority}(u') \geq \mathit{priority}(u)$, we conclude that $\mathit{priority}(u) = \mathit{priority}(u')$, and hence that M_0' is a greedy MWM of A. This completes the proof, since u is matched in M_0'. □

2.3 Finding a Greedy MWM

In this section, we briefly discuss how to efficiently compute a greedy MWM of a UAP via a slight modification of the classic Hungarian method for the assignment problem [10]. In the (maximization version of the) assignment problem, we are given a set of n agents, a set of n tasks, and a weight for each agent-task pair, and our objective is to find a perfect matching (i.e., every agent and task is required to be matched) of maximum total weight. The Hungarian method for the assignment problem proceeds as follows: a set of dual variables, namely a "price" for each task, and a possibly incomplete matching are maintained; an arbitrary unmatched agent u is chosen and a shortest augmenting path from u to an unmatched task is computed using "residual costs" as the edge weights; an augmentation is performed along the path to update the matching, and the dual variables are adjusted in order to maintain complementary slackness; the process repeats until a perfect matching is found.

Within our UAP setting, the set of bidders can be larger than the set of items, and some bidder-item pairs may not be matchable, i.e., the associated bipartite graph is not necessarily complete. In this setting, we can use an "incremental" version of the Hungarian method to find an (not necessarily greedy) MWM of a given UAP $A = (U, V)$ as follows. For the purpose of simplifying the presentation of our method, we enlarge the set of items by adding a dummy item v_0 such that v_0 is connected to each bidder u with an edge of weight $w(u, v_0) = 0$ and we always maintain

v_0 in the residual graph with a price of 0. We start with the empty matching M. Then, for each bidder u in U (in arbitrary order), we process u via an "incremental Hungarian step" as follows: let U' denote the set of bidders that are matched by M; let V' denote the set of items that are not matched by M; find the shortest paths from u to each item v in $V' + v_0$ in the residual graph; let W denote the minimum path weight among these shortest paths; choose a path P that is either (1) a shortest path of weight W from u to an item v in V', or (2) a shortest path from u to a bidder u' in $U' + u$ such that extending P with the edge (u', v_0) yields a shortest path of weight W from u to v_0; augment M along P; adjust the prices in order to maintain complementary slackness; update the residual graph. The algorithm terminates when every non-reserve bidder has been processed. The algorithm performs $|U|$ incremental Hungarian steps and each incremental Hungarian step can be implemented in $O(|V| \log |V| + m)$ time by utilizing Fibonacci heaps [6], where m denotes the number of edges in the residual graph, which is $O(|V|^2)$.

In order to find a greedy MWM, we slightly modify the implementation described in the previous paragraph. Lemmas 7 and 8 established below imply that choosing the path P in the following way results in a greedy MWM: if a path of type (1) exists, we arbitrarily choose such a path; if no path of type (1) exists, then we identify the nonempty set U'' of all bidders u' such that a path of type (2) exists, and we choose a shortest path P that terminates at a minimum priority bidder in U''. It is easy to see that the described modification does not increase the asymptotic time complexity of the algorithm. In the remainder of this section, we establish Lemmas 7, 8, and 9; Lemma 9 is used in Section 3.2.1 to prove Lemma 19. We start with some useful definitions.

Let $A = (U, V)$ and $A' = A + u$ be UAPs, and let M be an MWM of A. We define $\mathit{digraph}(A, u, M)$ as the edge-weighted digraph that may be obtained by modifying the subgraph of A induced by the set of vertices $(\mathit{matched}(M) + u) \cup V$ as follows: for each edge that belongs to M, we direct the edge from item to bidder and leave the weight unchanged; for each edge that does not belong to M, we direct the edge from bidder to item and negate the weight.

Lemma 5. Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be an MWM of A, and let G denote $\mathit{digraph}(A, u, M)$. Then G does not contain any negative-weight cycles.

Proof. Such a cycle could not involve u (since u only has outgoing edges) so it has to be a negative-weight cycle that already existed before u was added, a contradiction since M is an MWM of A. □

Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be an MWM of A, and let G denote $\mathit{digraph}(A, u, M)$. We define a set of items $\mathit{holes}(A, u, M)$, and a set of bidders $\mathit{candidates}(A, u, M)$, as follows. By Lemma 5, the shortest path distance in G from bidder u to any vertex reachable from u is well-defined. We define $\mathit{holes}(A, u, M)$ as the set of all items v in V such that v is unmatched in M and the weight of a shortest path in G from u to v is $w(A) - w(A')$. We define $\mathit{candidates}(A, u, M)$ as the set of all bidders u' such that the weight of a shortest path in G from u to u' is equal to $w(A) - w(A')$.

Let $A = (U, V)$ and $A' = A + u$ be UAPs, let M be an MWM of A, and let P be a directed path in $\mathit{digraph}(A, u, M)$ that starts at u, has weight $w(A) - w(A')$, and terminates at either an item in $\mathit{holes}(A, u, M)$ or a bidder in $\mathit{candidates}(A, u, M)$. (Note that P could be a path of length zero from u to u.) Let X denote the edges in M that correspond to item-to-bidder edges in P, and let Y denote the edges in A' that correspond to bidder-to-item edges in P. It is easy to see that the set of edges $(M \setminus X) \cup Y$ is an MWM of A'. We define this MWM of A' as $\mathit{augment}(A, u, M, P)$.

Lemma 6. Let $A = (U, V)$ be a UAP, let M be a greedy MWM of A, let u be a bidder that does not belong to U, let A' denote the UAP $(U + u, V)$, and let M' denote a greedy MWM of A' minimizing $|M \oplus M'|$. Then $digraph(A, u, M)$ contains a directed path P satisfying the following conditions: P has weight $w(A) - w(A')$; P starts at u; the bidder-to-item edges in P correspond to the edges in $M' \setminus M$; the item-to-bidder edges in P correspond to the edges in $M \setminus M'$; if $holes(A, u, M)$ is nonempty, then P terminates at an item in $holes(A, u, M)$; if $holes(A, u, M)$ is empty, then P terminates at a minimum-priority bidder in $candidates(A, u, M)$.

Proof. The edges of $M \oplus M'$ form a collection $\mathcal{S}$ of disjoint cycles and paths of positive length.

We begin by arguing that $\mathcal{S}$ does not contain any cycles. Suppose there is a cycle C in $\mathcal{S}$. Let X denote the edges of C that belong to $M \setminus M'$, and let Y denote the edges of C that belong to $M' \setminus M$. Let M'' denote $(M \cup Y) \setminus X$, which is a matching in A since u is unmatched in M and hence does not belong to C. Since M is an MWM of A and $w(M'') = w(M) + w(Y) - w(X)$, we conclude that $w(X) \geq w(Y)$. Let M''' denote $(M' \cup X) \setminus Y$, which is a matching in A'. Since M' is an MWM of A' and $w(M''') = w(M') + w(Y) - w(X)$, we conclude that $w(X) \leq w(Y)$. Thus $w(X) = w(Y)$ and hence $w(M''') = w(M')$, implying that M''' is an MWM of A'. Moreover, since M''' matches the same set of bidders as M', we find that M''' is a greedy MWM of A'. This contradicts the definition of M' since $|M \oplus M'''| < |M \oplus M'|$.

Next we argue that if Q is a path in $\mathcal{S}$, then u is an endpoint of Q. Suppose there is a path Q in $\mathcal{S}$ such that u is not an endpoint of Q. Thus u does not appear on Q since u is unmatched in M. Let X denote the edges of Q that belong to $M \setminus M'$, and let Y denote the edges of Q that belong to $M' \setminus M$. Let M'' denote $(M \cup Y) \setminus X$, which is a matching in A since u does not belong to Q. Since M is an MWM of A and $w(M'') = w(M) + w(Y) - w(X)$, we conclude that $w(X) \geq w(Y)$. Let M''' denote $(M' \cup X) \setminus Y$, which is a matching in A'. Since M' is an MWM of A' and $w(M''') = w(M') + w(Y) - w(X)$, we conclude that $w(X) \leq w(Y)$. Thus $w(X) = w(Y)$ and hence $w(M'') = w(M)$ and $w(M''') = w(M')$, implying that M'' is an MWM of A and M''' is an MWM of A'. Since M is a greedy MWM and hence an MCMWM of A, the set of bidders matched by M is not properly contained in the set of bidders matched by M''; we conclude that $|X| \geq |Y|$. Since M' is a greedy MWM and hence an MCMWM of A', the set of bidders matched by M' is not properly contained in the set of bidders matched by M'''; we conclude that $|X| \leq |Y|$. Thus $|X| = |Y|$, so the length of path Q is even. We consider two cases.

Case 1: The endpoints of Q are items. In this case, M' and M''' match the same set of bidders, and hence M''' is a greedy MWM of A'. This contradicts the definition of M', since Q has positive length and hence $|M \oplus M'''| < |M \oplus M'|$.

Case 2: The endpoints of Q are bidders. Since Q has positive length, one endpoint, call it u_0, is matched in M but not in M', and the other endpoint, call it u_1, is matched in M' but not in M. Since M is a greedy MWM of A and M'' is an MWM of A, we deduce that $priority(u_0) \geq priority(u_1)$. Since M' is a greedy MWM of A' and M''' is an MWM of A', we deduce that $priority(u_0) \leq priority(u_1)$. Thus $priority(u_0) = priority(u_1)$. It follows that $priority(M''') = priority(M')$. Hence M''' is a greedy MWM of A'. This contradicts the definition of M' since $|M \oplus M'''| < |M \oplus M'|$.

From the preceding arguments, we deduce that either $M = M'$ or $M \oplus M'$ corresponds to a positive-length path with u as an endpoint. Equivalently, $M \oplus M'$ is the edge set of a path that has u as an endpoint and may have length zero (i.e., the path may begin and end at u). We claim if the edges of this path are directed away from endpoint u, we obtain a directed path P satisfying the six conditions stated in the lemma. It is easy to see that P satisfies the first four of these conditions. It remains to establish that P satisfies the fifth and sixth conditions.

For the fifth condition, assume that $holes(A, u, M)$ is nonempty. We need to prove that P terminates at an item in $holes(A, u, M)$. Since $holes(A, u, M)$ is nonempty, we deduce that $|M'| = |M|+1$, and hence that P terminates at some item v. Since P has weight $w(A)-w(A')$, we deduce that v belongs to $holes(A, u, M)$, as required.

For the sixth condition, assume that $holes(A, u, M)$ is empty. We need to prove that P terminates at a minimum-priority bidder in $candidates(A, u, M)$. Suppose P terminates at some item v. Since P has weight $w(A) - w(A')$, we deduce that v belongs to $holes(A, u, M)$, a contradiction. Thus P terminates at some bidder u'. Since P has weight $w(A) - w(A')$, we deduce that u' belongs to $candidates(A, u, M)$. If u' is not a minimum-priority bidder in $candidates(A, u, M)$, it is easy to argue that M' is not a greedy MWM of A', a contradiction. Thus P terminates at a minimum-priority bidder in $candidates(A, u, M)$. □

Lemma 7. Let $A = (U, V)$ be a UAP, let M be a greedy MWM of A, let u be a bidder that does not belong to U, let A' denote the UAP $(U + u, V)$, let P be a directed path in $digraph(A, u, M)$ of weight $w(A)-w(A')$ from u to an item in $holes(A, u, M)$, and let M^* denote $augment(A, u, M, P)$. Then M^* is a greedy MWM of A'.

Proof. The definition of $augment(A, u, M, P)$ implies that M^* is an MWM of A'. Let M' denote a greedy MWM of A' minimizing $|M \oplus M'|$. Let U' denote the set of bidders in A matched by M. Since $holes(A, u, M)$ is nonempty, Lemma 6 implies that the set of bidders in A' matched by M' is $U' + u$. Since M^* is an MWM of A' that also matches the set of bidders $U' + u$, we deduce that M^* is a greedy MWM of A'. □

Lemma 8. Let $A = (U, V)$ be a UAP, let M be a greedy MWM of A, let u be a bidder that does not belong to U, and let A' denote the UAP $(U + u, V)$. Assume that $holes(A, u, M)$ is empty. Let u' denote a minimum-priority bidder in $candidates(A, u, M)$ (which is nonempty by Lemma 6), let P be a directed path in $digraph(A, u, M)$ of weight $w(A) - w(A')$ from u to u', and let M^* denote $augment(A, u, M, P)$. Then M^* is a greedy MWM of A'.

Proof. The definition of $augment(A, u, M, P)$ implies that M^* is an MWM of A'. Let M' denote a greedy MWM of A' minimizing $|M \oplus M'|$. Let U' denote the set of bidders in A matched by M. Since $holes(A, u, M)$ is empty, Lemma 6 implies that the set of bidders in A' matched by M' is $U' + u - u''$, where u'' is some minimum-priority bidder in $candidates(A, u, M)$. It is straightforward to check that M^* has the same weight, cardinality, and priority as M'. Thus M^* is a greedy MWM of A', as required. □

Lemma 9. Let A and A' be two UAPs such that A' extends A, let M be a greedy MWM of A, and let M' be a greedy MWM of A'. Then $|M'| \geq |M|$.

Proof. Immediate from Lemmas 7 and 8. □

2.4 Threshold of an Item

In this section, we define the notion of a "threshold" of an item in a UAP. This lays the groundwork for a corresponding IUAP definition in Section 3.2. Item thresholds play an important role in our strategyproofness results.

Lemma 10. Let $A = (U, V)$ be a UAP and let v be an item in V. Let U' be the set of bidders u such that $A + u$ is a UAP and $bid(u)$ is of the form $\{(v, x)\}$. Then there is a unique pair of reals (x^*, z^*) such that for any bidder u in U', the following conditions hold, where A' denotes $A + u$, x denotes $w(u, v)$, and z denotes $priority(u)$: (1) if $(x, z) > (x^*, z^*)$ then u is matched in every greedy MWM of A'; (2) if $(x, z) < (x^*, z^*)$ then u is not matched in any greedy MWM of A'; (3) if $(x, z) = (x^*, z^*)$ then u is matched in some but not all greedy MWMs of A'.

Proof. Let M be a greedy MWM of A, let W denote $w(M)$, and let Z denote $priority(M)$. Let $\mathcal{M}$ denote the set of matchings of A' that do not match v, let $\mathcal{M}'$ denote the maximum-weight elements of $\mathcal{M}$, let $\mathcal{M}''$ denote the maximum-cardinality elements of $\mathcal{M}'$, let $\mathcal{M}'''$ denote the maximum-priority elements of $\mathcal{M}''$, and observe that there is a unique pair of reals (W', Z') such that any matching M' in $\mathcal{M}'''$ has weight W' and priority Z'. It is straightforward to verify that the unique choice of (x^*, z^*) satisfying the conditions stated in the lemma is $(W - W', Z - Z')$. $\square$

For any UAP $A = (U, V)$ and any item v in V, we define the unique pair (x^*, z^*) of Lemma 10 as $threshold(A, v)$.

3 Iterated Unit-Demand Auctions with Priorities

In this section, we formally define the notion of an iterated unit-demand auction with priorities (IUAP). An IUAP allows the bidders, called "multibidders" in this context, to have a sequence of unit-demand bids instead of a single unit-demand bid. In Section 3.1, we define a mapping from an IUAP to a UAP by describing an algorithm that generalizes the DA algorithm, and we establish Lemma 15 that is useful for analyzing the matching produced by Algorithm 2 of Section 4. Lemma 15 is used to establish weak stability (Lemmas 27, 28, and 29) and Pareto-optimality (Lemma 30). In Section 3.2, we define the threshold of an item in an IUAP and we establish Lemma 18, which plays a key role in establishing our strategyproofness results. We start with some useful definitions.

A *multibidder t for a set of items V* is a pair (σ, z) where z is a real priority and σ is a sequence of bidders for V such that all the bidders in σ have distinct IDs and a common priority z. We define $priority(t)$ as z. For any integer i such that $1 \leq i \leq |\sigma|$, we define $bidder(t, i)$ as the bidder $\sigma(i)$. For any integer i such that $0 \leq i \leq |\sigma|$, we define $bidders(t, i)$ as $\{bidder(t, j) \mid 1 \leq j \leq i\}$. We define $bidders(t)$ as $bidders(t, |\sigma|)$.

An *iterated UAP (IUAP)* is a pair $B = (T, V)$ where V is a set of items and T is a set of multibidders for V. In addition, for any distinct multibidders t and t' in T, the following conditions hold: $priority(t) \neq priority(t')$; if u belongs to $bidders(t)$ and u' belongs to $bidders(t')$, then $id(u) \neq id(u')$. For any IUAP $B = (T, V)$, we define $bidders(B)$ as the union, over all t in T, of $bidders(t)$.

3.1 Mapping an IUAP to a UAP

Having defined the notion of an IUAP, we now describe an algorithm ToUap that maps a given IUAP to a UAP. Algorithm ToUap generalizes the DA algorithm. In each iteration of the DA algorithm, a single man is nondeterministically chosen, and this man reveals his next choice. In each iteration of ToUap, a single multibidder is nondeterministically chosen, and this multibidder reveals its next bid. We prove in Lemma 14 that, like the DA algorithm, algorithm ToUap is confluent: the output does not depend on the nondeterministic choices made during an execution. We conclude this section by establishing Lemma 15, which is useful for analyzing the matching produced by Algorithm 2 in Section 4. Lemma 15 is used to establish weak stability (Lemmas 27, 28, and 29) and Pareto-optimality (Lemma 30). We start with some useful definitions.

Let A be a UAP (U, V) and let B be an IUAP (T, V). The predicate $\mathit{prefix}(A, B)$ is said to hold if $U \subseteq \mathit{bidders}(B)$ and for any multibidder t in T, $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i)$ for some i.

A *configuration* C is a pair (A, B) where A is a UAP, B is an IUAP, and $\mathit{prefix}(A, B)$ holds.

Let $C = (A, B)$ be a configuration, where $A = (U, V)$ and $B = (T, V)$, and let u be a bidder in U. Then we define $\mathit{multibidder}(C, u)$ as the unique multibidder t in T such that u belongs to $\mathit{bidders}(t)$.

Let $C = (A, B)$ be a configuration where $A = (U, V)$ and $B = (T, V)$. For any t in T, we define $\mathit{bidders}(C, t)$ as $\{u \in U \mid \mathit{multibidder}(C, u) = t\}$.

Let $C = (A, B)$ be a configuration where $B = (T, V)$. We define $\mathit{ready}(C)$ as the set of all bidders u in $\mathit{bidders}(B)$ such that $\mathit{greedy}(A, \mathit{priority}(u)) = 0$ and $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)| + 1)$ where $t = \mathit{multibidder}(C, u)$.

Algorithm 1 ToUap(B)

Input: An IUAP $B = (T, V)$

$A \leftarrow (\emptyset, V)$
$C \leftarrow (A, B)$
while $\mathit{ready}(C)$ is nonempty **do**
  $A \leftarrow A+$ an arbitrary bidder in $\mathit{ready}(C)$
  $C \leftarrow (A, B)$
end while
return A

Our algorithm for mapping an IUAP to a UAP is Algorithm 1. The input is an IUAP B and the output is a UAP A such that $\mathit{prefix}(A, B)$ holds. The algorithm starts with the UAP consisting of all the items in V but no bidders. At this point, no bidder of any multibidder is "revealed". Then, the algorithm iteratively and nondeterministically chooses a "ready" bidder and "reveals" it by adding it to the UAP that is maintained in the program variable A. A bidder u associated with some multibidder $t = (\sigma, z)$ is ready if u is not revealed and for each bidder u' that precedes u in σ, u' is revealed and is not matched in any greedy MWM of A. It is easy to verify that the predicate $\mathit{prefix}(A, B)$ is an invariant of the algorithm loop: if a bidder u belonging to a multibidder t is to be revealed at an iteration, and $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i)$ for some integer i at the beginning of this iteration, then $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i+1)$ after revealing u, where (U, V) is the UAP that is maintained by the program variable A at the beginning of the iteration. No bidder can be revealed more than once since a bidder cannot be ready after it has been revealed; it follows that the algorithm terminates. We now argue that the output of the algorithm is uniquely determined (Lemma 14), even though the bidder that is revealed in each iteration is chosen nondeterministically.

For any configuration $C = (A, B)$, we define the predicate $\mathit{tail}(C)$ to hold if for any bidder u that is matched in some greedy MWM of A, we have $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)|)$ where t denotes $\mathit{multibidder}(C, u)$.

Lemma 11. Let $C = (A, B)$ be a configuration where $B = (T, V)$ and assume that $\mathit{tail}(C)$ holds. Then $\mathit{greedy}(A, \mathit{priority}(t)) \leq 1$ for each t in T.

Proof. The claim of the lemma easily follows from the definition of $\mathit{tail}(C)$. □

Lemma 12. The predicate $\mathit{tail}(C)$ is an invariant of the Algorithm 1 loop.

Proof. It is easy to see that $\mathit{tail}(C)$ holds when the loop is first encountered. Now consider an iteration of the loop that takes us from configuration $C = (A, B)$ where $A = (U, V)$ to configuration $C' = (A', B)$ where $A' = (U', V)$. We need to show that $\mathit{tail}(C')$ holds. Let u be a bidder that is matched in some greedy MWM M' of A'. Let u^* denote the bidder that is added to A in line 4, and consider the following three cases.

Case 1: $u = u^*$. Let t denote $\mathit{multibidder}(C', u^*)$. In this case, $|\mathit{bidders}(C, t)| + 1 = |\mathit{bidders}(C', t)|$, so $u^* = \mathit{bidder}(t, |\mathit{bidders}(C', t)|)$, as required.

Case 2: $u \neq u^*$ and $\mathit{priority}(u) \neq \mathit{priority}(u^*)$. Since U' contains U, Lemma 4 implies that u is matched in some greedy MWM of A. Since C is a configuration and $\mathit{tail}(C)$ holds, we deduce that $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)|)$ where t denotes $\mathit{multibidder}(C, u)$. Since $\mathit{multibidder}(C', u) = \mathit{multibidder}(C, u)$ and $\mathit{bidders}(C', t) = \mathit{bidders}(C, t)$, we conclude that u is equal to $\mathit{bidder}(t, |\mathit{bidders}(C', t)|)$ where t denotes $\mathit{multibidder}(C', u)$, as required.

Case 3: $u \neq u^*$ and $\mathit{priority}(u) = \mathit{priority}(u^*)$. Since u^* belongs to $\mathit{ready}(C)$, we know that $\mathit{greedy}(A, \mathit{priority}(u)) = 0$. Also, since u is not u^*, u belongs to U and we conclude that u is not matched in any greedy MWM of A. Since U' contains U, Lemma 4 implies that u is not matched in any greedy MWM of A', a contradiction. □

Lemma 13. Let $C = (A, B)$ be a configuration such that $\mathit{tail}(C)$ holds. Then $\mathit{unique}(A)$ holds.

Proof. Let M and M' be greedy MWMs of A, and let u be a bidder in $\mathit{matched}(M)$. To establish the lemma, it is sufficient to prove that u belongs to $\mathit{matched}(M')$. Let t denote $\mathit{multibidder}(C, u)$ and let z denote $\mathit{priority}(t)$. Since $\mathit{tail}(C)$ holds, we know that $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)|)$. Since u is matched by M and since $\mathit{tail}(C)$ holds, Lemma 11 implies that $\mathit{greedy}(A, z) = 1$. Thus Lemma 2 implies that M' matches one priority-z bidder. Since $\mathit{tail}(C)$ holds, this bidder is u. □

Lemma 14. Let $B = (T, V)$ be an IUAP. Then all executions of Algorithm 1 on input B produce the same output.

Proof. Suppose not, and let X_1 and X_2 denote two executions of Algorithm 1 on input B that produce distinct output UAPs $A_1 = (U_1, V)$ and $A_2 = (U_2, V)$. Without loss of generality, assume that $|U_1| \geq |U_2|$. Then there is a first iteration of execution X_1 in which the bidder added to A in line 4 belongs to $U_1 \setminus U_2$; let u' denote this bidder. Let $C' = (A', B)$ where $A' = (U', V)$ denote the configuration in program variable C at the start of this iteration, and let t' denote $\mathit{multibidder}(C', u')$. Let i be the integer such that $u' = \mathit{bidder}(t', i)$. We know that $i > 1$ because it is easy to see that U_2 contains $\mathit{bidder}(t', 1)$. Let u'' denote $\mathit{bidder}(t', i - 1)$. Since u' belongs to $\mathit{ready}(C')$, Lemmas 12 and 13 imply that u'' is not matched in any greedy MWM of A'. Since U' is contained in U_2, Lemma 4 implies that u'' is not matched in any greedy MWM of A_2. Let $C_2 = (A_2, B)$ denote the final configuration of execution X_2; thus $\mathit{ready}(C_2)$ is empty and $|\mathit{bidders}(C_2, t')| = i - 1$. By Lemma 12, we conclude that $\mathit{greedy}(A_2, \mathit{priority}(t')) = 0$, and hence that u'' is contained in $\mathit{ready}(C_2)$, a contradiction. □

For any IUAP B, we define $uap(B)$ as the unique (by Lemma 14) UAP returned by any execution of Algorithm 1 on input B.

We can use the modified incremental Hungarian step of Section 2.3 in each iteration of the loop of Algorithm 1 to maintain UAP A, and a greedy MWM of A, as follows: we maintain dual variables (a price for each item) and a residual graph; the initial greedy MWM is the empty matching; when a bidder u is added to A at line 4, we perform an incremental Hungarian step to process u to update the greedy MWM, the prices, and the residual graph. Since we maintain a greedy MWM of A at each iteration of the loop, it is easy to see that identifying a bidder in $ready(C)$ (or determining that this set is empty) takes $O(|V|)$ time. Thus the whole algorithm can be implemented in $O(|bidders(B)| \cdot |V|^2)$ time.

We now present a lemma that is used in Section 4 to establish weak stability (Lemmas 27, 28, and 29) and Pareto-optimality (Lemma 30).

Lemma 15. Let $B = (T, V)$ be an IUAP, let (σ, z) be a multibidder that belongs to T, let $uap(B)$ be (U, V), and let M be a greedy MWM of the UAP (U, V). Then the following claims hold: (1) if $\sigma(k)$ is matched in M for some k, then $\sigma(k') \in U$ if and only if $1 \leq k' \leq k$; (2) if $\sigma(k)$ is not matched in M for any k, then $\sigma(k) \in U$ for $1 \leq k \leq |\sigma|$.

Proof. Since $prefix(A, B)$ and $tail(C)$ hold at the end of Algorithm 1 by Lemma 12, the first claim follows. Since $ready(C)$ is empty at the end of Algorithm 1, the second claim follows. □

3.2 Threshold of an Item

In this section, we define the threshold of an item in an IUAP and we establish Lemma 18, which plays a key role in establishing our strategyproofness results. We start with some useful definitions.

For any IUAP B, Lemmas 12 and 13 imply that $unique(uap(B))$ holds, and thus that every greedy MWM of $uap(B)$ matches the same set of bidders. We define this set of matched bidders as $winners(B)$. For any IUAP B, we define $losers(B)$ as $U \setminus winners(B)$ where (U, V) is $uap(B)$.

Let $B = (T, V)$ be an IUAP and let $u = (\alpha, \beta, z)$ be a bidder for V. Then we define the IUAP $B + u$ as follows: if T contains a multibidder t of the form (σ, z) for some sequence of bidders σ, then we define $B + u$ as $(T - t + t', V)$ where $t' = (\sigma', z)$ and σ' is the sequence of bidders obtained by appending u to σ; otherwise, we define $B + u$ as $(T + t, V)$ where $t = (\langle u \rangle, z)$.

Lemma 16. Let $B = (T, V)$ and $B' = B + u$ be IUAPs. Then $losers(B) \subseteq losers(B')$.

Proof. Let u' be a bidder in $losers(B)$. Thus u' is not matched in any greedy MWM of $uap(B)$. Using Lemma 14, it is easy to see that $uap(B')$ extends $uap(B)$. Thus Lemma 4 implies that u' is not matched in any greedy MWM of $uap(B')$, and hence that u' belongs to $losers(B')$. □

Lemma 17. Let $B = (T, V)$ be an IUAP and let v be an item in V. For $i \in \{1, 2\}$, let $B_i = B + u_i$ be an IUAP where $u_i = (\alpha_i, \{(v, x_i)\}, z_i)$. Let $A_1 = (U_1, V)$ denote $uap(B_1)$ and let $A_2 = (U_2, V)$ denote $uap(B_2)$. Assume that $\alpha_1 \neq \alpha_2$, $z_1 \neq z_2$, and u_1 belongs to $winners(B_1)$. Then the following claims hold: if u_2 belongs to $winners(B_2)$ then $U_1 - u_1 = U_2 - u_2$; if u_2 belongs to $losers(B_2)$ then $U_1 - u_1$ contains $U_2 - u_2$.

Proof. Let B_3 denote the IUAP $B_1 + u_2$, which is equal to $B_2 + u_1$. For the first claim, assume that u_2 belongs to $winners(B_2)$. Using Lemma 14, it is straightforward to argue that $uap(B_3)$ is equal to $A_1 + u_2 = (U_1 + u_2, V)$ and is also equal to $A_2 + u_1 = (U_2 + u_1, V)$. Since u_1 belongs to U_1 and u_2 belongs to U_2, we conclude that $U_1 - u_1 = U_2 - u_2$, as required.

For the second claim, assume that u_2 belongs to $losers(B_2)$. Suppose $(x_1, z_1) < (x_2, z_2)$. Then Lemmas 3 and 14 imply that u_2 belongs to $winners(B_3)$. Since u_2 belongs to $losers(B_2)$, Lemma 16 implies that u_2 belongs to $losers(B_2 + u_1) = losers(B_3)$, a contradiction. Since $z_1 \neq z_2$, we conclude that $(x_1, z_1) > (x_2, z_2)$. Then, Lemma 14 implies that $uap(B_3) = uap(B_1) + u_2 = (U_1 + u_2, V)$. Since Lemma 14 also implies that $uap(B_3)$ extends $uap(B_2)$, it follows that $U_1 + u_2$ contains U_2, and hence that U_1 contains $U_2 - u_2$. Since u_1 does not belong to U_2, we conclude that $U_1 - u_1$ contains $U_2 - u_2$, as required. □

We are now ready to define the threshold of an item in an IUAP, and to state Lemma 18. In Section 4, Lemma 18 plays an important role in establishing that our SMIW mechanism is strategyproof (Lemma 32). The proof of Lemma 18 is provided in Section 3.2.1.

Let $B = (T, V)$ be an IUAP and let v be an item in V. By Lemma 17, there is a unique subset U of $bidders(B)$ such that the following condition holds: for any IUAP $B' = B + u$ where u is of the form $(\alpha, \{(v, x)\}, z)$ and u belongs to $winners(B')$, $uap(B')$ is equal to $(U + u, V)$. We define $uap(B, v)$ as the UAP (U, V), and we define $threshold(B, v)$ as $threshold(uap(B, v), v)$.

Lemma 18. Let $B = (T, V)$ be an IUAP, let $t = (\sigma, z)$ be a multibidder that belongs to T, and let B' denote the IUAP $(T - t, V)$. Suppose that $(\sigma(k), v)$ is matched in some greedy MWM of $uap(B)$ for some k. Then

$$(w(\sigma(k), v), z) \geq threshold(B', v). \tag{1}$$

Furthermore, for each k' and v' such that $1 \leq k' < k$ and v' belongs to $items(\sigma(k'))$, we have

$$(w(\sigma(k'), v'), z) < threshold(B', v'). \tag{2}$$

3.2.1 Proof of Lemma 18

The purpose of this section is to prove Lemma 18. We do so by establishing a stronger result, namely Lemma 26 below. We start with a useful definition.

For any IUAP B, we define $priorities(B)$ as $\{z \mid u \in winners(B) \text{ and } priority(u) = z\}$.

Lemma 19. Let $B = (T, V)$ and $B' = B + u = (T', V)$ be IUAPs, let Z denote $priorities(B)$, let Z' denote $priorities(B')$, and let z denote $priority(u)$. Then $|Z'| \geq |Z|$ and $Z' \subseteq Z + z$.

Proof. Consider running Algorithm 1 on input B', where we avoid selecting bidder u from $ready(C)$ unless it is the only bidder in $ready(C)$. (By Lemma 14, the final output is the same regardless of which bidder is selected from $ready(C)$ at each iteration.) If u never enters $ready(C)$, then $uap(B') = uap(B)$, and so $Z' = Z$, and the claim of the lemma holds.

Now suppose that u enters $ready(C)$ at some point. Let $A = (U, V)$ denote the UAP at the start of the iteration in which u is selected from $ready(C)$. Then A is equal to $uap(B)$, and we deduce that $uap(B')$ extends $uap(B)$. Lemma 11 implies that every greedy MWM of $A = uap(B)$ (resp., $uap(B')$) matches exactly one bidder of each priority in Z (resp., Z'). Then, since $uap(B')$ extends $uap(B)$, Lemma 9 implies that $|Z'| \geq |Z|$. Furthermore, letting U' denote the set of all bidders u' in $bidders(B)$ such that $priority(u')$ does not belong to $Z + z$, we deduce that U' is contained in $losers(B) = U \setminus winners(B)$. Then Lemma 16 implies that no bidder in U' is matched in any greedy MWM of $uap(B')$, and thus $Z' \subseteq Z + z$. □

Lemma 20. Let $A = (U, V)$ and $A' = A + u$ be UAPs, and let v be an item in V. Then $\mathit{threshold}(A, v) \leq \mathit{threshold}(A', v)$.

Proof. Assume for the sake of contradiction that $\mathit{threshold}(A, v) > \mathit{threshold}(A', v)$. Then there exists a bidder u' such that u' does not belong to $U + u$, $\mathit{bid}(u') = \{(v, x)\}$, $\mathit{priority}(u') = z$, and

$$\mathit{threshold}(A', v) < (x, z) < \mathit{threshold}(A, v).$$

Since $(x, z) < \mathit{threshold}(A, v)$, Lemma 10 implies that u' is not matched in any greedy MWM of $A + u'$. Thus Lemma 4 implies that u' is not matched in any greedy MWM of $A' + u'$. On the other hand, since $\mathit{threshold}(A', v) < (x, z)$, Lemma 10 implies that u' is matched in every greedy MWM of $A' + u'$, a contradiction. □

Lemma 21. Let $B = (T, V)$ and $B' = B + u$ be IUAPs where $u = (\alpha, \{(v, x)\}, z)$, v is an item in V, and z does not belong to $\mathit{priorities}(B)$. If u belongs to $\mathit{winners}(B')$, then $(x, z) > \mathit{threshold}(B, v)$. If u belongs to $\mathit{losers}(B')$, then $(x, z) < \mathit{threshold}(B, v)$.

Proof. First, assume that u belongs to $\mathit{winners}(B')$. Thus u is matched in every greedy MWM of $\mathit{uap}(B')$, which is equal to $\mathit{uap}(B, v) + u$ by definition. Lemma 10 implies that $(x, z) > \mathit{threshold}(\mathit{uap}(B, v), v) = \mathit{threshold}(B, v)$, as required.

Now assume that u belongs to $\mathit{losers}(B')$. Thus u is not matched in any greedy MWM of $\mathit{uap}(B')$. Define U so that $\mathit{uap}(B') = (U + u, V)$, and let A denote the UAP (U, V). Lemma 10 implies that $(x, z) < \mathit{threshold}(A, v)$. Lemma 17 implies that $\mathit{uap}(B, v) + u$ extends $\mathit{uap}(B')$, and hence that $\mathit{uap}(B, v)$ extends A. Lemma 20 therefore implies that

$$\mathit{threshold}(A, v) \leq \mathit{threshold}(\mathit{uap}(B, v), v) = \mathit{threshold}(B, v).$$

Thus $(x, z) < \mathit{threshold}(B, v)$, as required. □

Lemma 22. Let $B = (T, V)$ and $B' = B + u$ be IUAPs, and let v be an item in V. Then $\mathit{threshold}(B, v) \leq \mathit{threshold}(B', v)$.

Proof. Let (x, z) denote $\mathit{threshold}(B, v)$, let (x', z') denote $\mathit{threshold}(B', v)$, and assume for the sake of contradiction that $(x, z) > (x', z')$.

Let u' be a bidder $(\alpha, \{(v, x)\}, z'')$ such that z'' does not belong to $\mathit{priorities}(B) + \mathit{priority}(u)$, $z > z''$, and $(x, z'') > (x', z')$. Let B'' denote $B + u'$ and let B''' denote $B' + u'$. Since z'' does not belong to $\mathit{priorities}(B)$, we deduce that u' belongs to either $\mathit{winners}(B'')$ or $\mathit{losers}(B'')$. Then, by Lemma 21, u' belongs to $\mathit{losers}(B'')$, and hence by Lemma 16, u' belongs to $\mathit{losers}(B''')$. On the other hand, since z'' does not belong to $\mathit{priorities}(B) + \mathit{priority}(u)$, Lemma 19 implies that z'' does not belong to $\mathit{priorities}(B')$, and we deduce that u' belongs to either $\mathit{winners}(B''')$ or $\mathit{losers}(B''')$. Then, Lemma 21 implies that u' belongs to $\mathit{winners}(B''')$, a contradiction. □

Lemma 23. Let $B = (T, V)$ and $B' = B + u$ be IUAPs where u belongs to $\mathit{losers}(B')$, and let v be an item in V. Then $\mathit{threshold}(B', v) = \mathit{threshold}(B, v)$.

Proof. Suppose not. Then by Lemma 22, we have $\mathit{threshold}(B, v) < \mathit{threshold}(B', v)$. Let z denote $\mathit{priority}(u)$. Since $B' = B + u$ and u belongs to $\mathit{losers}(B')$, we deduce that z does not belong to $\mathit{priorities}(B)$. Since u belongs to $\mathit{losers}(B')$, we deduce that z does not belong to $\mathit{priorities}(B')$. Hence Lemma 19 implies that $\mathit{priorities}(B') = \mathit{priorities}(B)$.

Let B'' denote $B + u'$ where $u' = (\alpha, \{(v, x')\}, z')$ is a bidder such that z' does not belong to $\mathit{priorities}(B) + z$ and $\mathit{threshold}(B, v) < (x', z') < \mathit{threshold}(B', v)$.

Let B''' denote $B' + u'$. Since z' does not belong to $\mathit{priorities}(B) + z$, Lemma 19 implies that z' does not belong to $\mathit{priorities}(B')$, and we deduce that u' belongs to either $\mathit{winners}(B''')$ or $\mathit{losers}(B''')$. Since $(x', z') < \mathit{threshold}(B', v)$, Lemma 21 implies that u' belongs to $\mathit{losers}(B''')$. Hence Lemma 19 implies that $\mathit{priorities}(B''') = \mathit{priorities}(B')$. Since we have established above that $\mathit{priorities}(B') = \mathit{priorities}(B)$, we deduce that $\mathit{priorities}(B''') = \mathit{priorities}(B)$.

Since z' does not belong to $\mathit{priorities}(B)$, we deduce that u' belongs to either $\mathit{winners}(B'')$ or $\mathit{losers}(B'')$. Since $(x', z') > \mathit{threshold}(B, v)$, Lemma 21 implies that u' belongs to $\mathit{winners}(B'')$ and hence z' belongs to $\mathit{priorities}(B'')$. We consider two cases.

Case 1: $|\mathit{priorities}(B'')| \leq |\mathit{priorities}(B)|$. Lemma 19 implies that there exists a real z'' in $\mathit{priorities}(B)$ that does not belong to $\mathit{priorities}(B'')$. Since z does not belong to $\mathit{priorities}(B)$, we have $z \neq z''$. Since $B''' = B'' + u$ and $z \neq z''$, Lemma 19 implies that z'' does not belong to $\mathit{priorities}(B''')$, a contradiction since $\mathit{priorities}(B''') = \mathit{priorities}(B)$.

Case 2: $|\mathit{priorities}(B'')| > |\mathit{priorities}(B)|$. Since $\mathit{priorities}(B''') = \mathit{priorities}(B)$, we deduce that $|\mathit{priorities}(B'')| > |\mathit{priorities}(B''')|$. Since $B''' = B'' + u$, Lemma 19 implies that $|\mathit{priorities}(B''')| \geq |\mathit{priorities}(B'')|$, a contradiction. □

Lemma 24. Let $B = (T, V)$ and $B' = B + u$ be IUAPs where $u = (\alpha, \beta, z)$ and z does not belong to $\mathit{priorities}(B)$, and let v be an item in V. Assume that (v, x) belongs to β, and that $\mathit{threshold}(B, v) < (x, z)$. Then u belongs to $\mathit{winners}(B')$.

Proof. Suppose not. Let $A' = (U', V)$ denote $\mathit{uap}(B')$. Since z does not belong to $\mathit{priorities}(B)$, we deduce that u belongs to U'. Thus u belongs to $U' \setminus \mathit{winners}(B') = \mathit{losers}(B')$, and so $\mathit{threshold}(B', v) = \mathit{threshold}(B, v)$ by Lemma 23.

Let B'' denote $B' + u'$ where $u' = (\alpha, \{(v, x)\}, z')$ is a bidder such that z' does not belong to $\mathit{priorities}(B) + z$, $\mathit{threshold}(B, v) < (x, z')$, and $z' < z$. Since z' does not belong to $\mathit{priorities}(B) + z$, we deduce that u' belongs to either $\mathit{winners}(B'')$ or $\mathit{losers}(B'')$. Then, by Lemma 21, u' belongs to $\mathit{winners}(B'')$. Let $A'' = (U'', V)$ denote $\mathit{uap}(B'')$, and let M be a greedy MWM of A''. Since u' belongs to $\mathit{winners}(B'')$, the edge (u', v) belongs to M. Since u belongs to $\mathit{losers}(B')$, Lemma 16 implies that u belongs to $\mathit{losers}(B'')$, and hence that u is unmatched in M. By Lemma 3, we find that $(x, z) < (x, z')$ and hence $z < z'$, a contradiction. □

Lemma 25. Let $B = (T, V)$ and $B_0 = B + u$ be IUAPs where $u = (\alpha, \beta, z)$, z does not belong to $\mathit{priorities}(B)$, and $\beta = \{(v_1, x_1), \ldots, (v_k, x_k)\}$. Assume that $(x_i, z) < \mathit{threshold}(B, v_i)$ holds for all i such that $1 \leq i \leq k$. Then u belongs to $\mathit{losers}(B_0)$.

Proof. Suppose not. Since z does not belong to $\mathit{priorities}(B)$, we deduce that u belongs to $\mathit{winners}(B_0)$, and hence that z belongs to $\mathit{priorities}(B_0)$.

For i ranging from 1 to k, let B_i denote the IUAP $B_{i-1} + u_i$ where $u_i = (\alpha_i, \{(v_i, x_i)\}, z_i)$ and z_i is a real number satisfying the following conditions: z_i does not belong to $\mathit{priorities}(B_{i-1})$; $z < z_i$; $(x_i, z_i) < \mathit{threshold}(B, v_i)$. Since z_i does not belong to $\mathit{priorities}(B_{i-1})$, we deduce that u_i belongs to either $\mathit{winners}(B_i)$ or $\mathit{losers}(B_i)$ for $1 \leq i \leq k$. Then, by Lemmas 21 and 22, we deduce that u_i belongs to $\mathit{losers}(B_i)$ for $1 \leq i \leq k$. By repeated application of Lemma 19, we find that $\mathit{priorities}(B_i) = \mathit{priorities}(B_0)$ for $1 \leq i \leq k$, and hence that z belongs to $\mathit{priorities}(B_k)$.

We claim that u belongs to $\mathit{winners}(B_k)$. To prove this claim, let t denote the unique multi-bidder in B_k for which $\mathit{priority}(t) = \mathit{priority}(u)$. Let ℓ denote $|\mathit{bidders}(t)|$, and observe that $u =$

$bidder(t, \ell)$. Furthermore, since z does not belong to $priorities(B)$, we deduce that $bidder(t, i)$ belongs to $losers(B)$ for $1 \leq i < \ell$. By repeated application of Lemma 16, we deduce that $bidder(t, i)$ belongs to $losers(B_k)$ for $1 \leq i < \ell$. Since z belongs to $priorities(B_k)$, the claim follows.

Let M denote a greedy MWM of $uap(B_k)$. Since u belongs to $winners(B_k)$, there is a unique integer i, $1 \leq i \leq k$, such that M contains edge (u, v_i). Let i denote this integer. Since z_i does not belong to $priorities(B_k)$, we know that u_i belongs to $losers(B_k)$ and hence that u_i is not matched in any greedy MWM of $uap(B_k)$. By Lemma 3, we deduce that $(x_i, z_i) < (x_i, z)$. Hence $z_i < z$, contradicting the definition of z_i. □

Lemma 26. Let $B_0 = (T, V)$ be an IUAP, let z be a real that is not equal to the priority of any multibidder in T, let k be a nonnegative integer, and for i ranging from 1 to k, let B_i denote the IUAP $B_{i-1} + u_i$, where $priority(u_i) = z$. Let I denote the set of all integers i in $\{1, \ldots, k\}$ such that there exists an item v in V for which $(w(u_i, v), z) > threshold(B_0, v)$. If I is empty, then z does not belong to $priorities(B_k)$. Otherwise, u_j belongs to $winners(B_k)$, where j denotes the minimum integer in I.

Proof. If I is empty, then by repeated application of Lemmas 23 and 25, we find that u_i belongs to $losers(B_i)$ for $1 \leq i \leq k$. By repeated application of Lemma 16, we deduce that u_i belongs to $losers(B_k)$ for $1 \leq i \leq k$. It follows that z does not belong to $priorities(B_k)$, as required.

Now assume that I is nonempty, and let j denote the minimum integer in I. Arguing as in the preceding paragraph, we find that z does not belong to $priorities(B_{j-1})$. By repeated application of Lemma 23, we deduce that $threshold(B_{j-1}, v) = threshold(B_0, v)$ for all items v in V. Thus Lemma 24 implies that u_j belongs to $winners(B_j)$. Then, since $u_{j+1}, \ldots, u_k$ all have the same priority as u_j, it is easy to argue by Lemma 14 that $uap(B_k) = uap(B_j)$, and hence u_j belongs to $winners(B_k)$, as required. □

Proof of Lemma 18. It is easy to see that the claims of the lemma follow from Lemma 26. □

4 Stable Marriage with Indifferences

The *stable marriage model with incomplete and weak preferences (SMIW)* involves a set P of men and a set Q of women. The preference relation of each man p in P is specified as a binary relation $\succeq_p$ over $Q \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. Similarly, the preference relation of each woman q in Q is specified as a binary relation $\succeq_q$ over $P \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. To allow indifferences, the preference relations are not required to satisfy antisymmetry. We will use $\succ_p$ and $\succ_q$ to denote the asymmetric part of $\succeq_p$ and $\succeq_q$ respectively.

A matching is a function μ from P to $Q \cup \{\varnothing\}$ such that for any woman q in Q, there exists at most one man p in P for which $\mu(p) = q$. Given a matching μ and a woman q in Q, we denote

$$\mu(q) = \begin{cases} p & \text{if } \mu(p) = q \\ \varnothing & \text{if there is no man } p \text{ in } P \text{ such that } \mu(p) = q \end{cases}$$

A matching μ is *individually rational* if for any man p in P and woman q in Q such that $\mu(p) = q$, we have $q \succeq_p \varnothing$ and $p \succeq_q \varnothing$. A pair (p, q') in $P \times Q$ is said to form a *strongly blocking* *pair* for a matching μ if $q' \succ_p \mu(p)$ and $p \succ_{q'} \mu(q')$. A matching is *weakly stable* if it is individually rational and does not admit a strongly blocking pair.

For any matching μ and μ', we say that the binary relation $\mu \succeq \mu'$ holds if for every man p in P and woman q in Q, we have $\mu(p) \succeq_p \mu'(p)$ and $\mu(q) \succeq_q \mu'(q)$. We let $\succ$ denote the asymmetric part of $\succeq$. We say that a matching μ *Pareto-dominates* another matching μ' if $\mu \succ \mu'$. We say that a matching is *Pareto-optimal* if it is not Pareto-dominated by any other matching. A matching is *Pareto-stable* if it is Pareto-optimal and weakly stable.

A *mechanism* is an algorithm that, given $(P, Q, (\succeq_p)_{p \in P}, (\succeq_q)_{q \in Q})$, produces a matching μ. A mechanism is said to be *strategyproof (for the men)* if for any man p in P expressing preference $\succeq'_p$ instead of his true preference $\succeq_p$, we have $\mu(p) \succeq_p \mu'(p)$, where μ and μ' are the matchings produced by the mechanism given $\succeq_p$ and $\succeq'_p$, respectively, when all other inputs are fixed.

By introducing extra men or women who prefer being unmatched to being matched with any potential partner, we may assume without loss of generality that the number of men is equal to the number of women. So, $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$.

4.1 Algorithm

The computation of a matching for SMIW is shown in Algorithm 2. We construct an item for each woman in line 4, and a multibidder for each man in line 13 by examining the tiers of preferences of the men and the utilities of the women. Together with dummy items constructed in line 8, this forms an IUAP, from which we obtain a UAP and a greedy MWM M_0. Using Lemma 15, we argue that for any man p_i, exactly one of the bidders associated with p_i is matched in M_0; see the proof of Lemma 27. Finally, in line 18, we use M_0 to determine the match of a man p_i as follows, where u denotes the unique bidder associated with p_i that is matched in M_0: if u is matched in M_0 to the item corresponding to a woman q_j, then we match p_i to q_j; otherwise, u is matched to a dummy item in M_0, and we leave p_i unmatched.

In Lemma 28, we prove individually rationality by arguing that the dummy items ensure that no man or woman is matched to an unacceptable partner. In Lemma 29, we prove weak stability using the properties of a greedy MWM. In Lemmas 30 and 31, we prove Pareto-optimality by showing that any matching that Pareto-dominates the output matching induces another MWM that contradicts the greediness of the MWM produced by the algorithm. In Lemma 32, we establish two properties of IUAP thresholds that are used to show strategyproofness in Theorem 1.

Lemma 27. Algorithm 2 produces a valid matching.

Proof. First, we show that for any man p_i where $1 \leq i \leq n$, there exists at most one j in $\{1, \ldots, 2n\}$ such that bidder $\sigma_i(k)$ is matched to item v_j in M_0 for some k. For the sake of contradiction, suppose bidder $\sigma_i(k)$ is matched to item v_j and bidder $\sigma_i(k')$ is matched to item $v_{j'}$ in M_0 for some k and k' where $j \neq j'$. By Lemma 15, we have $k \leq k'$ and $k' \leq k$. Therefore, bidder $\sigma_i(k) = \sigma_i(k')$ is matched in M_0 to both v_j and $v_{j'}$, which is a contradiction.

Next, we show that for any man p_i where $1 \leq i \leq n$, there exists at least one j in $\{1, \ldots, 2n\}$ such that bidder $\sigma_i(k)$ is matched to item v_j in M_0 for some k. For the sake of contradiction, suppose bidder $\sigma_i(k)$ is unmatched in M_0 for all k. Let j denote $n + i$ and let k denote $\kappa_i(q_j)$. By Lemma 15, the set U contains bidder $\sigma_i(k)$. Since both bidder $\sigma_i(k)$ and item v_j are unmatched by M_0, adding the pair $(\sigma_i(k), v_j)$ to M_0 gives a matching of (U, V) with the same weight and larger cardinality. This contradicts the fact that M_0 is a greedy MWM of (U, V).

Algorithm 2

Let p_0 denote $\varnothing$.
for all $1 \le j \le n$ **do**
    Convert the preference relation $\succeq_{q_j}$ of woman q_j into utility function $\psi_{q_j} : P \cup \{\varnothing\} \to \mathbb{R}$ that satisfies the followings: $\psi_{q_j}(\varnothing) = 0$; for any i and i' in $\{0, 1, \ldots, n\}$, we have $p_i \succeq_{q_j} p_{i'}$ if and only if $\psi_{q_j}(p_i) \ge \psi_{q_j}(p_{i'})$. This utility assignment should not depend on the preferences of the men.
    Construct an item v_j corresponding to woman q_j.
end for
for all $n < j \le 2n$ **do**
    Let q_j denote $\varnothing$.
    Construct a dummy item v_j corresponding to q_j.
end for
for all $1 \le i \le n$ **do**
    Partition the set $\{1, \ldots, n\} \cup \{n+i\}$ of woman indices into tiers $\tau_i(1), \ldots, \tau_i(K_i)$ according to the preference relation of man p_i, such that for any j in $\tau_i(k)$ and j' in $\tau_i(k')$, we have $q_j \succeq_{p_i} q_{j'}$ if and only if $k \le k'$.
    For j in $\{1, \ldots, n\} \cup \{n + i\}$, denote tier number $\kappa_i(q_j)$ as the unique k such that j in $\tau_i(k)$.
    Construct a multibidder $t_i = (\sigma_i, z_i)$ with priority $z_i = i$ corresponding to man p_i. The multibidder t_i has K_i bidders. For each bidder $\sigma_i(k)$ we define $\mathit{items}(\sigma_i(k))$ as $\{v_j \mid j \in \tau_i(k)\}$ and $w(\sigma_i(k), q_j)$ as $\psi_{q_j}(p_i)$, where $\psi_{q_{n+i}}(p_i)$ is defined to be 0.
end for
$(T, V) = (\{t_i \mid 1 \le i \le n\}, \{v_j \mid 1 \le j \le 2n\})$.
$(U, V) = uap(T, V)$.
Compute a greedy MWM M_0 of UAP (U, V) as described in Section 2.3.
Output matching μ such that for all $1 \le i \le n$ and $1 \le j \le 2n$, we have $\mu(p_i) = q_j$ if and only if $\sigma_i(k)$ is matched to item v_j in M_0 for some k.

This shows that $\mu(p_i)$ is well-defined for all men p_i where $1 \le i \le n$. Furthermore, since each item v_j where $1 \le j \le n$ is matched to at most one bidder in M_0, each woman q_j is matched to at most one man p_i in μ where $1 \le i \le n$. Hence, μ is a valid matching. □

Lemma 28. Algorithm 2 produces an individually rational matching.

Proof. We have shown in Lemma 27 that μ is a valid matching. Consider man p_i and woman q_j such that $\mu(p_i) = q_j$, where i and j belong to $\{1, \ldots, n\}$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{n+i})$. It suffices to show that $k \le k'$ and $\psi_{q_j}(p_i) \ge 0$.

Since $\mu(p_i) = q_j$, bidder $\sigma_i(k)$ is matched to item v_j in M_0. Since M_0 is an MWM, we have $\psi_{q_j}(p_i) = w(\sigma_i(k), v_j) \ge 0$.

It remains to show that $k \le k'$. For the sake of contradiction, suppose $k > k'$. Since bidder $\sigma_i(k)$ is matched to item v_j in M_0, by Lemma 15 the set U contains bidder $\sigma_i(k')$. Since bidder $\sigma_i(k')$ is not matched in M_0, the dummy item v_{n+i} is also not matched in M_0. Hence, adding the pair $(\sigma_i(k'), v_{n+i})$ to M_0 gives a matching in (U, V) with the same weight and larger cardinality. This contradicts the fact that M_0 is a greedy MWM of (U, V). □

Lemma 29. Algorithm 2 produces a weakly stable matching.

Proof. By Lemma 28, it remains only to show that μ does not admit a strongly blocking pair. Consider man p_i and woman $q_{j'}$, where i and j' belong to $\{1, \ldots, n\}$. We want to show that $(p_i, q_{j'})$ does not form a strongly blocking pair. Let q_j denote $\mu(p_i)$ and let $p_{i'}$ denote $\mu(q_{j'})$, where j belongs to $\{1, \ldots, n\} \cup \{n + i\}$ and i' belongs to $\{0, 1, \ldots, n\}$. It suffices to show that either $\kappa_i(q_j) \le \kappa_i(q_{j'})$ or $\psi_{q_{j'}}(p_{i'}) \ge \psi_{q_{j'}}(p_i)$. For the sake of contradiction, suppose $\kappa_i(q_j) > \kappa_i(q_{j'})$ and $\psi_{q_{j'}}(p_{i'}) < \psi_{q_{j'}}(p_i)$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{j'})$. Since $\sigma_i(k)$ is matched in M_0 and $k' < k$, Lemma 15 implies that the set U contains bidder $\sigma_i(k')$ and that $\sigma_i(k')$ is unmatched in M_0. We consider two cases.

Case 1: $i' = 0$. Then $\psi_{q_{j'}}(p_i) > \psi_{q_{j'}}(p_{i'}) = 0$. Since neither bidder $\sigma_i(k')$ nor item $v_{j'}$ is matched in M_0, adding the pair $(\sigma_i(k'), v_{j'})$ to M_0 gives a matching of (U, V) with a larger weight. This contradicts the fact that M_0 is an MWM of (U, V).

Case 2: $i' \neq 0$. Since $p_{i'} = \mu(q_{j'})$, there exists k'' such that bidder $\sigma_{i'}(k'')$ is matched to $v_{j'}$ in M_0. Since $\sigma_i(k')$ is unmatched in M_0, the matching $M_0 - (\sigma_{i'}(k''), v_{j'}) + (\sigma_i(k'), v_{j'})$ is a matching of (U, V) with weight $w(M_0) - \psi_{q_{j'}}(p_{i'}) + \psi_{q_{j'}}(p_i)$, which is greater than $w(M_0)$. This contradicts the fact that M_0 is an MWM of (U, V). □

Lemma 30. Let μ be the matching produced by Algorithm 2 and let μ' be a matching such that $\mu'(p) \succeq_p \mu(p)$ for every man p in P and

$$\sum_{q \in Q} \psi_q(\mu'(q)) \ge \sum_{q \in Q} \psi_q(\mu(q)).$$

Then $\mu(p) \succeq_p \mu'(p)$ for every man p in P and

$$\sum_{q \in Q} \psi_q(\mu'(q)) = \sum_{q \in Q} \psi_q(\mu(q)).$$

Proof. For any i such that $1 \le i \le n$, let k_i denote $\kappa_i(\mu(p_i))$ and let k'_i denote $\kappa_i(\mu'(p_i))$.

Below we use μ' to construct an MWM M'_0 of (U, V). We give the construction of M'_0 first, and then argue that M'_0 is an MWM of (U, V). Let M'_0 denote the set of bidder-item pairs $(\sigma_i(k'_i), v_j)$ such that $\mu'(p_i) = q_j$ where i in $\{1, \ldots, n\}$ and j in $\{1, \ldots, n\} \cup \{n + i\}$. It is easy to see that M'_0 is a valid matching. Notice that for any $1 \le i \le n$, since $\mu'(p_i) \succeq_{p_i} \mu(p_i)$, we have $k'_i \le k_i$. So, by Lemma 15, the set U contains all bidders $\sigma_i(k'_i)$. Hence, M'_0 is a matching of (U, V). Furthermore, it is easy to see that

$$w(M'_0) = \sum_{1 \le j \le n} \psi_{q_j}(\mu'(q_j)) \ge \sum_{1 \le j \le n} \psi_{q_j}(\mu(q_j)) = w(M_0).$$

Thus M'_0 is an MWM of (U, V), and we have

$$\sum_{1 \le j \le n} \psi_{q_j}(\mu'(q_j)) = \sum_{1 \le j \le n} \psi_{q_j}(\mu(q_j)).$$

Furthermore, M'_0 is an MCMWM of (U, V) because both M'_0 and M_0 have cardinality equal to n. Also, M'_0 is a greedy MWM of (U, V), because both M'_0 and M_0 have priorities equal to $\sum_{1 \le i \le n} z_i$. Hence, for each $1 \le i \le n$, we have $k_i \le k'_i$ by Lemma 15. Thus, $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for all $1 \le i \le n$. □

Lemma 31. Let μ be the matching produced by Algorithm 2 and μ' be a matching such that $\mu' \succeq \mu$. Then, $\mu \succeq \mu'$.

Proof. Since $\mu' \succeq \mu$, we have $\mu'(p_i) \succeq_{p_i} \mu(p_i)$ and $\psi_{q_j}(\mu'(q_j)) \geq \psi_{q_j}(\mu(q_j))$ for every i and j in $\{1, \ldots, n\}$. So, by Lemma 30, we have $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for every i in $\{1, \ldots, n\}$ and

$$\sum_{1 \leq j \leq n} \psi_{q_j}(\mu'(q_j)) = \sum_{1 \leq j \leq n} \psi_{q_j}(\mu(q_j)).$$

Therefore, $\psi_{q_j}(\mu'(q_j)) = \psi_{q_j}(\mu(q_j))$ for every j in $\{1, \ldots, n\}$. This shows that $\mu \succeq \mu'$. □

Lemma 32. Consider Algorithm 2. Suppose $\mu(p_i) = q_j$, where $1 \leq i \leq n$ and j belongs to $\{1, \ldots, n\} \cup \{n + i\}$. Then, we have

$$(\psi_{q_j}(p_i), i) \geq \textit{threshold}((T - t_i, V), v_j). \tag{3}$$

Furthermore, for all j' in $\{1, \ldots, n\} \cup \{n + i\}$ such that $\kappa_i(q_{j'}) < \kappa_i(q_j)$, we have

$$(\psi_{q_{j'}}(p_i), i) < \textit{threshold}((T - t_i, V), v_{j'}). \tag{4}$$

Proof. Let k denote $\kappa_i(q_j)$. Since $\mu(p_i) = q_j$, we know that bidder $\sigma_i(k)$ is matched to item v_j in M_0. So, inequality (1) of Lemma 18 implies inequality (3), because $w(\sigma_i(k), v_j) = \psi_{q_j}(p_i)$ and $z_i = i$.

Now, suppose $\kappa_i(q_{j'}) < \kappa_i(q_j)$. Let k' denote $\kappa_i(q_{j'})$. Since $k' < k$, inequality (2) of Lemma 18 implies inequality (4), because $w(\sigma_i(k'), v_{j'}) = \psi_{q_{j'}}(p_i)$ and $z_i = i$. □

Theorem 1. Algorithm 2 is a strategyproof Pareto-stable mechanism for the stable marriage problem with incomplete and weak preferences (for any fixed choice of utility assignment).

Proof. We have shown in Lemma 29 that the algorithm produces a weakly stable matching. Moreover, Lemma 31 shows that the weakly stable matching produced is not Pareto-dominated by any other matching. Hence, the algorithm produces a Pareto-stable matching. It remains to show that the algorithm is a strategyproof mechanism.

Suppose man p_i expresses $\succeq'_{p_i}$ instead of his true preference relation $\succeq_{p_i}$, where $1 \leq i \leq n$. Let μ and μ' be the resulting matchings given $\succeq_{p_i}$ and $\succeq'_{p_i}$, respectively. Let q_j denote $\mu(p_i)$ and let $q_{j'}$ denote $\mu'(p_i)$, where j and j' belong to $\{1, \ldots, n\} \cup \{n + i\}$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{j'})$, where $\kappa_i(\cdot)$ denotes the tier number with respect to $\succeq_{p_i}$. It suffices to show that $k \leq k'$. For the sake of contradiction, suppose $k > k'$.

Let (T, V) be the IUAP, let t_i be the multibidder corresponding to man p_i, and let $v_{j'}$ be the item corresponding to woman $q_{j'}$ constructed in the algorithm given input $\succeq_{p_i}$. Since $\mu(p_i) = q_j$, by inequality (4) of Lemma 32, we have

$$(\psi_{q_{j'}}(p_i), i) < \textit{threshold}((T - t_i, V), v_{j'}).$$

Now, consider the behavior of the algorithm when preference relation $\succeq_{p_i}$ is replaced with $\succeq'_{p_i}$. Let (T', V') be the IUAP, let t'_i be the multibidder corresponding to man p_i, and let $v'_{j'}$ be the item corresponding to woman $q_{j'}$ constructed in the algorithm given input $\succeq'_{p_i}$. Since $\mu'(p_i) = q_{j'}$, by inequality (3) of Lemma 32, we have

$$(\psi_{q_{j'}}(p_i), i) \geq \textit{threshold}((T' - t'_i, V'), v'_{j'}).$$

Notice that in Algorithm 2, the only part of the IUAP instance that depends on the preferences of man p_i is the multibidder corresponding to man p_i. In particular, we have $T - t_i = T' - t'_i$, $V = V'$, and $v_{j'} = v'_{j'}$. Hence, we get

$$\begin{aligned}(\psi_{q_{j'}}(p_i), i) &< \mathit{threshold}((T - t_i, V), v_{j'}) \\ &= \mathit{threshold}((T' - t'_i, V'), v'_{j'}) \\ &\leq (\psi_{q_{j'}}(p_i), i),\end{aligned}$$

which is a contradiction. □

5 College Admissions with Indifferences

The *college admissions model with weak preferences (CAW)* involves a set P of students and a set Q of colleges. The preference relation of each student p in P is specified as a binary relation $\succeq_p$ over $Q \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. The preference relation of each college q in Q over individual students is specified as a binary relation $\succeq_q$ over $P \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. Each college q in Q has an associated integer capacity $c_q > 0$. We will use $\succ_p$ and $\succ_q$ to denote the asymmetric parts of $\succeq_p$ and $\succeq_q$, respectively.

The colleges' preference relation over individual students can be extended to group preference using responsiveness. We say that a transitive and reflexive relation $\succeq'_q$ over the power set 2^P is *responsive to the preference relation* $\succeq_q$ if the following conditions hold: for any $S \subseteq P$ and p in $P \setminus S$, we have $p \succeq_q \varnothing$ if and only if $S \cup \{p\} \succeq'_q S$; for any $S \subseteq P$ and any p and p' in $P \setminus S$, we have $p \succeq_q p'$ if and only if $S \cup \{p\} \succeq'_q S \cup \{p'\}$. Furthermore, we say that a relation $\succeq'_q$ is *minimally responsive to the preference relation* $\succeq_q$ if it is responsive to the preference relation $\succeq_q$ and does not strictly contain another relation that is responsive to the preference relation $\succeq_q$.

A *(capacitated) matching* is a function μ from P to $Q \cup \{\varnothing\}$ such that for any college q in Q, there exists at most c_q students p in P for which $\mu(p) = q$. Given a matching μ and a college q in Q, we let $\mu(q)$ denote $\{p \in P \mid \mu(p) = q\}$.

A matching μ is *individually rational* if for any student p in P and college q in Q such that $\mu(p) = q$, we have $q \succeq_p \varnothing$ and $p \succeq_q \varnothing$. A pair (p', q) in $P \times Q$ is said to form a *strongly blocking pair* for a matching μ if $q \succ_{p'} \mu(p')$ and at least one of the following two conditions holds: (1) there exists a student p in P such that $\mu(p) = q$ and $p' \succ_q p$; (2) $|\mu(q)| < c_q$ and $p' \succ_q \varnothing$. A matching is *weakly stable* if it is individually rational and does not admit a strongly blocking pair.

Let $\succeq'_q$ be the group preference associated with college q in Q. For any matching μ and μ', we say that the binary relation $\mu \succeq \mu'$ holds if for every student p in P and college q in Q, we have $\mu(p) \succeq_p \mu'(p)$ and $\mu(q) \succeq'_q \mu'(q)$. We let $\succ$ denote the asymmetric part of $\succeq$. We say that a matching μ *Pareto-dominates* another matching μ' if $\mu \succ \mu'$. We say that a matching is *Pareto-optimal* if it is not Pareto-dominated by any other matching. A matching is *Pareto-stable* if it is Pareto-optimal and weakly stable.

A *mechanism* is an algorithm that, given $(P, Q, (\succeq_p)_{p \in P}, (\succeq_q)_{q \in Q}, (c_q)_{q \in Q})$, produces a matching μ. A mechanism is said to be *strategyproof (for the students)* if for any student p in P expressing preference $\succeq'_p$ instead of their true preference $\succeq_p$, we have $\mu(p) \succeq_p \mu'(p)$, where μ and μ' are the matchings produced by the mechanism given $\succeq_p$ and $\succeq'_p$, respectively, when all other inputs are fixed.

Without loss of generality, we may assume that the number of students equals the total capacity of the colleges. So, $P = \{p_i\}_{1 \le i \le |P|}$ and $Q = \{q_j\}_{1 \le j \le |Q|}$ such that $|P| = \sum_{1 \le j \le |Q|} c_{q_j}$.

5.1 Algorithm

The computation of a matching for CAW is shown in Algorithm 3. We transform each student to a man in line 1, and each slot of a college to a woman in line 2. This forms an SMIW. Using this SMIW, we produce a matching by invoking Algorithm 2 in lines 8 and 9.

Algorithm 3

1: For each $1 \le i \le |P|$, construct man p'_i corresponding to student p_i.
2: For each $1 \le j \le |Q|$, construct women $q'_{j1}, \ldots, q'_{jc}$ corresponding to college q_j with capacity $c = c_{q_j}$.
3: $(P', Q') = (\{p'_i \mid 1 \le i \le |P|\}, \{q'_{jk} \mid 1 \le j \le |Q| \text{ and } 1 \le k \le c_{q_j}\})$.
4: Let p_0 denote $\varnothing$. Let p'_0 denote $\varnothing$.
5: Let q_0 denote $\varnothing$. Let q'_{00} denote $\varnothing$.
6: For each $1 \le i \le |P|$, define the preference relation $\succeq_{p'_i}$ over $Q' \cup \{q'_{00}\}$ for man p'_i using the preference relation of his corresponding student, such that $q'_{jk} \succeq_{p'_i} q'_{j'k'}$ if and only if $q_j \succeq_{p_i} q_{j'}$.
7: For each $1 \le j \le |Q|$ and $1 \le k \le c_{q_j}$, define the preference relation $\succeq_{q'_{jk}}$ over $P' \cup \{p'_0\}$ for woman q'_{jk} using the preference relation of her corresponding college, such that $p'_i \succeq_{q'_{jk}} p'_{i'}$ if and only if $p_i \succeq_{q_j} p_{i'}$.
8: Compute matching μ_0 for SMIW $(P', Q', (\succeq_{p'})_{p' \in P'}, (\succeq_{q'})_{q' \in Q'})$ using Algorithm 2, where we require the utility functions associated with the same college to be the same.
9: Output matching μ, such that for all $1 \le i \le |P|$ and $0 \le j \le |Q|$, we have $\mu(p_i) = q_j$ if and only if $\mu_0(p'_i) = q'_{jk}$ for some k.

Lemma 33. Algorithm 3 produces an individually rational matching.

Proof. It is easy to see that μ satisfies the capacity constraints because each college q_j is associated with c_{q_j} women q'_{jk} and each woman can be matched with at most one man in μ_0 by Lemma 27.

The individual rationality of μ follows from the individual rationality of μ_0. Let p_i in P and q_j in Q such that $\mu(p_i) = q_j$. Then $\mu_0(p'_i) = q'_{jk}$ for some k. By Lemma 28, we have $q'_{jk} \succeq_{p'_i} \varnothing$ and $p'_i \succeq_{q'_{jk}} \varnothing$. Hence, $q_j \succeq_{p_i} \varnothing$ and $p_i \succeq_{q_j} \varnothing$. □

Lemma 34. Algorithm 3 produces a weakly stable matching.

Proof. By Lemma 33, it remains only to show that μ does not admit a strongly blocking pair. Consider student $p_{i'}$ in P and college q_j in Q. In what follows, we use the weak stability of μ_0 to show that $(p_{i'}, q_j)$ does not form a strongly blocking pair.

Let $q'_{j'k'}$ denote $\mu_0(p'_{i'})$. It is possible that $q'_{j'k'} = \varnothing$, in which case $j' = k' = 0$. For $1 \le k \le c_{q_j}$, let p'_{i_k} denote $\mu_0(q'_{jk})$, where p'_{i_k} belongs to $P' \cup \{p'_0\}$. By Lemma 29, for any $1 \le k \le c_{q_j}$, either $q'_{j'k'} \succeq_{p'_{i'}} q'_{jk}$ or $p'_{i_k} \succeq_{q'_{jk}} p'_{i'}$, for otherwise $(p'_{i'}, q'_{jk})$ forms a strongly blocking pair.

Suppose $q'_{j'k'} \succeq_{p'_{i'}} q'_{jk}$ for some $1 \leq k \leq c_{q_j}$. Then $q_{j'} \succeq_{p_{i'}} q_j$, and hence $(p_{i'}, q_j)$ does not form a strongly blocking pair.

Otherwise, $p'_{i_k} \succeq_{q'_{jk}} p'_{i'}$ for all $1 \leq k \leq c_{q_j}$. Then $p_{i_k} \succeq_{q_j} p_{i'}$ for all $1 \leq k \leq c_{q_j}$. In particular, we have $p_{i_k} \succeq_{q_j} p_{i'}$ for all students p_{i_k} in P such that $\mu(p_{i_k}) = q_j$. Furthermore, if $|\mu(q_j)| < c_{q_j}$, then $p_{i_k} = \varnothing$ for some $1 \leq k \leq c_{q_j}$. Hence $\varnothing \succeq_{q_j} p_{i'}$. It follows that $(p_{i'}, q_j)$ does not form a strongly blocking pair. □

Lemma 35. Suppose that for every college q in Q, the group preference relation $\succeq'_q$ is minimally responsive to $\succeq_q$. Let μ be the matching produced by Algorithm 3 and let μ' be a matching such that $\mu' \succeq \mu$. Then $\mu \succeq \mu'$.

Proof. Since μ' is a matching that satisfies the capacity constraints, we can construct an SMIW matching $\mu'_0 \colon P' \to Q' \cup \{q'_{00}\}$ such that for all $1 \leq i \leq |P|$ and $0 \leq j \leq |Q|$, we have $\mu'(p_i) = q_j$ if and only if $\mu_0(p'_i) = q'_{jk}$ for some k.

Since $\mu' \succeq \mu$, we have $\mu'(p_i) \succeq_{p_i} \mu(p_i)$ for every $1 \leq i \leq |P|$ and $\mu'(q_j) \succeq'_{q_j} \mu(q_j)$ for every $1 \leq j \leq |Q|$. Thus $\mu'_0(p'_i) \succeq_{p'_i} \mu_0(p'_i)$ for every $1 \leq i \leq |P|$ and

$$\sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu'_0(q'_{jk})) \geq \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu_0(q'_{jk}))$$

for every $1 \leq j \leq |Q|$. Hence, by Lemma 30, we have $\mu_0(p'_i) \succeq_{p'_i} \mu'_0(p'_i)$ for every $1 \leq i \leq |P|$ and

$$\sum_{1 \leq j \leq |Q|} \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu'_0(q'_{jk})) = \sum_{1 \leq j \leq |Q|} \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu_0(q'_{jk})).$$

Therefore, we have $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for every $1 \leq i \leq |P|$ and

$$\sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu'_0(q'_{jk})) = \sum_{1 \leq k \leq c_{q_j}} \psi_{q'_{jk}}(\mu_0(q'_{jk}))$$

for every $1 \leq j \leq |Q|$. We conclude that $\mu(q_j) \succeq'_{q_j} \mu'(q_j)$ for every $1 \leq j \leq |Q|$. Thus $\mu \succeq \mu'$. □

Theorem 2. Suppose that for every college q in Q, the group preference relation $\succeq'_q$ is minimally responsive to $\succeq_q$. Algorithm 3 is a strategyproof Pareto-stable mechanism for the college admissions problem with weak preferences (for any fixed choice of utility assignment).

Proof. We have shown in Lemma 34 that Algorithm 3 produces a weakly stable matching. Moreover, Lemma 35 shows that the weakly stable matching produced is not Pareto-dominated by any other matching. Hence, Algorithm 3 produces a Pareto-stable matching.

To show that Algorithm 3 provides a strategyproof mechanism, suppose student p_i expresses $\succeq'_{p_i}$ instead of their true preference relation $\succeq_{p_i}$, where $1 \leq i \leq |P|$. Let μ and μ' be the matchings produced by Algorithm 3 given $\succeq_{p_i}$ and $\succeq'_{p_i}$, respectively. Let μ_0 and μ'_0 be the SMIW matching produced by the call to Algorithm 2 (line 8 of Algorithm 3) given $\succeq_{p_i}$ and $\succeq'_{p_i}$, respectively.

Notice that in Algorithm 3, the only part of the stable marriage instance that depends on the preferences of student p_i is the preference relation corresponding to man p'_i. Since Algorithm 2 is strategyproof by Theorem 1, we have $\mu_0(p'_i) \succeq_{p'_i} \mu'_0(p'_i)$ where $\succeq_{p'_i}$ is the preference relation of man p'_i in the algorithm given $\succeq_{p_i}$. Hence, $\mu(p_i) \succeq_{p_i} \mu'(p_i)$. □

We remark that our algorithm admits an $O(n^4)$-time implementation, where n is the sum of the number of students and the total capacities of all the colleges, because the reduction from CAW to IUAP takes $O(n^2)$ time, and lines 16 and 17 of Algorithm 2 can be implemented in $O(n^4)$ time using the version of the incremental Hungarian method discussed in Sections 2.3 and 3.1.

5.2 Further Discussion

In our SMIW and CAW algorithms, we transform the preference relations of the women and colleges into real-valued utility functions. One way to do this is to take

$$\psi_q(p) = |\{p' \in P \cup \{\varnothing\} : p \succeq_q p'\}| - |\{p' \in P \cup \{\varnothing\} : \varnothing \succeq_q p'\}|.$$

This is by no means the only way. In fact, different ways of assigning the utilities can affect the outcome. Nonetheless, our mechanisms remain strategyproof for the men as long as the utility assignment is fixed and independent of the preferences of the men, as shown in Theorems 1 and 2.

We can also consider the scenario where each college expresses their preferences directly in terms of a utility function instead of a preference relation. Such utility functions provide another way to extend preferences over individuals to group preferences. If a college q expresses the utility function ψ_q over individual students in $P \cup \{\varnothing\}$, we can define the *group preference induced by additive utility* ψ_q as a binary relation $\succeq'_q$ over 2^P such that $S \succeq'_q S'$ if and only if

$$\sum_{p \in S} \psi_q(p) \geq \sum_{p \in S'} \psi_q(p).$$

Our algorithm can accept such utility functions as input in lieu of constructing them by some utility assignment method. It is not hard to see that the mechanism remains Pareto-stable and strategyproof when the group preferences of the colleges are induced by additive utilities.

References

[1] N. Chen. On computing Pareto stable assignments. In *Proceedings of the 29th International Symposium on Theoretical Aspects of Computer Science*, pages 384–395, March 2012.

[2] N. Chen and A. Ghosh. Algorithms for Pareto stable assignment. In *Proceedings of the Third International Workshop on Computational Social Choice*, pages 343–354, September 2010.

[3] N. O. Domaniç, C.-K. Lam, and C. G. Plaxton. Strategyproof Pareto-stable mechanisms for two-sided matching with indifferences. Technical Report TR–16–19, Department of Computer Science, University of Texas at Austin, November 2016.

[4] A. Erdil and H. Ergin. Two-sided matching with indifferences. Working paper.

[5] A. Erdil and H. Ergin. What's the matter with tie-breaking? Improving efficiency in school choice. *American Economic Review*, 98:669–689, 2008.

[6] M. L. Fredman and R. E. Tarjan. Fibonacci heaps and their uses in improved network optimization algorithms. *J. ACM*, 34:596–615, 1987.

[7] D. Gale and L. S. Shapley. College admissions and the stability of marriage. *American Mathematical Monthly*, 69:9–15, 1962.

[8] N. Kamiyama. A new approach to the Pareto stable matching problem. *Mathematics of Operations Research*, 39:851–862, 2014.

[9] O. Kesten. School choice with consent. *The Quarterly Journal of Economics*, 125(3):1297–1348, 2010.

[10] H. W. Kuhn. The Hungarian method for the assignment problem. *Naval Research Logistics Quarterly*, 2:83–97, 1955.

[11] D. F. Manlove. *Algorithmics of Matching Under Preferences*. World Scientific, Singapore, 2013.

[12] A. E. Roth. The economics of matching: Stability and incentives. *Mathematics of Operations Research*, 7:617–628, 1982.

[13] A. E. Roth. The college admissions problem is not equivalent to the marriage problem. *Journal of Economic Theory*, 36:277–288, 1985.

[14] A. E. Roth and M. Sotomayor. *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press, New York, 1990.

[15] L. S. Shapley and M. Shubik. The assignment game I: The core. *International Journal of Game Theory*, 1:111–130, 1972.

[16] M. Sotomayor. The Pareto-stability concept is a natural solution concept for discrete matching markets with indifferences. *International Journal of Game Theory*, 40:631–644, 2011.

A Discussion of the Two-Phase Approach

As discussed in Section 1, Erdil and Ergin [4] present a polynomial-time algorithm for computing a Pareto-stable matching for a given instance of SMCW (and also its generalizations to SMIW and CAW). Their algorithm uses a two-phase approach: in the first phase, ties are broken arbitrarily, and the Gale-Shapley DA algorithm is used to obtain a weakly stable matching; in the second phase, the matching is repeatedly updated via a sequence of Pareto improvements until no such improvement is possible. This two-phase framework was previously proposed by Sotomayor [16], who argued its correctness by observing that when we apply a Pareto improvement to a weakly stable matching, we obtain another weakly stable matching.

It is natural to ask whether there is a strategyproof Pareto-stable mechanism for SMCW based on the foregoing two-phase approach. More precisely, suppose the men and women are indexed from 1 to n, and assume that we break ties in the first phase in favor of higher-indexed agents. Is there a way to implement the second phase so that the resulting two-phase algorithm corresponds to a strategyproof Pareto-stable mechanism? The example presented below provides a negative answer to this question.

Consider an SMCW instance I with men $\{p_1, p_2, p_3\}$ and women $\{q_1, q_2, q_3\}$, and where the preferences of the agents are as follows: p_1 prefers q_2, then q_3, then q_1; p_2 prefers q_1, then q_3, then q_2; p_3 is indifferent between q_1 and q_2, and prefers q_1 and q_2 to q_3; q_1 prefers p_3, then p_1, then p_2; q_2 is indifferent between all of the men; q_3 prefers p_3, then p_2, then p_1. Let M_1 through M_6 denote the six possible matchings: $M_1 = \{(p_1, q_1), (p_2, q_2), (p_3, q_3)\}$; $M_2 = \{(p_1, q_1), (p_2, q_3), (p_3, q_2)\}$; $M_3 = \{(p_1, q_2), (p_2, q_1), (p_3, q_3)\}$; $M_4 = \{(p_1, q_2), (p_2, q_3), (p_3, q_1)\}$; $M_5 = \{(p_1, q_3), (p_2, q_1), (p_3, q_2)\}$; $M_6 = \{(p_1, q_3), (p_2, q_2), (p_3, q_1)\}$. It is easy to verify that $\{M_2, M_4, M_5\}$ is the set of weakly stable matchings for I. (Matchings M_1 and M_3 are blocked by (p_3, q_1), and matching M_6 is blocked by (p_2, q_3).) Furthermore, the set of Pareto-stable matchings for I is $\{M_4, M_5\}$. (Matching M_2 is Pareto-dominated by matching M_4.) If we break ties in favor of the agents with higher indices, then it is easy to verify that the first phase produces matching M_5. Since M_5 is Pareto-stable, the second phase does not update the matching, and hence M_5 is the final output.

Now suppose man p_1 lies by stating that he prefers q_2, then q_1, then q_3, and let I' denote the resulting SMCW instance. It is easy to verify that $\{M_2, M_4\}$ is the set of weakly stable matchings for I'. (Matchings M_1 and M_3 are blocked by (p_3, q_1), matching M_5 is blocked by (p_1, q_1), and matching M_6 is blocked by (p_2, q_3).) Furthermore, the set of Pareto-stable matchings for I' is $\{M_4\}$. (Matching M_2 is Pareto-dominated by matching M_4.) Thus M_4 is the only possible output of the second phase. Since man p_1 prefers his match under M_4 to his match under M_5, strategyproofness is violated.