Strategyproof Pareto-Stable Mechanisms for Two-Sided Matching with Indifferences*

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Abstract

We study variants of the stable marriage and college admissions models in which the agents are allowed to express weak preferences over the set of agents on the other side of the market and the option of remaining unmatched. For the problems that we address, previous authors have presented polynomial-time algorithms for computing a "Pareto-stable" matching. In the case of college admissions, these algorithms require the preferences of the colleges over groups of students to satisfy a technical condition related to responsiveness. We design new polynomial-time Pareto-stable algorithms for stable marriage and college admissions that correspond to strategyproof mechanisms. For stable marriage, it is known that no Pareto-stable mechanism is strategyproof for all of the agents; our algorithm provides a mechanism that is strategyproof for the agents on one side of the market. For college admissions, it is known that no Pareto-stable mechanism can be strategyproof for the colleges; our algorithm provides a mechanism that is strategyproof for the students.

Keywords: stable matching, strategyproofness, Pareto-optimality.

1 Introduction

Gale and Shapley [6] introduced the stable marriage model and its generalization to the college admissions model. Their work spawned a vast literature on two-sided matching; see Manlove [9] for a recent survey. The present paper is primarily concerned with variants of the stable marriage and college admissions models where the agents have weak preferences, i.e., where indifferences are allowed.

In the most basic stable marriage model, we are given an equal number of men and women, where each man (resp., woman) has complete, strict preferences over the set of women (resp., men); we refer to this model as SMCS. For SMCS, an outcome is a matching that pairs up all of the men and women into disjoint man-woman pairs. A man-woman pair (p, q) is said to form a *blocking pair* for a matching M if p prefers q to his partner in M and q prefers p to her partner in M. A matching is *stable* if it does not have a blocking pair. It is straightforward to prove that any stable matching is also Pareto-optimal. Gale and Shapley presented the deferred acceptance (DA) algorithm for the SMCS problem and proved that the man-proposing version of the DA algorithm produces the unique man-optimal (and woman-pessimal) stable matching. Roth [10] showed that the associated mechanism, which we refer to as the *man-proposing DA mechanism*, is *strategyproof* for the men, i.e., it is a weakly dominant strategy for each man to declare his true preferences. Unfortunately, the man-proposing DA mechanism is not strategyproof for the women. In fact, Roth [10] showed that no stable mechanism for SMCS is strategyproof for all of the agents.

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The SMCW model is the generalization of the SMCS model in which each man (resp., woman) has weak preferences over the set of women (resp., men). When indifferences are allowed, we need to refine our notion of a blocking pair. A man-woman pair (p, q) is said to form a *strongly blocking pair* for a matching M if p prefers q to his partner in M and q prefers p to her partner in M. A matching is *weakly stable* if it is individually rational and does not have a strongly blocking pair. Two other natural notions of stability, namely strong stability and super-stability, have been investigated in the literature (see Manlove [9, Chapter 3] for a survey of these results). We focus on weak stability because every SMCW instance admits a weakly stable matching (this follows from the existence of stable matchings for SMCS, coupled with arbitrary tie-breaking), but not every SMCW instance admits a strongly stable or super-stable matching. It is straightforward to exhibit SMCW instances (with as few as two men and two women) for which some weakly stable matching is not Pareto-optimal. Sotomayor [14] argues that *Pareto-stability* (i.e., Pareto-optimality plus weak stability) is an appropriate solution concept for SMCW and certain other matching models with weak preferences, and proves that every SMCW instance admits a Pareto-stable matching.

Erdil and Ergin [5] and Chen and Ghosh [2] present polynomial-time algorithms for computing a Pareto-stable matching of a given SMCW instance; in fact, these algorithms are applicable to certain more general models to be discussed shortly. Given the existence of a stable mechanism for SMCS that is strategyproof for the men (or, symmetrically, for the women), it is natural to ask whether there is a Pareto-stable mechanism for SMCW that is strategyproof for the men. We cannot hope to find a Pareto-stable mechanism for SMCW that is strategyproof for all agents, since that would imply a stable mechanism for SMCS that is strategyproof for all agents. A similar statement holds for the SMIW model, the generalization of the SMCW model in which the agents are allowed to express incomplete preferences. See Section 4 for a formal definition of the SMIW model and the associated notions of weak stability and Pareto-stability. Throughout the remainder of the paper, when we say that a mechanism for a stable marriage model is strategyproof, we mean that it is strategyproof for the agents on one side of the market; moreover, unless otherwise specified, it is to be understood that the mechanism is strategyproof for the men. The Pareto-stable algorithms of Erdil and Ergin, and of Chen and Ghosh, are based on a two-phase approach where the first phase runs the Gale-Shapley DA algorithm after breaking all ties arbitrarily. In Appendix A of the full version of this paper [3], we show that this approach does not provide a strategyproof mechanism.

This paper provides the first Pareto-stable mechanism for SMIW (and also SMCW) that is shown to be strategyproof. We present a nondeterministic algorithm for SMIW that generalizes Gale and Shapley's DA algorithm as follows: in each iteration, an arbitrarily chosen unmatched man "proposes" simultaneously to all of the women in his next-highest tier of preference (i.e., the highest tier to which he has not already proposed); the women respond to this proposal by solving a certain maximum-weight matching problem to determine which man becomes unmatched (i.e., the man making the proposal or one of the tentatively matched men). Our generalization of the DA mechanism admits a polynomial-time implementation.

The college admissions model with weak preferences, which we denote CAW, is a further generalization of the SMIW model. In the CAW model, students and colleges are being matched rather than men and women, and each college has a positive integer capacity representing the number of students that it can accommodate. See Section 5 for a formal definition of the CAW model and the associated notions of weak stability and Pareto-stability.

A key difference between CAW and SMIW is that in addition to expressing preferences over individual students, the colleges have preferences over *groups* of students. This characteristic is shared by the CAS model, which is the restriction of the CAW model to strict preferences. It is known that no stable mechanism for CAS is strategyproof for the colleges [11]; the proof makes use of the fact that the colleges do not (in general) have unit demand. It follows that no Pareto-stable mechanism for CAW is strategyproof for the colleges. Throughout the remainder of the paper, when we say that a mechanism for a college admissions model is strategyproof, we mean that it is strategyproof for the students.

Gale and Shapley's DA algorithm generalizes easily to the CAS model. Roth [11] has shown that the student-proposing DA algorithm provides a strategyproof stable mechanism for CAS when the preferences of the colleges are *responsive*. When the colleges have responsive preferences, the student-proposing DA mechanism is also known to be student-optimal for CAS [11].

Erdil and Ergin [5] consider the special case of the CAW model where the following restrictions hold for all students x and colleges y: x is not indifferent between being assigned to y and being left unassigned; y is not indifferent between having one of its slots assigned to x and having that slot left unfilled. We remark that this special case of CAW corresponds to the HRT problem discussed in Manlove [9, Chapter 3].[1] For this special case, Erdil and Ergin present a polynomial-time algorithm for computing a Pareto-stable matching when the preferences of the colleges satisfy a technical restriction related to responsiveness. We consider the same class of preferences, which we refer to as *minimally responsive*; see Section 5 for a formal definition. The algorithm of Erdil and Ergin does not provide a strategyproof mechanism. Chen and Ghosh [2] build on the results of Erdil and Ergin by considering the many-to-many generalization of HRT in which the agents on both sides of the market have capacities (and the agent preferences are minimally responsive). For this generalization, Chen and Ghosh provide a *strongly* polynomial-time algorithm. No strategyproof mechanism (even for the agents on one side of the market) is possible in the many-to-many setting, since it is a generalization of CAS. We provide the first Pareto-stable mechanism for CAW that is shown to be strategyproof. As in the work of Erdil-Ergin and Chen-Ghosh, we assume that the preferences of the colleges are minimally responsive. We can also handle the class of college preferences "induced by additive utility" that is defined in Section 5.2 in the full version of this paper [3].

In the many-to-many matching setting addressed by Chen and Ghosh [2], a pair of agents (on opposite sides of the market) can be matched with arbitrary multiplicity, as long as the capacity constraints are respected. Chen [1] presents a polynomial-time algorithm for the variation of many-to-many matching in which a pair of agents can only be matched with multiplicity one. Kamiyama [7] addresses the same problem using a different algorithmic approach. (The algorithms of Chen and Kamiyama are strongly polynomial, since we can assume without loss of generality that the capacity of any agent is at most the number of agents on the other side of the market.) Since this variation of the many-to-many setting also generalizes CAS, it does not admit a strategyproof mechanism, even for the agents on one side of the market.

Erdil and Ergin [4, 5] and Kesten [8] consider a second natural solution concept in addition to Pareto-stability. In the context of SMIW (or its special case SMCW), this second solution concept seeks a weakly stable matching M that is "man optimal" in the following sense: for all weakly stable matchings M', either all of the men are indifferent between M and M', or at least one man prefers M to M'. Erdil and Ergin [5] present a polynomial-time algorithm to compute such a man optimal weakly stable matching for SMIW; in fact, their algorithm is presented for the generalization of SMIW to CAW. Erdil and Ergin [4] and Kesten [8] prove that no strategyproof man optimal weakly stable mechanism exists for SMCW. Prior to our work, it was unclear whether such an impossibility result might hold for strategyproof Pareto-stable mechanisms for SMCW (or its generalizations to SMIW and CAW).

The assignment game of Shapley and Shubik [13] can be viewed as an auction with multiple distinct items where each bidder is seeking to acquire at most one item. This class of *unit-demand auctions* has been heavily studied in the literature (see, e.g., Roth and Sotomayor [12, Chapter 8]). In Section 2, we define the notion of a "unit-demand auction with priorities" (UAP) and establish a number of useful properties of UAPs; these are straightforward generalizations of corresponding properties of unit-demand auctions.

[1] In the model of Erdil and Ergin, which is stated using worker-firm terminology rather than student-college terminology, a "no indifference to unemployment/vacancy" assumption makes the aforementioned restrictions explicit. In the HRT model of Manlove, which is stated using resident-hospital terminology rather than student-college terminology, it is assumed that a set of acceptable resident-hospital pairs is given, and that each agent specifies weak preferences over the set of agents with whom they form an acceptable pair. We consider the approach of Erdil-Ergin — where the starting point is the preferences of the individual agents, and the "acceptability" of a given pair of agents may be deduced from those preferences — to be more natural, but the resulting models are equivalent.

Section 3 builds on the UAP notion to define the notion of an "iterated UAP" (IUAP), and establishes a number of important properties of IUAPs; these results are nontrivial to prove and provide the technical foundation for our main results. Section 4 presents our first main result, a polynomial-time algorithm for SMIW that provides a strategyproof Pareto-stable mechanism. Section 5 presents our second main result, a polynomial-time algorithm for CAW that provides a strategyproof Pareto-stable mechanism assuming that the preferences of the colleges are minimally responsive.

Due to space limitations, some of the proofs are omitted from this paper. See the full version [3] for all of the proof details.

2 Unit-Demand Auctions with Priorities

In this section, we first formally define the notion of a unit-demand auction with priorities (UAP). Then, we describe an associated matroid for a given UAP and we use this matroid to define the notion of a "greedy MWM". Finally, we introduce a key definition that is helpful for establishing our strategyproofness results. We start with some useful definitions.

A *(unit-demand) bid β for a set of items V* is a subset of $V \times \mathbb{R}$ such that no two pairs in β share the same first component. (So β may be viewed as a partial function from V to $\mathbb{R}$.)

A *bidder u for a set of items V* is a triple (α, β, z) where α is an integer ID, β is a bid for V, and z is a real priority. For any bidder $u = (\alpha, \beta, z)$, we define $id(u)$ as α, $bid(u)$ as β, $priority(u)$ as z, and $items(u)$ as the union, over all (v, x) in β, of $\{v\}$.

A *unit-demand auction with priorities (UAP)* is a pair $A = (U, V)$ satisfying the following conditions: V is a set of items; U is a set of bidders for V; each bidder in U has a distinct ID.

A UAP $A = (U, V)$ may be viewed as an edge-weighted bipartite graph, where the set of edges incident on bidder u correspond to $bid(u)$: for each pair (v, x) in $bid(u)$, there is an edge (u, v) of weight x. We refer to a matching (resp., maximum-weight matching (MWM), maximum-cardinality MWM (MCMWM)) in the associated edge-weighted bipartite graph as a matching (resp., MWM, MCMWM) of A. For any edge $e = (u, v)$ in a given UAP, the associated weight is denoted $w(e)$ or $w(u, v)$. For any set of edges E, we define $w(E)$ as $\sum_{e \in E} w(e)$. For any UAP A, we let $w(A)$ denote the weight of an MWM of A.

Lemma 1. Let $A = (U, V)$ be a UAP, and let $\mathcal{I}$ denote the set of all subsets U' of U such that there exists an MWM of A that matches every bidder in U'. Then $(U, \mathcal{I})$ is a matroid.

For any UAP A, we define $matroid(A)$ as the matroid of Lemma 1.

For any UAP $A = (U, V)$ and any independent set U' of $matroid(A)$, we define the *priority of U'* as the sum, over all bidders u in U', of $priority(u)$. For any UAP A, the matroid greedy algorithm can be used to compute a maximum-priority maximal independent set of $matroid(A)$.

For any matching M of a UAP $A = (U, V)$, we define $matched(M)$ as the set of all bidders in U that are matched in M. We say that an MWM M of a UAP A is *greedy* if $matched(M)$ is a maximum-priority maximal independent set of $matroid(A)$. For any UAP A, we define the predicate $unique(A)$ to hold if $matched(M) = matched(M')$ for all greedy MWMs M and M' of A.

For any matching M of a UAP, we define the *priority of M*, denoted $priority(M)$, as the sum, over all bidders u in $matched(M)$, of $priority(u)$. Thus an MWM is greedy if and only if it is a maximum-priority MCMWM.

Lemma 2. All greedy MWMs of a given UAP have the same distribution of priorities.

For any UAP A and any real priority z, we define $greedy(A, z)$ as the (uniquely defined, by Lemma 2) number of matched bidders with priority z in any greedy MWM of A.

Now, we define the notion of a "threshold" of an item in a UAP. This lays the groundwork for a corresponding IUAP definition in Section 3.2. Item thresholds play an important role in our strategyproofness results. We start with some useful definitions.

Let $A = (U, V)$ be a UAP and let u be a bidder such that $id(u)$ is not equal to the ID of any bidder in U. Then we define $A + u$ as the UAP $(U + u, V)$. For any UAPs $A = (U, V)$ and $A' = (U', V')$, we say that A' *extends* A if $U \subseteq U'$ and $V = V'$.

Lemma 3. Let $A = (U, V)$ be a UAP, let u be a bidder in U that is not matched in any greedy MWM of A, and let $A' = (U', V)$ be a UAP that extends A. Then u is not matched in any greedy MWM of A'.

Lemma 4. Let $A = (U, V)$ be a UAP and let v be an item in V. Let U' be the set of bidders u such that $A + u$ is a UAP and $bid(u)$ is of the form $\{(v, x)\}$. Then there is a unique pair of reals (x^*, z^*) such that for any bidder u in U', the following conditions hold, where A' denotes $A + u$, x denotes $w(u, v)$, and z denotes $priority(u)$: (1) if $(x, z) > (x^*, z^*)$ then u is matched in every greedy MWM of A'; (2) if $(x, z) < (x^*, z^*)$ then u is not matched in any greedy MWM of A'; (3) if $(x, z) = (x^*, z^*)$ then u is matched in some but not all greedy MWMs of A'.[2]

For any UAP $A = (U, V)$ and any item v in V, we define the unique pair (x^*, z^*) of Lemma 4 as $threshold(A, v)$.

3 Iterated Unit-Demand Auctions with Priorities

In this section, we formally define the notion of an iterated unit-demand auction with priorities (IUAP). An IUAP allows the bidders, called "multibidders" in this context, to have a sequence of unit-demand bids instead of a single unit-demand bid. In Section 3.1, we define a mapping from an IUAP to a UAP by describing an algorithm that generalizes the DA algorithm, and we establish Lemma 8 that is useful for analyzing the matching produced by Algorithm 2 of Section 4. Lemma 8 is used to establish weak stability (Lemmas 11, 12, and 13) and Pareto-optimality (Lemma 14). In Section 3.2, we define the threshold of an item in an IUAP and we establish Lemma 10, which plays a key role in establishing our strategyproofness results. We start with some useful definitions.

A *multibidder* t *for a set of items* V is a pair (σ, z) where z is a real priority and σ is a sequence of bidders for V such that all the bidders in σ have distinct IDs and a common priority z. We define $priority(t)$ as z. For any integer i such that $1 \le i \le |\sigma|$, we define $bidder(t, i)$ as the bidder $\sigma(i)$. For any integer i such that $0 \le i \le |\sigma|$, we define $bidders(t, i)$ as $\{bidder(t, j) \mid 1 \le j \le i\}$. We define $bidders(t)$ as $bidders(t, |\sigma|)$.

An *iterated UAP (IUAP)* is a pair $B = (T, V)$ where V is a set of items and T is a set of multibidders for V. In addition, for any distinct multibidders t and t' in T, the following conditions hold: $priority(t) \neq priority(t')$; if u belongs to $bidders(t)$ and u' belongs to $bidders(t')$, then $id(u) \neq id(u')$. For any IUAP $B = (T, V)$, we define $bidders(B)$ as the union, over all t in T, of $bidders(t)$.

3.1 Mapping an IUAP to a UAP

Having defined the notion of an IUAP, we now describe an algorithm ToUap that maps a given IUAP to a UAP. Algorithm ToUap generalizes the DA algorithm. In each iteration of the DA algorithm, an arbitrary single man is chosen, and this man reveals his next choice. In each iteration of ToUap, an arbitrary single multibidder is chosen, and this multibidder reveals its next bid. We prove in Lemma 7 that, like the DA algorithm, algorithm ToUap is confluent: the output does not depend on the nondeterministic choices made during an execution. We conclude this section by establishing Lemma 8, which is useful for analyzing the

[2]Throughout this paper, comparisons of pairs are to be performed lexicographically.

matching produced by Algorithm 2 in Section 4. Lemma 8 is used to establish weak stability (Lemmas 11, 12, and 13) and Pareto-optimality (Lemma 14). We start with some useful definitions.

Let A be a UAP (U, V) and let B be an IUAP (T, V). The predicate $\mathit{prefix}(A, B)$ is said to hold if $U \subseteq \mathit{bidders}(B)$ and for any multibidder t in T, $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i)$ for some i.

A *configuration* C is a pair (A, B) where A is a UAP, B is an IUAP, and $\mathit{prefix}(A, B)$ holds.

Let $C = (A, B)$ be a configuration, where $A = (U, V)$ and $B = (T, V)$, and let u be a bidder in U. Then we define $\mathit{multibidder}(C, u)$ as the unique multibidder t in T such that u belongs to $\mathit{bidders}(t)$.

Let $C = (A, B)$ be a configuration where $A = (U, V)$ and $B = (T, V)$. For any t in T, we define $\mathit{bidders}(C, t)$ as $\{u \in U \mid \mathit{multibidder}(C, u) = t\}$.

Let $C = (A, B)$ be a configuration where $B = (T, V)$. We define $\mathit{ready}(C)$ as the set of all bidders u in $\mathit{bidders}(B)$ such that $\mathit{greedy}(A, \mathit{priority}(u)) = 0$ and $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)| + 1)$ where $t = \mathit{multibidder}(C, u)$.

Algorithm 1 TOUAP(B)

Input: An IUAP $B = (T, V)$
1: $A \leftarrow (\emptyset, V)$
2: $C \leftarrow (A, B)$
3: **while** $\mathit{ready}(C)$ is nonempty **do**
4: $\quad A \leftarrow A+$ an arbitrary bidder in $\mathit{ready}(C)$
5: $\quad C \leftarrow (A, B)$
6: **end while**
7: **return** A

Our algorithm for mapping an IUAP to a UAP is Algorithm 1. The input is an IUAP B and the output is a UAP A such that $\mathit{prefix}(A, B)$ holds. The algorithm starts with the UAP consisting of all the items in V but no bidders. At this point, no bidder of any multibidder is "revealed". Then, the algorithm iteratively and chooses an arbitrary "ready" bidder and "reveals" it by adding it to the UAP that is maintained in the program variable A. A bidder u associated with some multibidder $t = (\sigma, z)$ is ready if u is not revealed and for each bidder u' that precedes u in σ, u' is revealed and is not matched in any greedy MWM of A. It is easy to verify that the predicate $\mathit{prefix}(A, B)$ is an invariant of the algorithm loop: if a bidder u belonging to a multibidder t is to be revealed at an iteration, and $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i)$ for some integer i at the beginning of this iteration, then $U \cap \mathit{bidders}(t) = \mathit{bidders}(t, i + 1)$ after revealing u, where (U, V) is the UAP that is maintained by the program variable A at the beginning of the iteration. No bidder can be revealed more than once since a bidder cannot be ready after it has been revealed; it follows that the algorithm terminates. We now argue that the output of the algorithm is uniquely determined (Lemma 7), even though the bidder that is revealed in each iteration is chosen nondeterministically.

For any configuration $C = (A, B)$, we define the predicate $\mathit{tail}(C)$ to hold if for any bidder u that is matched in some greedy MWM of A, we have $u = \mathit{bidder}(t, |\mathit{bidders}(C, t)|)$ where $t = \mathit{multibidder}(C, u)$.

Lemma 5. The predicate $\mathit{tail}(C)$ is an invariant of the Algorithm 1 loop.

Lemma 6. Let $C = (A, B)$ be a configuration such that $\mathit{tail}(C)$ holds. Then $\mathit{unique}(A)$ holds.

Lemma 7. Let $B = (T, V)$ be an IUAP. Then all executions of Algorithm 1 on input B produce the same output.

Proof. Suppose not, and let X_1 and X_2 denote two executions of Algorithm 1 on input B that produce distinct output UAPs $A_1 = (U_1, V)$ and $A_2 = (U_2, V)$. Without loss of generality, assume that $|U_1| \geq |U_2|$. Then there is a first iteration of execution X_1 in which the bidder added to A in line 4 belongs to $U_1 \setminus U_2$;

let u' denote this bidder. Let $C' = (A', B)$ where $A' = (U', V)$ denote the configuration in program variable C at the start of this iteration, and let t' denote $\mathit{multibidder}(C', u')$. Let i be the integer such that $u' = \mathit{bidder}(t', i)$. We know that $i > 1$ because it is easy to see that U_2 contains $\mathit{bidder}(t', 1)$. Let u'' denote $\mathit{bidder}(t', i-1)$. Since u' belongs to $\mathit{ready}(C')$, Lemmas 5 and 6 imply that u'' is not matched in any greedy MWM of A'. Since U' is contained in U_2, Lemma 3 implies that u'' is not matched in any greedy MWM of A_2. Let $C_2 = (A_2, B)$ denote the final configuration of execution X_2; thus $\mathit{ready}(C_2)$ is empty and $|\mathit{bidders}(C_2, t')| = i - 1$. By Lemma 5, we conclude that $\mathit{greedy}(A_2, \mathit{priority}(t')) = 0$, and hence that u'' is contained in $\mathit{ready}(C_2)$, a contradiction. □

For any IUAP B, we define $\mathit{uap}(B)$ as the unique (by Lemma 7) UAP returned by any execution of Algorithm 1 on input B.

We can use the modified incremental Hungarian step of [3, Section 2.3] in each iteration of the loop of Algorithm 1 to maintain UAP A, and a greedy MWM of A, as follows: we maintain dual variables (a price for each item) and a residual graph; the initial greedy MWM is the empty matching; when a bidder u is added to A at line 4, we perform an incremental Hungarian step to process u to update the greedy MWM, the prices, and the residual graph. Since we maintain a greedy MWM of A at each iteration of the loop, it is easy to see that identifying a bidder in $\mathit{ready}(C)$ (or determining that this set is empty) takes $O(|V|)$ time. Thus the whole algorithm can be implemented in $O(|\mathit{bidders}(B)| \cdot |V|^2)$ time.

We now present a lemma that is used in Section 4 to establish weak stability (Lemmas 11, 12, and 13) and Pareto-optimality (Lemma 14).

Lemma 8. Let $B = (T, V)$ be an IUAP, let (σ, z) be a multibidder that belongs to T, let $\mathit{uap}(B)$ be (U, V), and let M be a greedy MWM of the UAP (U, V). Then the following claims hold: (1) if $\sigma(k)$ is matched in M for some k, then $\sigma(k') \in U$ if and only if $1 \leq k' \leq k$; (2) if $\sigma(k)$ is not matched in M for any k, then $\sigma(k) \in U$ for $1 \leq k \leq |\sigma|$.

3.2 Threshold of an Item

In this section, we define the threshold of an item in an IUAP and we establish Lemma 10, which plays a key role in establishing our strategyproofness results. We start with some useful definitions.

For any IUAP B, Lemmas 5 and 6 imply that $\mathit{unique}(\mathit{uap}(B))$ holds, and thus that every greedy MWM of $\mathit{uap}(B)$ matches the same set of bidders. We define this set of matched bidders as $\mathit{winners}(B)$. For any IUAP B, we define $\mathit{losers}(B)$ as $U \setminus \mathit{winners}(B)$ where (U, V) is $\mathit{uap}(B)$.

Let $B = (T, V)$ be an IUAP and let $u = (\alpha, \beta, z)$ be a bidder for V. Then we define the IUAP $B + u$ as follows: if T contains a multibidder t of the form (σ, z) for some sequence of bidders σ, then we define $B + u$ as $(T - t + t', V)$ where $t' = (\sigma', z)$ and σ' is the sequence of bidders obtained by appending u to σ; otherwise, we define $B + u$ as $(T + t, V)$ where $t = (\langle u \rangle, z)$.

Lemma 9. Let $B = (T, V)$ be an IUAP and let v be an item in V. For $i \in \{1, 2\}$, let $B_i = B + u_i$ be an IUAP where $u_i = (\alpha_i, \{(v, x_i)\}, z_i)$. Let $A_1 = (U_1, V)$ denote $\mathit{uap}(B_1)$ and let $A_2 = (U_2, V)$ denote $\mathit{uap}(B_2)$. Assume that $\alpha_1 \neq \alpha_2$, $z_1 \neq z_2$, and u_1 belongs to $\mathit{winners}(B_1)$. Then the following claims hold: if u_2 belongs to $\mathit{winners}(B_2)$ then $U_1 - u_1 = U_2 - u_2$; if u_2 belongs to $\mathit{losers}(B_2)$ then $U_1 - u_1$ contains $U_2 - u_2$.

We are now ready to define the threshold of an item in an IUAP, and to state Lemma 10. In Section 4, Lemma 10 plays an important role in establishing that our SMIW mechanism is strategyproof (Lemma 16).

Let $B = (T, V)$ be an IUAP and let v be an item in V. By Lemma 9, there is a unique subset U of $\mathit{bidders}(B)$ such that the following condition holds: for any IUAP $B' = B + u$ where u is of the form $(\alpha, \{(v, x)\}, z)$ and u belongs to $\mathit{winners}(B')$, $\mathit{uap}(B')$ is equal to $(U + u, V)$. We define $\mathit{uap}(B, v)$ as the UAP (U, V), and we define $\mathit{threshold}(B, v)$ as $\mathit{threshold}(\mathit{uap}(B, v), v)$.

Lemma 10. Let $B = (T, V)$ be an IUAP, let $t = (\sigma, z)$ be a multibidder that belongs to T, and let B' denote the IUAP $(T - t, V)$. Suppose that $(\sigma(k), v)$ is matched in some greedy MWM of $uap(B)$ for some k. Then

$$(w(\sigma(k), v), z) \geq \mathit{threshold}(B', v). \tag{1}$$

Furthermore, for each k' and v' such that $1 \leq k' < k$ and v' belongs to $\mathit{items}(\sigma(k'))$, we have

$$(w(\sigma(k'), v'), z) < \mathit{threshold}(B', v'). \tag{2}$$

4 Stable Marriage with Indifferences

The *stable marriage model with incomplete and weak preferences (SMIW)* involves a set P of men and a set Q of women. The preference relation of each man p in P is specified as a binary relation $\succeq_p$ over $Q \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. Similarly, the preference relation of each woman q in Q is specified as a binary relation $\succeq_q$ over $P \cup \{\varnothing\}$ that satisfies transitivity and totality, where $\varnothing$ denotes being unmatched. To allow indifferences, the preference relations are not required to satisfy antisymmetry. We will use $\succ_p$ and $\succ_q$ to denote the asymmetric part of $\succeq_p$ and $\succeq_q$ respectively.

A matching is a function μ from P to $Q \cup \{\varnothing\}$ such that for any woman q in Q, there exists at most one man p in P for which $\mu(p) = q$. Given a matching μ and a woman q in Q, we denote

$$\mu(q) = \begin{cases} p & \text{if } \mu(p) = q \\ \varnothing & \text{if there is no man } p \text{ in } P \text{ such that } \mu(p) = q \end{cases}$$

A matching μ is *individually rational* if for any man p in P and woman q in Q such that $\mu(p) = q$, we have $q \succeq_p \varnothing$ and $p \succeq_q \varnothing$. A pair (p, q') in $P \times Q$ is said to form a *strongly blocking pair* for a matching μ if $q' \succ_p \mu(p)$ and $p \succ_{q'} \mu(q')$. A matching is *weakly stable* if it is individually rational and does not admit a strongly blocking pair.

For any matching μ and μ', we say that the binary relation $\mu \succeq \mu'$ holds if for every man p in P and woman q in Q, we have $\mu(p) \succeq_p \mu'(p)$ and $\mu(q) \succeq_q \mu'(q)$. We let $\succ$ denote the asymmetric part of $\succeq$. We say that a matching μ *Pareto-dominates* another matching μ' if $\mu \succ \mu'$. We say that a matching is *Pareto-optimal* if it is not Pareto-dominated by any other matching. A matching is *Pareto-stable* if it is Pareto-optimal and weakly stable.

A *mechanism* is an algorithm that, given $(P, Q, (\succeq_p)_{p \in P}, (\succeq_q)_{q \in Q})$, produces a matching μ. A mechanism is said to be *strategyproof (for the men)* if for any man p in P expressing preference $\succeq'_p$ instead of his true preference $\succeq_p$, we have $\mu(p) \succeq_p \mu'(p)$, where μ and μ' are the matchings produced by the mechanism given $\succeq_p$ and $\succeq'_p$, respectively, when all other inputs are fixed.

By introducing extra men or women who prefer being unmatched to being matched with any potential partner, we may assume without loss of generality that the number of men is equal to the number of women. So, $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_n\}$.

4.1 Algorithm

The computation of a matching for SMIW is shown in Algorithm 2. We construct an item for each woman in line 4, and a multibidder for each man in line 13 by examining the tiers of preferences of the men and the utilities of the women. Together with dummy items constructed in line 8, this forms an IUAP, from which we obtain a UAP and a greedy MWM M_0. Using Lemma 8, we argue that for any man p_i, exactly one of the bidders associated with p_i is matched in M_0; see the proof of Lemma 11. Finally, in line 18, we use M_0 to determine the match of a man p_i as follows, where u denotes the unique bidder associated with p_i that is

Algorithm 2

Let p_0 denote $\varnothing$.
for all $1 \leq j \leq n$ **do**
  Convert the preference relation $\succeq_{q_j}$ of woman q_j into utility function $\psi_{q_j} \colon P \cup \{\varnothing\} \to \mathbb{R}$ that satisfies the followings: $\psi_{q_j}(\varnothing) = 0$; for any i and i' in $\{0, 1, \ldots, n\}$, we have $p_i \succeq_{q_j} p_{i'}$ if and only if $\psi_{q_j}(p_i) \geq \psi_{q_j}(p_{i'})$. This utility assignment should not depend on the preferences of the men.
  Construct an item v_j corresponding to woman q_j.
end for
for all $n < j \leq 2n$ **do**
  Let q_j denote $\varnothing$.
  Construct a dummy item v_j corresponding to q_j.
end for
for all $1 \leq i \leq n$ **do**
  Partition the set $\{1, \ldots, n\} \cup \{n+i\}$ of woman indices into tiers $\tau_i(1), \ldots, \tau_i(K_i)$ according to the preference relation of man p_i, such that for any j in $\tau_i(k)$ and j' in $\tau_i(k')$, we have $q_j \succeq_{p_i} q_{j'}$ if and only if $k \leq k'$.
  For j in $\{1, \ldots, n\} \cup \{n+i\}$, denote tier number $\kappa_i(q_j)$ as the unique k such that j in $\tau_i(k)$.
  Construct a multibidder $t_i = (\sigma_i, z_i)$ with priority $z_i = i$ corresponding to man p_i. The multibidder t_i has K_i bidders. For each bidder $\sigma_i(k)$ we define $\mathit{items}(\sigma_i(k))$ as $\{v_j \mid j \in \tau_i(k)\}$ and $w(\sigma_i(k), q_j)$ as $\psi_{q_j}(p_i)$, where $\psi_{q_{n+i}}(p_i)$ is defined to be 0.
end for
$(T, V) = (\{t_i \mid 1 \leq i \leq n\}, \{v_j \mid 1 \leq j \leq 2n\})$.
$(U, V) = \mathit{uap}(T, V)$.
Compute a greedy MWM M_0 of UAP (U, V) as described in [3, Section 2.3].
Output matching μ such that for all $1 \leq i \leq n$ and $1 \leq j \leq 2n$, we have $\mu(p_i) = q_j$ if and only if $\sigma_i(k)$ is matched to item v_j in M_0 for some k.

matched in M_0: if u is matched in M_0 to the item corresponding to a woman q_j, then we match p_i to q_j; otherwise, u is matched to a dummy item in M_0, and we leave p_i unmatched.

In Lemma 12, we prove individually rationality by arguing that the dummy items ensure that no man or woman is matched to an unacceptable partner. In Lemma 13, we prove weak stability using the properties of a greedy MWM. In Lemmas 14 and 15, we prove Pareto-optimality by showing that any matching that Pareto-dominates the output matching induces another MWM that contradicts the greediness of the MWM produced by the algorithm. In Lemma 16, we establish two properties of IUAP thresholds that are used to show strategyproofness in Theorem 1.

Lemma 11. Algorithm 2 produces a valid matching.

Lemma 12. Algorithm 2 produces an individually rational matching.

Proof. We have shown in Lemma 11 that μ is a valid matching. Consider man p_i and woman q_j such that $\mu(p_i) = q_j$, where i and j belong to $\{1, \ldots, n\}$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{n+i})$. It suffices to show that $k \leq k'$ and $\psi_{q_j}(p_i) \geq 0$.

Since $\mu(p_i) = q_j$, bidder $\sigma_i(k)$ is matched to item v_j in M_0. Since M_0 is an MWM, we have $\psi_{q_j}(p_i) = w(\sigma_i(k), v_j) \geq 0$.

It remains to show that $k \leq k'$. For the sake of contradiction, suppose $k > k'$. Since bidder $\sigma_i(k)$ is matched to item v_j in M_0, by Lemma 8 the set U contains bidder $\sigma_i(k')$. Since bidder $\sigma_i(k')$ is not matched in M_0, the dummy item v_{n+i} is also not matched in M_0. Hence, adding the pair $(\sigma_i(k'), v_{n+i})$ to M_0 gives a matching in (U, V) with the same weight and larger cardinality. This contradicts the fact that M_0 is a greedy MWM of (U, V). $\square$

Lemma 13. Algorithm 2 produces a weakly stable matching.

Proof. By Lemma 12, it remains only to show that μ does not admit a strongly blocking pair. Consider man p_i and woman $q_{j'}$, where i and j' belong to $\{1, \ldots, n\}$. We want to show that $(p_i, q_{j'})$ does not form a strongly blocking pair. Let q_j denote $\mu(p_i)$ and let $p_{i'}$ denote $\mu(q_{j'})$, where j belongs to $\{1, \ldots, n\} \cup \{n+i\}$ and i' belongs to $\{0, 1, \ldots, n\}$. It suffices to show that either $\kappa_i(q_j) \leq \kappa_i(q_{j'})$ or $\psi_{q_{j'}}(p_{i'}) \geq \psi_{q_{j'}}(p_i)$. For the sake of contradiction, suppose $\kappa_i(q_j) > \kappa_i(q_{j'})$ and $\psi_{q_{j'}}(p_{i'}) < \psi_{q_{j'}}(p_i)$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{j'})$. Since $\sigma_i(k)$ is matched in M_0 and $k' < k$, Lemma 8 implies that the set U contains bidder $\sigma_i(k')$ and that $\sigma_i(k')$ is unmatched in M_0. We consider two cases.

Case 1: $i' = 0$. Then $\psi_{q_{j'}}(p_i) > \psi_{q_{j'}}(p_{i'}) = 0$. Since neither bidder $\sigma_i(k')$ nor item $v_{j'}$ is matched in M_0, adding the pair $(\sigma_i(k'), v_{j'})$ to M_0 gives a matching of (U, V) with a larger weight. This contradicts the fact that M_0 is an MWM of (U, V).

Case 2: $i' \neq 0$. Since $p_{i'} = \mu(q_{j'})$, there exists k'' such that bidder $\sigma_{i'}(k'')$ is matched to $v_{j'}$ in M_0. Since $\sigma_i(k')$ is unmatched in M_0, the matching $M_0 - (\sigma_{i'}(k''), v_{j'}) + (\sigma_i(k'), v_{j'})$ is a matching of (U, V) with weight $w(M_0) - \psi_{q_{j'}}(p_{i'}) + \psi_{q_{j'}}(p_i)$, which is greater than $w(M_0)$. This contradicts the fact that M_0 is an MWM of (U, V). $\square$

Lemma 14. Let μ be the matching produced by Algorithm 2 and let μ' be a matching such that $\mu'(p) \succeq_p \mu(p)$ for every man p in P and $\sum_{q \in Q} \psi_q(\mu'(q)) \geq \sum_{q \in Q} \psi_q(\mu(q))$. Then $\mu(p) \succeq_p \mu'(p)$ for every man p in P and $\sum_{q \in Q} \psi_q(\mu'(q)) = \sum_{q \in Q} \psi_q(\mu(q))$.

Proof. For any i such that $1 \leq i \leq n$, let k_i denote $\kappa_i(\mu(p_i))$ and let k'_i denote $\kappa_i(\mu'(p_i))$.

Below we use μ' to construct an MWM M'_0 of (U, V). We give the construction of M'_0 first, and then argue that M'_0 is an MWM of (U, V). Let M'_0 denote the set of bidder-item pairs $(\sigma_i(k'_i), v_j)$ such that $\mu'(p_i) = q_j$ where i in $\{1, \ldots, n\}$ and j in $\{1, \ldots, n\} \cup \{n+i\}$. It is easy to see that M'_0 is a valid matching. Notice that for any $1 \leq i \leq n$, since $\mu'(p_i) \succeq_{p_i} \mu(p_i)$, we have $k'_i \leq k_i$. So, by Lemma 8, the set U contains all bidders $\sigma_i(k'_i)$. Hence, M'_0 is a matching of (U, V). Furthermore, it is easy to see that

$$w(M'_0) = \sum_{1 \leq j \leq n} \psi_{q_j}(\mu'(q_j)) \geq \sum_{1 \leq j \leq n} \psi_{q_j}(\mu(q_j)) = w(M_0).$$

Thus M'_0 is an MWM of (U, V), and we have $\sum_{1 \leq j \leq n} \psi_{q_j}(\mu'(q_j)) = \sum_{1 \leq j \leq n} \psi_{q_j}(\mu(q_j))$.

Furthermore, M'_0 is an MCMWM of (U, V) because both M'_0 and M_0 have cardinality equal to n. Also, M'_0 is a greedy MWM of (U, V), because both M'_0 and M_0 have priorities equal to $\sum_{1 \leq i \leq n} z_i$. Hence, for each $1 \leq i \leq n$, we have $k_i \leq k'_i$ by Lemma 8. Thus, $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for all $1 \leq i \leq n$. $\square$

Lemma 15. Let μ be the matching produced by Algorithm 2 and μ' be a matching such that $\mu' \succeq \mu$. Then, $\mu \succeq \mu'$.

Proof. Since $\mu' \succeq \mu$, we have $\mu'(p_i) \succeq_{p_i} \mu(p_i)$ and $\psi_{q_j}(\mu'(q_j)) \geq \psi_{q_j}(\mu(q_j))$ for every i and j in $\{1, \ldots, n\}$. So, by Lemma 14, we have $\mu(p_i) \succeq_{p_i} \mu'(p_i)$ for every i in $\{1, \ldots, n\}$ and

$$\sum_{1 \leq j \leq n} \psi_{q_j}(\mu'(q_j)) = \sum_{1 \leq j \leq n} \psi_{q_j}(\mu(q_j)).$$

Therefore, $\psi_{q_j}(\mu'(q_j)) = \psi_{q_j}(\mu(q_j))$ for every j in $\{1, \ldots, n\}$. This shows that $\mu \succeq \mu'$. $\square$

Lemma 16. Consider Algorithm 2. Suppose $\mu(p_i) = q_j$, where $1 \leq i \leq n$ and j belongs to $\{1, \ldots, n\} \cup \{n + i\}$. Then, we have

$$(\psi_{q_j}(p_i), i) \geq \mathit{threshold}((T - t_i, V), v_j). \tag{3}$$

Furthermore, for all j' in $\{1, \ldots, n\} \cup \{n + i\}$ such that $\kappa_i(q_{j'}) < \kappa_i(q_j)$, we have

$$(\psi_{q_{j'}}(p_i), i) < \mathit{threshold}((T - t_i, V), v_{j'}). \tag{4}$$

Proof. Let k denote $\kappa_i(q_j)$. Since $\mu(p_i) = q_j$, we know that bidder $\sigma_i(k)$ is matched to item v_j in M_0. So, inequality (1) of Lemma 10 implies inequality (3), because $w(\sigma_i(k), v_j) = \psi_{q_j}(p_i)$ and $z_i = i$.

Now, suppose $\kappa_i(q_{j'}) < \kappa_i(q_j)$. Let k' denote $\kappa_i(q_{j'})$. Since $k' < k$, inequality (2) of Lemma 10 implies inequality (4), because $w(\sigma_i(k'), v_{j'}) = \psi_{q_{j'}}(p_i)$ and $z_i = i$. $\square$

Theorem 1. Algorithm 2 is a strategyproof Pareto-stable mechanism for the stable marriage problem with incomplete and weak preferences (for any fixed choice of utility assignment).

Proof. We have shown in Lemma 13 that the algorithm produces a weakly stable matching. Moreover, Lemma 15 shows that the weakly stable matching produced is not Pareto-dominated by any other matching. Hence, the algorithm produces a Pareto-stable matching. It remains to show that the algorithm is a strategyproof mechanism.

Suppose man p_i expresses $\succeq'_{p_i}$ instead of his true preference relation $\succeq_{p_i}$, where $1 \leq i \leq n$. Let μ and μ' be the resulting matchings given $\succeq_{p_i}$ and $\succeq'_{p_i}$, respectively. Let q_j denote $\mu(p_i)$ and let $q_{j'}$ denote $\mu'(p_i)$, where j and j' belong to $\{1, \ldots, n\} \cup \{n + i\}$. Let k denote $\kappa_i(q_j)$ and let k' denote $\kappa_i(q_{j'})$, where $\kappa_i(\cdot)$ denotes the tier number with respect to $\succeq_{p_i}$. It suffices to show that $k \leq k'$. For the sake of contradiction, suppose $k > k'$.

Let (T, V) be the IUAP, let t_i be the multibidder corresponding to man p_i, and let $v_{j'}$ be the item corresponding to woman $q_{j'}$ constructed in the algorithm given input $\succeq_{p_i}$. Since $\mu(p_i) = q_j$, by inequality (4) of Lemma 16, we have

$$(\psi_{q_{j'}}(p_i), i) < \mathit{threshold}((T - t_i, V), v_{j'}).$$

Now, consider the behavior of the algorithm when preference relation $\succeq_{p_i}$ is replaced with $\succeq'_{p_i}$. Let (T', V') be the IUAP, let t'_i be the multibidder corresponding to man p_i, and let $v'_{j'}$ be the item corresponding to woman $q_{j'}$ constructed in the algorithm given input $\succeq'_{p_i}$. Since $\mu'(p_i) = q_{j'}$, by inequality (3) of Lemma 16, we have

$$(\psi_{q_{j'}}(p_i), i) \geq \mathit{threshold}((T' - t'_i, V'), v'_{j'}).$$

Notice that in Algorithm 2, the only part of the IUAP instance that depends on the preferences of man p_i is the multibidder corresponding to man p_i. In particular, we have $T - t_i = T' - t'_i$, $V = V'$, and $v_{j'} = v'_{j'}$. Hence, we get

$$(\psi_{q_{j'}}(p_i), i) < \mathit{threshold}((T - t_i, V), v_{j'}) = \mathit{threshold}((T' - t'_i, V'), v'_{j'}) \leq (\psi_{q_{j'}}(p_i), i),$$

which is a contradiction. $\square$

5 College Admissions with Indifferences

Our strategyproof mechanism can be generalized to the college admissions model with weak preferences. In this model, students and colleges play the roles of men and women respectively, and colleges are allowed to be matched with multiple students up to their capacities. Colleges' preferences over groups of students can be obtained from preferences over individual students using minimal responsiveness. The formal definitions are given in [3, Section 5].

We can apply our mechanism for SMIW by transforming each student to a man and each slot of a college to a woman in a standard fashion. In [3, Section 5.1], we show that this yields a strategyproof Pareto-stable mechanism when colleges' group preferences are minimally responsive. In [3, Section 5.2], we introduce the notion of group preferences induced by additive utilities, for which our results also hold.

Our algorithm admits an $O(n^4)$-time implementation, where n is the sum of the number of students and the total capacities of all the colleges, because the reduction from CAW to IUAP takes $O(n^2)$ time, and lines 16 and 17 of Algorithm 2 can be implemented in $O(n^4)$ time using the version of the incremental Hungarian method discussed in Section 3.1.

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