A $(\ln 4)$-Approximation Algorithm for Maximum Stable Matching with One-Sided Ties and Incomplete Lists*

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Abstract

We study the problem of finding large weakly stable matchings with one-sided ties and incomplete lists. Computing maximum weakly stable matchings is known to be NP-hard. We present a polynomial-time algorithm that achieves an improved approximation ratio of $\ln 4$. Some of the previous approximation algorithms for this problem maintain a non-decreasing priority for each agent on one side of the market, and use these priorities for tie-breaking purposes. Our algorithm is motivated by the idea of incrementing the priorities by a small step size, and our analysis involves an associated infinite-dimensional optimization problem. We also show that the integrality gap is at most $\ln 4$.

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1 Introduction

The stable matching problem, introduced by Gale and Shapley [2], involves two disjoint sets of agents, typically called men and women in the literature. Each agent has ordinal preferences over the agents of the opposite sex. The objective is to find a set of disjoint man-woman pairs, called a *matching*, such that no man and woman prefer each other to their partners. Matchings satisfying this property are said to be stable and can computed efficiently by the Gale-Shapley algorithm. Stable matchings have applications such as centralized schemes for recruiting residents to hospitals [18].

Ties and incomplete lists arise naturally in real-world problems. The preference list of an agent is said to contain a tie when the agent is indifferent between two or more agents of the opposite sex. The preference list of an agent is said to be incomplete when one or more agents of the opposite sex are unacceptable to the agent. For such variants, the notion of stability can be generalized to weak stability, strong stability, or super-stability [7]. In this paper, we focus on weak stability since weakly stable matchings, unlike strongly stable or super-stable matchings, always exist. A weakly stable matching can be obtained by arbitrarily breaking all ties before invoking the Gale-Shapley algorithm. When either ties or incomplete lists are absent, all weakly stable matchings have the same size [3, 18]. However, when both ties and incomplete lists are present, the resulting weakly stable matchings can have different sizes, depending on the way ties are broken.

The problem of finding large weakly stable matchings with ties and incomplete lists has been theoretically challenging. Iwama et al. [9] first demonstrated that finding a maximum weakly stable matching with ties and incomplete lists is NP-hard. Results by Yanagisawa [21] imply that getting an approximation ratio of $(\frac{33}{29} - \varepsilon)$ (≈ 1.1379) is NP-hard, and achieving a ratio of $(\frac{4}{3} - \varepsilon)$ (≈ 1.3333) is UG-hard. Notwithstanding these hardness results, it is straightforward to see that any weakly stable matching is a 2-approximate solution [14]. Using a local search approach, Iwama et al. [10] gave an algorithm with an approximation ratio of $\frac{15}{8}$ ($= 1.875$). Király [12] improved the approximation ratio to $\frac{5}{3}$ (≈ 1.6667) by introducing the idea of promoting unmatched agents to higher priorities for tie-breaking. The current best approximation ratio for two-sided ties and incomplete lists is $\frac{3}{2}$ ($= 1.5$), which is attained by the polynomial-time algorithm of McDermid [15], and the linear-time algorithms of Paluch [16] and Király [13]. This ratio coincides with a lower bound of the integrality gap of an associated linear programming formulation [11], indicating that there seems to be a strong barrier for further improvements.

Ties often appear only on one side of the market, especially in settings where institutions need to evaluate a large number of candidates. For example, in the Scottish Foundation Allocation Scheme, residents have strict preferences but the preferences of the hospitals may contain ties [8]. With one-sided ties and incomplete lists, the problem of finding a maximum weakly stable matching remains NP-hard [14]. Results by Halldórsson et al. [4] imply that getting an approximation ratio of $(\frac{21}{19} - \varepsilon)$ (≈ 1.105) is NP-hard, and achieving a ratio of $(\frac{5}{4} - \varepsilon)$ (≈ 1.25) is UG-hard. Király, who showed an approximation ratio of $\frac{3}{2}$ ($= 1.5$) for an algorithm based on the idea of promotion, conjectured that a $(\frac{3}{2} - \varepsilon)$-approximation is UG-hard even for one-sided ties [12]. However, Iwama et al. [11] later presented an algorithm based on linear programming with an approximation ratio of $\frac{25}{17}$ (≈ 1.4706). For the same problem, Huang and Kavitha [5] gave an algorithm based on rounding half-integral stable matchings that has an approximation ratio of $\frac{22}{15}$ (≈ 1.4667). An improved analysis of their algorithm by Radnai [17] establishes an approximation ratio of $\frac{41}{28}$ (≈ 1.4643). Dean and Jalasutram [1] revisited the linear programming approach and improved the approximation ratio to $\frac{19}{13}$ (≈ 1.4615) by solving a large factor-revealing linear program with a computer. It is known that $(1 + \frac{1}{e})$ (≈ 1.3679) is a lower bound for the integrality gap of the linear programming formulation associated with one-sided ties and incomplete lists [11]. In a paper by Huang et al. [6], the integrality gap is claimed to be at least $\frac{3}{2}$, but their proof contains an error[1].

In this paper, we focus on the maximum stable matching problem with one-sided ties and incomplete lists. We improve the linear programming approach by adopting a different way of manipulating the priorities. Our algorithm is motivated by a process in which the priorities are incremented by an infintesimally small step size. We show that our algorithm has an approximation ratio of $\ln 4$ (≈ 1.386) by considering to an

[1] In the proof of this claim [6, Theorem 19], Huang et al. exhibit a family of instances with $2k$ men and $2k$ women such that the corresponding linear program has a feasible fractional value of $(3/2 - o(1))k$. It is asserted that a certain weakly stable matching of size k is a maximum weakly stable matching, but this assertion is incorrect. For the case when $k = 2$, there exists a weakly stable matching of size 3. Similarly, when $k > 2$, it can be shown that the maximum size of weakly stable matching is greater than k. In a personal communication, Huang et al. have acknowledged this error.

associated infinite-dimensional optimization problem. Our analysis also shows that the integrality gap is at most $\ln 4$.

2 Stable Matching with One-Sided Ties and Incomplete Lists

2.1 The Model

The stable matching problem with one-sided ties and incomplete lists (SMOTI) involves a set I of men and a set J of women. We assume that the sets I and J are disjoint and do not contain the element 0, which we use to denote being unmatched. Each man $i \in I$ has a preference relation $\geq_i$ over the set $J \cup \{0\}$ that satisfies antisymmetry, transitivity, and totality. Each woman $j \in J$ has a preference relation $\geq_j$ over the set $I \cup \{0\}$ that satisfies transitivity and totality. We denote this SMOTI instance as $(I, J, \{\geq_i\}_{i \in I}, \{\geq_j\}_{j \in J})$.

Notice that the preference relations $\{\geq_j\}_{j \in J}$ of the women are not required to be antisymmetric but the preference relations $\{\geq_i\}_{i \in I}$ of the men are required to be antisymmetric. So ties are allowed in the preferences of the women, but not in the preferences of the men. For every woman $j \in J$, we denote $>_j$ and $=_j$ as the asymmetric part and the symmetric part of $\geq_j$, respectively. Similarly, for every man $i \in I$, we denote $>_i$ as the asymmetric part of $\geq_i$. Notice that preference lists are allowed to be incomplete. For example, if a woman j is unacceptable to a man i, then we have $0 >_i j$. Similarly, if a man i is unacceptable to a woman j, then we have $0 >_j i$.

A matching is a subset $\mu \subseteq I \times J$ such that for every $(i, j), (i', j') \in \mu$, we have $i = i'$ if and only if $j = j'$. For every man $i \in I$, if $(i, j) \in \mu$ for some woman $j \in J$, we say that man i is matched to woman j in matching μ, and we write $\mu(i) = j$. Otherwise, we say that man i is unmatched in matching μ, and we write $\mu(i) = 0$. Similarly, for every woman $j \in J$, if $(i, j) \in \mu$ for some man $i \in I$, we say that woman j is matched to man i in matching μ, and we write $\mu(j) = i$. Otherwise, we say that woman j is unmatched in matching μ, and we write $\mu(j) = 0$.

A matching μ is *individually rational* if for every $(i, j) \in \mu$, we have $j \geq_i 0$ and $i \geq_j 0$. An individually rational matching μ is *weakly stable* if for every man $i \in I$ and woman $j \in J$, either $\mu(i) \geq_i j$ or $\mu(j) \geq_j i$. Otherwise, (i, j) forms a *strongly blocking pair*.

The goal of the maximum stable matching problem with one-sided ties and incomplete lists (MAX-SMOTI) is to find a weakly stable matching μ with the largest cardinality given an SMOTI instance.

2.2 The Linear Programming Formulation

The following linear programming formulation is based on that of Rothblum [19], which extends that of Vande Vate [20].

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(i,j) \in I \times J} x_{i,j} & & \\
\text{subject to} \quad & \sum_{j \in J} x_{i,j} \leq 1 & \forall i \in I & \qquad (1)\\
& \sum_{i \in I} x_{i,j} \leq 1 & \forall j \in J & \qquad (2)\\
& \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'} + \sum_{\substack{i' \in I \\ i' \geq_j i}} x_{i',j} \geq 1 & \forall (i,j) \in I \times J \text{ such that } j >_i 0 \text{ and } i >_j 0 & \qquad (3)\\
& x_{i,j} = 0 & \forall (i,j) \in I \times J \text{ such that } 0 >_i j \text{ or } 0 >_j i & \qquad (4)\\
& x_{i,j} \geq 0 & \forall (i,j) \in I \times J & \qquad (5)
\end{aligned}
$$

In Lemmas 1 and 2, we present two straightforward properties of the linear programming formulation. Vande Vate [20] used constraint (6) in Lemma 2 together with constraints (1), (2), and (5) to characterize stable matchings for the special case where all preference lists are complete and the number of men equals the number of women. Rothblum [19] extended the result of Vande Vate and used constraints (1), (2), (3), (4), and (5) to characterize stable matchings for the model with strict preferences and incomplete lists, and where the number of men is not necessarily equal to the number of women. This formulation was adapted to study maximum weakly stable matching with one-sided ties and incomplete lists by Iwama et al. [11], and by Dean and Jalasutram [1]. Our model also allows a woman to be indifferent between being unmatched and being matched with some of the men. Accordingly, for the sake of completeness, we provide the proofs of Lemmas 1 and 2 in Appendix A.1.

Lemma 1. *An integral solution $\{x_{i,j}\}_{(i,j)\in I\times J}$ corresponds to the indicator variables of a weakly stable matching if and only if it satisfies constraints* (1), (2), (3), (4), *and* (5).

Lemma 2. *Let $\{x_{i,j}\}_{(i,j)\in I\times J}$ be a fractional solution that satisfies constraints* (2), (3), (4), *and* (5). *Then, the following constraint is also satisfied.*

$$\sum_{\substack{j'\in J\\ j'>_i j}} x_{i,j'} \geq \sum_{\substack{i'\in I\\ i>_j i'}} x_{i',j} \qquad \forall (i,j)\in I\times J \text{ such that } j \geq_i 0 \text{ and } i \geq_j 0 \tag{6}$$

3 The Algorithm

3.1 Deferred Acceptance with Priorities

In this subsection, we describe a deferred acceptance process with priorities which is similar to that of Iwama et al. [11], and that of Dean and Jalasutram [1]. The actual implementation of our algorithm is motivated by this process, and is presented in the next subsection.

Our deferred acceptance process with priorities takes an SMOTI instance and a step size parameter $\delta > 0$ as input, and produces a weakly stable matching μ as output. In the preprocessing phase, we computes an optimal fractional solution $\{x_{i,j}\}_{(i,j)\in I\times J}$ to the associated linear program. Then, in the initialization phase, we assign the empty matching to μ and each man i gets a priority p_i equal to 0. For each man i, we also maintain a set L_i of women, which is initialized to the empty set. We use the set L_i to store the women to whom man i will propose before his priority p_i is increased by δ. After that, the process enters the proposal phase and proceeds iteratively.

In each iteration, we pick an unmatched man i with priority $p_i < 1 + \delta$. If the set L_i is empty, we increment his priority p_i by δ and then update L_i to the set

$$\Big\{ j \in J \colon j \geq_i 0 \text{ and } \sum_{\substack{j'\in J\\ j'>_i j}} x_{i,j'} \leq p_i \Big\}.$$

Otherwise, the man i that we pick has a non-empty set L_i of women. Let j denote his most preferred woman j in L_i. We remove j from L_i and have man i propose to woman j. When woman j receives the proposal from man i, she tentatively accepts him if she is currently unmatched and he is acceptable to her. Otherwise, if woman j is currently matched to another man i', she tentatively accepts her preferred choice between men i and i', and rejects the other. In case of a tie, she compares the current priorities p_i and $p_{i'}$ of the men and accepts the one with a higher priority. (If the priorities of i and i' are equal, she breaks the tie arbitrarily.) If man i is temporarily accepted by woman j, we update the matching μ accordingly.

When every unmatched man i has priority $p_i \geq 1 + \delta$, the process terminates and outputs the final matching μ.

Our process is similar to that of Iwama et al. [11], and that of Dean and Jalasutram [1], which also use a proposal scheme with priorities. In particular, the way we populate the set L_i with a subset of women by referring to the solution of the linear program is based on their methods. The major difference is that, in our process, priorities only increase by a small step size δ, whereas in their algorithms, the priorities may increase by a possibly larger amount, essentially to ensure that a new woman is included into L_i.

As in their algorithms, for every woman j, as our process executes, the temporary partner $\mu(j)$ of woman j satisfies a natural monotonicity property. Woman j is initially unmatched, and becomes matched the first time she receives a proposal from a man who is acceptable to her. In each subsequent iteration, she either keeps her current partner or gets a weakly preferred partner. Furthermore, if she is indifferent between her new partner and her old partner, then the new partner has a weakly larger priority. When the process terminates, we have the following observations, which are analogous to properties of the algorithms of Iwama et al. [11] and Dean and Jalasutram [1].

(O1) Let $i \in I$ be a man such that $\mu(i) \neq 0$. Then $\sum_{\substack{j \in J \\ j >_i \mu(i)}} x_{i,j} \leq p_i \leq 1 + \delta$.

(O2) Let $(i, j) \in \mu$. Then $j \geq_i 0$ and $i \geq_j 0$.

(O3) Let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i \mu(i)$ and $i \geq_j 0$. Then $\mu(j) \neq 0$ and $\mu(j) \geq_j i$.

(O4) Let $i \in I$ be a man such that $\mu(i) \neq 0$, and let $j \in J$ be a woman such that $j \geq_i 0$ and $i \geq_j 0$. If $p_i - \delta > \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$, then $\mu(j) \neq 0$ and $\mu(j) \geq_j i$.

(O5) Let $i \in I$ be a man such that $\mu(i) \neq 0$, and let $j \in J$ be a woman such that $j \geq_i 0$ and $\mu(j) =_j i$. If $p_i - \delta > \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$, then $\mu(j) \neq 0$ and $p_{\mu(j)} \geq p_i - \delta$.

(O6) Let $i \in I$ be a man such that $\mu(i) = 0$, and let $j \in J$ be a woman such that $j \geq_i 0$ and $\mu(j) =_j i$. Then $\mu(j) \neq 0$ and $p_{\mu(j)} \geq 1$.

For (O1), it is easy to see that the priority p_i of man i lies within the specified range when he proposes to woman $\mu(i)$.

For (O2), it is easy to see that man i proposes to woman j only if she is acceptable to him, and woman j accepts a proposal from man i only if he is acceptable to her.

For (O3), if man i weakly prefers woman j to $\mu(i)$ and is acceptable to woman j, then man i has proposed to woman j. Thus the monotonicity property implies that $\mu(j) \neq 0$ and $\mu(j) \geq_j i$.

For (O4), if man i and woman j satisfy the stated assumptions, then man i proposed to woman j when his priority was equal to $p_i - \delta$, and this proposal was eventually rejected. Immediately after this proposal was rejected, woman j was matched with a man i' such that $i' \neq i$ and $i' \geq_j i$. The monotonicity property implies that $\mu(j) \neq 0$ and $\mu(j) \geq_j i' \geq_j i$.

For (O5), if man i and woman j satisfy the stated assumptions, then man i proposed to woman j when his priority was equal to $p_i - \delta$, and this proposal was eventually rejected. Immediately after this proposal was rejected, woman j was matched with a man i' such that $i' \neq i$ and $i' \geq_j i$. The monotonicity property implies that $\mu(j) \neq 0$ and $\mu(j) \geq_j i' \geq_j i$. Since $\mu(j) =_j i$, we conclude that $\mu(j) =_j i' =_j i$. Since $i' =_j i$, we conclude that the priority of i' was at least $p_i - \delta$ when the aforementioned proposal was rejected. Since $\mu(j) =_j i'$, the monotonicity property implies that $p_{\mu(j)} \geq p_i - \delta$.

For (O6), if man i and woman j satisfy the stated assumptions, then man i proposed to woman j when his priority was at least 1, and this proposal was eventually rejected. Immediately after this proposal was rejected, woman j was matched with a man i' such that $i' \neq i$ and $i' \geq_j i$. Arguing as in the preceding paragraph, we deduce that $\mu(j) \neq 0$ and $\mu(j) =_j i' =_j i$. Since $i' =_j i$, the priority of i' was at least 1 when the aforementioned proposal was rejected. Since $\mu(j) =_j i'$, the monotonicity property implies that $p_{\mu(j)} \geq 1$.

3.2 The Implementation

The deferred acceptance process with priorities that we present in Section 3.1 depends on a step size parameter $\delta > 0$. To obtain a good approximation ratio, we would like the step size parameter δ to be small. However, the running time of a naive implementation grows in proportion to $\frac{1}{\delta}$. We can imagine that if we take an infinitesimally small step size, conditions (O1), (O2), (O3), (O4), (O5), and (O6) can be satisfied with $\delta = 0$. In this subsection, we present a polynomial-time algorithm which is motivated by the idea of simulating the process of Section 3.1 with an infinitesimally small step size.

For any man i in I, we define J_i as the set of all women j in J such that $j >_i 0$ and $i \geq_j 0$. For any man i in I, and any integer k such that $0 \leq k \leq |J_i|$, we define $J_{i,k}$ as the top k choices of man i out of the women

Algorithm 1

compute an optimal fractional solution $\{x_{i,j}\}_{(i,j)\in I\times J}$ to the associated linear program
initialize bipartite graph G to $(I, J, \emptyset)$ and μ to the empty matching
initialize counter $c(G,i)$ to 0 and priority p_i to $w(i,0)$ for each man $i \in I$
while there exists a man $i \in I$ such that $\mu(i) = 0$ and $c(G,i) < |J_i|$ **do**
  let i_0 be such a man, and increment $c(G,i_0)$
  let j_0 denote the woman in the singleton set $J_{i_0,c(G,i_0)} \setminus J_{i_0,c(G,i_0)-1}$
  if $\mu(j_0) = 0$ **then**
    add edge (i_0, j_0) to G and μ
  else
    let i_1 denote $\mu(j_0)$ if $i_0 >_{j_0} \mu(j_0)$, and i_0 otherwise
    if $i_0 =_{j_0} \mu(j_0)$ **then**
      add edge (i_0, j_0) to G
    else if $i_0 >_{j_0} \mu(j_0)$ **then**
      remove all edges incident to j_0 from G and μ, and add edge (i_0, j_0) to G and μ
    end if
    use alternating breadth-first search to identify the set I' of all men reachable from i_1 via a μ-alternating path in G of length zero or larger
    let i_2 be a man in I' with minimum weight
    let P be a μ-alternating path from i_1 to i_2
    update p_i to $\max(p_i, w(G,i_2))$ for each man i in I'
    update μ to $\mu \oplus P$
  end if
end while
return matching μ

in J_i. For any man i in I, and any integer k such that $0 \le k < |J_i|$, we define $w(i,k)$ as $\sum_{j\in J_{i,k}} x_{i,j}$. In addition, we define $w(i,|J_i|)$ as 1. Our algorithm maintains a bipartite graph G with vertex set $I \cup J$. Each man i in G has an associated integer counter. We write $c(G,i)$ as a shorthand for "the counter of man i in bipartite graph G". We define the *weight* of man i in G, denoted $w(G,i)$, as $w(i, c(G,i))$. For any matching μ of G, we define a μ-alternating path in G as a path that alternates between edges in μ and not in μ.

The details of the implementation are presented in Algorithm 1, and the proof of Lemma 3 is given in Appendix A.2.

Lemma 3. *When Algorithm 1 terminates, conditions (O1), (O2), (O3), (O4), (O5), and (O6) hold with* $\delta = 0$.

Lemma 4. *Algorithm 1 produces a weakly stable matching in polynomial time.*

Proof. It is easy to see that Algorithm 1 runs in polynomial time since the number of iterations is at most $|I| \times |J|$. Let μ be the matching produced by Algorithm 1. Condition (O2) of Lemma 3 implies that μ is individually rational. To show weak stability, consider $(i,j) \in I \times J$. It suffices to show that (i,j) is not a strongly blocking pair. For the sake of contradiction, suppose $j >_i \mu(i)$ and $i >_j \mu(j)$. We consider two cases.

Case 1: $0 >_j i$. Then $0 >_j i >_j \mu(j)$, which contradicts the individual rationality of μ.

Case 2: $i \ge_j 0$. Since $j >_i \mu(i)$ and $i \ge_j 0$, condition (O3) implies that $\mu(j) \ge_j i$, which contradicts the assumption that $i >_j \mu(j)$. □

4 The Analysis

In this section, we analyze the approximation ratio and the integrality gap. Throughout this section, whenever we mention $\{x_{i,j}\}_{(i,j)\in I\times J}$, $\{p_i\}_{i\in I}$, and μ, unless otherwise specified, we are referring to their values when Algorithm 1 terminates. We also denote $I^* = \{i \in I : \mu(i) \ne 0\}$ as the set of matched men and $J^* = \{j \in J : \mu(j) \ne 0\}$ as the set of matched women in the output matching μ.

4.1 Auxiliary Charges

For every man $i \in I^*$ and woman $j \in J$, we define the auxiliary charge

$$\widetilde{x}_{i,j} = \min\Big(x_{i,j}, \max\Big(0, p_i - \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}\Big)\Big).$$

So, for every man $i \in I^*$, the auxiliary charges $\{\widetilde{x}_{i,j}\}_{j \in J}$ correspond to the following charging process: Go through the list of all woman j from the most preferred woman to the least preferred woman under the preference relation $\geq_i$, and charge an amount of at most $x_{i,j}$ until a total charge of p_i is reached or the list of women is exhausted. The quantities $\{\widetilde{x}_{i,j}\}_{j \in J}$ correspond to the charged amount.

For every man $i \in I^*$, we define the auxiliary quantity

$$q_i = \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j}.$$

For every woman $j \in J^*$, we define the auxiliary quantities

$$y_j = \sum_{\substack{i \in I^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} \qquad \text{and} \qquad z_j = p_{\mu(j)} - \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j}.$$

We also denote

$$\mathbb{1}_{[0,\infty)}(\xi) = \begin{cases} 1 & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}$$

as the Heaviside step function.

In the definitions of $\{q_i\}_{i \in I^*}$, every term in the summation corresponds to a tie-related charge associated with a pair (i, j), since woman j is indifferent between i and $\mu(j)$. So for every man $i \in I^*$, the quantity q_i corresponds to the total tie-related charge associated with man i and all matched women. Similarly, for every woman $j \in J^*$, the quantity y_j corresponds to the total tie-related charge associated with woman j and all matched men. The auxiliary quantities $\{z_j\}_{j \in J^*}$ will be useful in Section 4.2.

In Lemma 5, we present some simple properties of the charges $\{\widetilde{x}_{i,j}\}_{(i,j) \in I^* \times J}$. In Lemma 6, we present some bounds for the tie-related charges $\{q_i\}_{i \in I^*}$ and $\{y_j\}_{j \in J^*}$. In Lemma 7, we show that the difference between the size of the matching μ produced by the algorithm and the optimal fractional value of the linear program can be bounded by an expression involving the auxiliary quantities. The proofs are given in Appendix A.3.

Lemma 5. *Let $i \in I^*$ be a man. Then the following properties of the charges $\{\widetilde{x}_{i,j}\}_{j \in J}$ hold.*

(1) For every woman $j \in J$, if $x_{i,j} = 0$ or $p_i \leq \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$, then $\widetilde{x}_{i,j} = 0$.

(2) For all women $j, j' \in J$ such that $\widetilde{x}_{i,j} > 0$ and $j' >_i j$, we have $\widetilde{x}_{i,j'} = x_{i,j'}$.

(3) $\sum_{j \in J} \widetilde{x}_{i,j} \leq p_i$.

(4) $\sum_{j \in J} (x_{i,j} - \widetilde{x}_{i,j}) \leq 1 - p_i$.

Lemma 6. *The following properties of the tie-related charges $\{q_i\}_{i \in I^*}$ and $\{y_j\}_{j \in J^*}$ hold.*

(1) For every man $i \in I^*$, we have $0 \leq q_i \leq 1$.

(2) For every woman $j \in J^*$, we have $0 \leq y_j \leq 1 - p_{\mu(j)} + z_j - \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} x_{i,j} \leq \min(1 - q_{\mu(j)} + z_j, 1)$.

Lemma 7. $\Big(\sum_{i \in I} \sum_{j \in J} x_{i,j}\Big) - |I^*| \leq \sum_{i \in I^*} (q_i - \min(y_{\mu(i)}, z_{\mu(i)}))$.

4.2 The Stability Constraint and The Tie-Breaking Criterion

In this subsection, we develop the key ingredients underlying our analysis by carefully utilizing the stability constraint for matched women and the tie-breaking criterion for matched men. The stability constraint, which says that no matched woman can participate in a strongly blocking pair, corresponds to Lemma 2. Meanwhile, the tie-breaking criterion for a matched man corresponds to (O5).

In Lemma 8, we consider the stability constraint associated with a matched pairs $(\mu(j), j) \in \mu$ and use it to bound the auxiliary quantity z_j.

Lemma 8. *Let $j \in J^*$ be a woman. Then $0 \leq z_j \leq 1$.*

Proof. Let $i = \mu(j)$. Since $\mu(i) \neq 0$, condition (O1) of Lemma 3 implies

$$\sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'} \leq p_i \leq 1.$$

Hence

$$z_j = p_i - \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j} \leq 1 - \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j} \leq 1.$$

Also, since $(i, j) \in \mu$, condition (O2) of Lemma 3 implies $j \geq_i 0$ and $i \geq_j 0$. Hence

$$z_j = p_i - \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j} \geq \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'} - \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j} \geq 0,$$

where the last inequality follows from Lemma 2. □

In Lemma 9, we consider pairs $(i, j) \in I^* \times J^*$ where the associated auxiliary charge $\widetilde{x}_{i,j}$ is positive and woman j is indifferent between man i and the man $\mu(j)$ to whom she is matched. These are the pairs that contribute to the tie-related charges $\{q_i\}_{i \in I^*}$ and $\{y_j\}_{j \in J^*}$. By apply the tie-breaking criterion and the stability constraint to these pairs (i, j), we show that the sum of the tie-related charges associated with man i and all of the women j' such that $j \geq_i j'$ is at most z_j.

Lemma 9. *Let $i \in I^*$ be a man and $j \in J^*$ be a woman such that $\widetilde{x}_{i,j} > 0$ and $\mu(j) =_j i$. Then*

$$\sum_{\substack{j' \in J \\ \mu(j') =_{j'} i \\ j \geq_i j'}} \widetilde{x}_{i,j'} \leq z_j.$$

Proof. Since $\widetilde{x}_{i,j} > 0$, part (1) of Lemma 5 implies $x_{i,j} > 0$ and $p_i > \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$. Since $x_{i,j} > 0$, constraint (4) implies $j \geq_i 0$. So condition (O5) of Lemma 3 implies

$$p_i \leq p_{\mu(j)} = z_j + \sum_{\substack{i' \in I \\ \mu(j) >_j i'}} x_{i',j} = z_j + \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j}, \tag{7}$$

where the first equality follows from the definition of z_j, and the second equality follows from $\mu(j) =_j i$. Since $x_{i,j} > 0$, constraint (4) implies $j \geq_i 0$ and $i \geq_j 0$. Thus Lemma 2 implies

$$\sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'} \geq \sum_{\substack{i' \in I \\ i >_j i'}} x_{i',j}. \tag{8}$$

Since $\widetilde{x}_{i,j} > 0$, parts (3) and (2) of Lemma 5 imply

$$p_i - \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'} \geq \sum_{j' \in J} \widetilde{x}_{i,j'} - \sum_{\substack{j' \in J \\ j' >_i j}} \widetilde{x}_{i,j'} = \sum_{\substack{j' \in J \\ j \geq_i j'}} \widetilde{x}_{i,j'} \geq \sum_{\substack{j' \in J \\ \mu(j') =_{j'} i \\ j \geq_i j'}} \widetilde{x}_{i,j'}. \tag{9}$$

Combining (7), (8), and (9) gives the desired inequality. ☐

Lemma 10 below aggregates the inequalities of Lemma 9. Since there are many scenarios in which a tie-related charge is incurred, this results in a family of inequalities parameterized by ξ. The proof is given in Appendix A.4.

Lemma 10. *If I^* is non-empty, then for every $0 \leq \xi \leq 1$,*

$$\frac{1}{|I^*|} \sum_{i \in I^*} \max(q_i - \xi, 0) \leq \frac{1}{|I^*|} \sum_{i \in I^*} y_{\mu(i)} \cdot \mathbb{1}_{[0,\infty)}(z_{\mu(i)} - \xi).$$

4.3 Approximation Ratio

In order to obtain the approximate ratio, we use Lemmas 6, 8, and 10 to derive a bound for the right-hand side of Lemma 7. To take into account the unbounded number of inequalities in Lemma 10, we appeal to an associated optimization problem, the result of which is summarized in Lemma 11. The details of the proof are presented in Section 4.4 and Appendix A.5.

Lemma 11. *Let Q, Y, Z be real-valued random variables such that*

$$\Pr[0 \leq Q \leq 1 \text{ and } 0 \leq Z \leq 1 \text{ and } 0 \leq Y \leq \min(1 - Q + Z, 1)] = 1 \tag{10}$$

and

$$\mathrm{E}[\max(Q - \xi, 0)] \leq \mathrm{E}[Y \cdot \mathbb{1}_{[0,\infty)}(Z - \xi)] \qquad \forall 0 \leq \xi \leq 1 \tag{11}$$

Then $\mathrm{E}[Q - \min(Y, Z)] \leq \ln 4 - 1$*.*

In Lemma 12, we use Lemma 11 to bound the ratio of the fractional solution to the size of the matching. In Theorem 13, we give our main result of a $(\ln 4)$-approximation algorithm. In Theorem 14, we establish an upper bound on the integrality gap.

Lemma 12. $\sum_{i \in I} \sum_{j \in J} x_{i,j} \leq (\ln 4)|\mu|$.

Proof. We may assume that I^* is non-empty, for otherwise Lemma 7 implies $\sum_{i \in I} \sum_{j \in J} x_{i,j} = 0$. Consider picking a man $\tilde{\imath}$ uniformly at random from I^*. Let $(Q, Y, Z) = (q_{\tilde{\imath}}, y_{\mu(\tilde{\imath})}, z_{\mu(\tilde{\imath})})$. Then Lemmas 6 and 8 imply that (Q, Y, Z) satisfies (10). Also, Lemma 10 implies that (Q, Y, Z) satisfies (11). Hence Lemma 11 implies

$$\ln 4 - 1 \geq \mathrm{E}[Q - \min(Y, Z)] = \frac{1}{|I^*|} \sum_{i \in I^*} (q_i - \min(y_{\mu(i)}, z_{\mu(i)})).$$

Thus Lemma 7 implies

$$\begin{aligned} \sum_{i \in I} \sum_{j \in J} x_{i,j} &\leq |I^*| + \sum_{i \in I^*} (q_i - \min(y_{\mu(i)}, z_{\mu(i)})) \\ &\leq |I^*| + (\ln 4 - 1)|I^*| \\ &= (\ln 4)|\mu|. \end{aligned}$$

☐

Theorem 13. *Algorithm 1 is a polynomial-time $(\ln 4)$-approximation algorithm for the maximum stable matching problem with one-sided ties and incomplete lists.*

Proof. By Lemma 4, Algorithm 1 runs in polynomial time and produces a weakly stable matching μ. Let μ' be a maximum weakly stable matching, and $\{x'_{i,j}\}_{(i,j)\in I\times J}$ be the indicator variables of μ'. Since μ' is weakly stable, Lemma 1 implies that $\{x'_{i,j}\}_{(i,j)\in I\times J}$ satisfies constraints (1), (2), (3), (4), and (5). Hence

$$(\ln 4)|\mu| \geq \sum_{i\in I}\sum_{j\in J} x_{i,j} \geq \sum_{i\in I}\sum_{j\in J} x'_{i,j} = |\mu'|,$$

where the first inequality follows from Lemma 12, and the second inequality follows from the optimality of $\{x_{i,j}\}_{(i,j)\in I\times J}$. □

Theorem 14. *The integrality gap of the linear programming formulation in Section 2.2 is at most* $\ln 4$.

Proof. By Lemmas 4 and 12, there exists a weakly stable matching μ such that $(\ln 4)|\mu| \geq \sum_{i\in I}\sum_{j\in J} x_{i,j}$. Let $\{x'_{i,j}\}_{(i,j)\in I\times J}$ be the indicator variables of μ. Since μ is weakly stable, Lemma 1 implies that $\{x'_{i,j}\}_{(i,j)\in I\times J}$ is an integral solution satisfying constraints (1), (2), (3), (4), and (5). Since

$$(\ln 4)\sum_{i\in I}\sum_{j\in J} x'_{i,j} = (\ln 4)|\mu| \geq \sum_{i\in I}\sum_{j\in J} x_{i,j},$$

the integrality gap is at most $\ln 4$. □

4.4 The Associated Optimization Problem

The goal of this subsection is to prove Lemma 11. It is useful to rewrite constraint (11) as

$$\mathrm{E}[\max(Q - f(u), 0)] \leq \mathrm{E}[Y \cdot \mathbb{1}_{[0,\infty)}(Z - f(u))] \qquad \forall 0 \leq u \leq 1, \tag{12}$$

where

$$f(u) = \begin{cases} 2u - \ln(1+2u) & \text{if } 0 \leq u \leq \frac{1}{2} \\ 1 - (1-u)\ln 4 & \text{if } \frac{1}{2} < u \leq 1 \end{cases} \tag{13}$$

To prove Lemma 11, we will solve the associated optimization problem of maximizing $\mathrm{E}[Q - \min(Y, Z)]$ over all random variables (Q, Y, Z) subject to constraint (12).

For intuition, it is useful to treat (Q, Y, Z) as continuous random variables with a well-defined joint probability density function $\rho_{Q,Y,Z}$, even though our proofs do not require this assumption. Then one can optimize over $\rho_{Q,Y,Z}$ subject to constraint (12), together with the constraints that probability is non-negative and sums to 1. All of these constraints as well as the objective function turn out to be linear in $\rho_{Q,Y,Z}$. This results in an infinite-dimensional linear programming problem, where both the degrees of freedom and the number of constraints are infinite. Nevertheless, we show that it can be solved analytically.

Lemma 15 below can be regarded as a dual feasibility result. The probability density function ρ_U plays the role of the dual variables corresponding to constraint (12), while the dual variable λ corresponds to the constraint that probability sums to 1. The proof involves computing the expectation by integration and case analysis, and is given in Appendix A.5.

Lemma 15. *Let* $0 \leq q \leq 1$ *and* $0 \leq z \leq 1$*. Let* $0 \leq y \leq \min(1 - q + z, 1)$*. Let* U *be a continuous random variable with probability density function*

$$\rho_U(u) = \begin{cases} 2(1+2u)^{-2} & \textit{if } 0 < u < \frac{1}{2} \\ 4^{u-1}\ln 4 & \textit{if } \frac{1}{2} < u < 1 \\ 0 & \textit{otherwise} \end{cases} \tag{14}$$

and $\lambda = \ln 4 - 1$*. Then*

$$\mathrm{E}\left[\max(q - f(U), 0) - y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \geq q - \min(y, z) - \lambda.$$

We prove Lemma 11 using Lemma 15. Our proof is analogous to that of the weak-duality theorem for finite-dimensional linear programming, in which any feasible primal solution is bounded by any feasible dual solution. We stress that our proof works for both continuous and discrete random variables (Q, Y, Z), even though a well-defined joint density function $\rho_{Q,Y,Z}$ does not exist for discrete random variables.

Proof of Lemma 11. Pick U independent of (Q, Y, Z) according to the probability density function ρ_U defined by (14). Lemma 15 implies

$$\mathrm{E}\left[\max(Q - f(U), 0) - Y \cdot \mathbb{1}_{[0,\infty)}(Z - f(U))\right] \geq \mathrm{E}[Q - \min(Y, Z)] + 1 - \ln 4.$$

Since constraint (11) is satisfied by (Q, Y, Z), constraint (12) is also satisfied by (Q, Y, Z). Hence

$$\mathrm{E}\left[\max(Q - f(U), 0) - Y \cdot \mathbb{1}_{[0,\infty)}(Z - f(U))\right] \leq 0$$

Thus $\mathrm{E}[Q - \min(Y, Z)] \leq \ln 4 - 1$. □

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A Omitted Proofs

A.1 The Linear Programming Formulation

In this subsection, we prove Lemmas 1 and 2.

Proof of Lemma 1. Suppose $\{x_{i,j}\}_{(i,j)\in I\times J}$ satisfies constraints (1), (2), (3), (4), and (5). Constraints (1), (2), and (5) imply that $\{x_{i,j}\}_{(i,j)\in I\times J}$ corresponds to a valid matching μ. Constraint (4) implies that μ is individually rational. To show the weak stability of μ, consider man $i \in I$ and woman $j \in J$. It suffices to show that (i, j) is not a strongly blocking pair. We may assume that $j >_i 0$ and $i >_j 0$, for otherwise individual rationality implies $\mu(i) \geq_i 0 \geq_i j$ or $\mu(j) \geq_j 0 \geq_j i$. Consider constraint (3) associated with (i, j). At least one of the two summations is equal to 1. If the first summation equals 1, then $\mu(i) >_i j$. If the second summation equals 1, then $\mu(j) \geq_j i$. Thus, μ is a weakly stable matching.

Conversely, suppose $\{x_{i,j}\}_{(i,j)\in I\times J}$ corresponds to a weakly stable matching μ. Since μ is a valid matching, constraints (1), (2), and (5) are satisfied. Also, the individual rationality of μ implies that constraint (4) is satisfied. To show that constraint (3) is satisfied, consider $(i, j) \in I \times J$ such that $j >_i 0$ and $i >_j 0$. It suffices to show that at least one of the two summations in constraint (3) associated with (i, j) equals 1. By the weak stability of μ, we have either $\mu(i) \geq_i j$ or $\mu(j) \geq_j i$. We consider two cases.

Case 1: $\mu(j) \geq_j i$. Since $\mu(j) \geq_j i >_j 0$, the second summation equals 1.

Case 2: $i >_j \mu(j)$ and $\mu(i) \geq_i j$. Since $i >_j \mu(j)$, we have $(i, j) \notin \mu$. Since $\mu(i) \geq_i j$ and $(i, j) \notin \mu$, we have $\mu(i) >_i j$. Since $\mu(i) >_i j >_i 0$, the first summation equals 1. □

Proof of Lemma 2. Let $(i, j) \in I \times J$ with $j \geq_i 0$ and $i \geq_j 0$. We consider two cases.

Case 1: $i =_j 0$. Then

$$\sum_{\substack{i'\in I \\ i>_j i'}} x_{i',j} = \sum_{\substack{i'\in I \\ 0>_j i'}} x_{i',j} = 0 \leq \sum_{\substack{j'\in J \\ j'>_i j}} x_{i,j'},$$

where the second equality follows from constraint (4) and the inequality follows from constraint (5).

Case 2: $i >_j 0$. Since $j \in J$ and $j \geq_i 0$, we have $j >_i 0$. Since $j >_i 0$ and $i >_j 0$, constraint (3) implies

$$\sum_{\substack{j'\in J \\ j'>_i j}} x_{i,j'} \geq 1 - \sum_{\substack{i'\in I \\ i'\geq_j i}} x_{i',j} \geq \sum_{i'\in I} x_{i',j} - \sum_{\substack{i'\in I \\ i'\geq_j i}} x_{i',j} = \sum_{\substack{i'\in I \\ i>_j i'}} x_{i',j},$$

where the second inequality follows from constraint (2). □

A.2 The Implementation

The goal of this subsection is to analyze Algorithm 1 and prove Lemma 3. It is useful to define the following predicates.

(P1) μ is a matching of G that matches every woman with non-zero degree in G.

(P2) For every man $i \in I$ and woman $j \in J$, edge (i,j) belongs to G if and only if (1) j belongs to $J_{i,c(G,i)}$ and (2) there is no man i' such that j belongs to $J_{i',c(G,i')}$ and $i' >_j i$.

(P3) For any man i, $c(G,i) = 0$ implies $p_i = w(i,0)$, and $c(G,i) > 0$ implies $w(i, c(G,i) - 1) \leq p_i \leq w(i, c(G,i))$.

(P4) For any man i, $\mu(i) = 0$ implies $p_i = w(i, c(G,i))$.

(P5) For any edge (i,j) in G, $\mu(j) \neq 0$ and $p_i \leq p_{\mu(j)}$.

Lemma 16. *Consider an iteration of the loop in Algorithm 1. Suppose (P1) holds at the start of the iteration. Then (P1) holds before and after each line is executed.*

Proof. Let state 0 refer to the initial state before the execution of line 5, let state 1 refer to the intermediate state before the execution of line 20, and let state 2 refer to the final state after the execution of line 21. For each k in $\{0,1,2\}$, let $G^{(k)}$ and $\mu^{(k)}$ denote the values of G and μ in state k, and let $X^{(k)}$ denote the set of all women with non-zero degree in $G^{(k)}$. (If $\mu^{(0)}(j_0) = 0$, then we leave state 1 and the associated symbols undefined.) The predicate (P1) depends only on μ and the edge set of G. By identifying the lines that update either μ or the edge set of G, we find that it is sufficient to prove that (P1) holds in states 1 (if defined) and 2.

Case 1: $\mu^{(0)}(j_0) = 0$. Since (P1) holds in state 0, $G^{(2)} = G^{(0)} + (i_0, j_0)$, $\mu^{(2)} = \mu^{(0)} + (i_0, j_0)$ and $X^{(2)} = X^{(0)} + j_0$, we deduce that (P1) holds in state 2.

Case 2: $\mu^{(0)}(j_0) \neq 0$. Let i denote $\mu^{(0)}(j_0)$. We argue that (P1) holds in state 1 by considering three subcases.

Case 2.1: $i_0 <_{j_0} i$. Since (P1) holds in state 0, the edge set of $G^{(1)}$ is equal to that of $G^{(0)}$, $\mu^{(1)} = \mu^{(0)}$, and $X^{(1)} = X^{(0)}$, we deduce that (P1) holds in state 1.

Case 2.2: $i_0 =_{j_0} i$. Since (P1) holds in state 0, the edge set of $G^{(1)}$ is equal to that of $G^{(0)}$ plus (i_0, j_0), $\mu^{(1)} = \mu^{(0)}$, and $X^{(1)} = X^{(0)}$, we deduce that (P1) holds in state 1.

Case 2.3: $i_0 >_{j_0} i$. Since (P1) holds in state 0, the edge set of $G^{(1)}$ is the equal to that of $G^{(0)}$ except the set of edges incident on woman j_0 is $\{(i_0, j_0)\}$, $\mu^{(1)} = \mu^{(0)} - (i_1, j_0) + (i_0, j_0)$, and $X^{(1)} = X^{(0)}$, we deduce that (P1) holds in state 1.

Since (P1) holds in state 1, $G^{(1)} = G^{(2)}$, and the alternating path P used to update $\mu^{(1)}$ to $\mu^{(2)}$ is a man-to-man path, we deduce that (P1) holds in state 2. ☐

Lemma 17. *Consider an iteration of the loop in Algorithm 1. Suppose that (P1) and (P2) hold at the start of the iteration. Then (P2) holds at the end of the iteration.*

Proof. By Lemma 16, (P2) holds at the end of the iteration. Let $G^{(0)}$ and $\mu^{(0)}$ denote the values of G and μ at the start of the iteration, and let $G^{(1)}$ denote the value of G at the end of the iteration.

Fix an arbitrary woman j. For each k in $\{0,1\}$, let $X^{(k)}$ denote the set of all men i such that j belongs to $J_{i,c(G^{(k)},i)}$, and let $Y^{(k)}$ denote the set of all men i in $X^{(k)}$ such that $i \geq_j i'$ holds for all men i' in $X^{(k)}$. Since (P2) holds for $G^{(0)}$, the set of men adjacent to j in $G^{(0)}$ is $Y^{(0)}$. It remains to prove that the set of men adjacent to j in $G^{(1)}$ is $Y^{(1)}$.

Case 1: $j \neq j_0$. Since the body of the loop only modifies the set of edges incident on woman j_0, the set of men adjacent to j in $G^{(1)}$ is the same as the set of men adjacent to j in $G^{(0)}$, and hence is equal to $Y^{(0)}$. Moreover, $X^{(1)} = X^{(0)}$ and hence $Y^{(1)} = Y^{(0)}$. The desired result follows.

Case 2: $j = j_0$. In this case, $X^{(1)} = X^{(0)} + i_0$. We consider two subcases.

Case 2.1: $\mu^{(0)}(j_0) = 0$. Since (P2) holds at the start of the iteration, we deduce that the degree of j_0 in $G^{(0)}$ is zero, i.e., that $Y^{(0)}$ is empty. This in turn implies that $X^{(0)}$ is empty. Thus $X^{(1)} = \{i_0\}$ and hence $Y^{(1)} = \{i_0\}$. Since Algorithm 1 adds (i_0, j_0) to the edge set of G, the set of men adjacent to j_0 in $G^{(1)}$ is $Y^{(1)}$, as required.

Case 2.2: $\mu^{(0)}(j_0) \neq 0$. Let i denote $\mu^{(0)}(j_0)$. There are three subcases to consider. If $i_0 <_{j_0} i$, then $Y^{(1)} = Y^{(0)}$; the desired result follows since Algorithm 1 makes no change to the edge set of G. If $i_0 =_{j_0} i$ then $Y^{(1)} = Y^{(0)} + i_0$; the desired result follows since Algorithm 1 adds (i_0, j_0) to the edge set of G. If $i_0 >_{j_0} i$ then $Y^{(1)} = \{i_0\}$; the desired result follows since Algorithm 1 ensures that the set of men adjacent to j_0 in $G^{(1)}$ is $\{i_0\}$. □

Lemma 18. *Consider an iteration of the loop in Algorithm 1. Suppose (P1), (P3), and (P4) hold at the start of the iteration. Then (P3) and (P4) hold at the end of the iteration.*

Proof. Let state 0 refer to the initial state before the execution of line 5, let state 1 refer to the intermediate state after the execution of line 5, let state 2 refer to the intermediate state before the execution of line 19, and let state 3 refer to the final state after the execution of line 21. For any state k, we write $G^{(k)}$ and $\mu^{(k)}$ to refer to the values of G and μ in state k. Since (P1) holds in state 0, Lemma 16 implies that $\mu^{(k)}$ is a matching of $G^{(k)}$ in any state k. For any man i, we write $p_i^{(k)}$ to denote the priority of man i in state k. (If $\mu^{(1)}(j_0) = 0$, then line 19 is not executed; in that case, we leave state 2 and the associated symbols undefined.) We need to show that (P3) and (P4) holds in state 3. We begin by proving that (P3) holds in states 1, 2 (if defined), and 3.

Claim 1: (P3) holds in state 1. Since (P3) holds in state 0, and the priorities and weights of the men are the same in states 0 and 1 except that the weight of man i_0 might be higher in state 1, we only need to verify the inequalities associated with man i_0. Let k denote $c(G^{(0)}, i_0)$. Since (P4) holds in state 0, we have $p_{i_0}^{(0)} = w(i_0, k)$. Furthermore, $p_{i_0}^{(1)} = p_{i_0}^{(0)}$ and $c(G^{(1)}, i_0) = k + 1$. It follows that $w(i, k) = p_i^{(1)} \leq w(i, k + 1)$, so the inequalities associated with man i_0 are satisfied in state 1.

Claim 2: (P3) holds in states 2 (if defined) and 3. We consider two cases.

Case 1: $\mu^{(1)}(j_0) = 0$. Since the priorities and weights of the men are the same in states 1 and 3, Claim 1 implies that (P3) holds in state 3.

Case 2: $\mu^{(1)}(j_0) \neq 0$. Since the priorities and weights of the men are the same in states 1 and 2, Claim 1 implies that (P3) holds in state 2. Now we argue that (P3) holds in state 3. Since (P3) holds in state 2, the priorities do not decrease, and the weights of the men are the same in states 2 and 3, we deduce that the lower bounds associated with (P3) hold in state 3. It remains to prove that the upper bounds hold in state 3. Let i be an arbitrary man in I'. Since $G^{(3)} = G^{(2)}$, we have $w(G^{(3)}, i) = w(G^{(2)}, i)$. Since (P3) holds in state 2, we have $p_i^{(2)} \leq w(G^{(3)}, i)$. The definition of i_2 implies that $w(G^{(2)}, i_2) \leq w(G^{(3)}, i)$. Since $p_i^{(3)} = \max(p_i^{(2)}, w(G^{(2)}, i_2))$, we conclude that $p_i^{(3)} \leq w(G^{(3)}, i)$, as required.

We now address predicate (P4). For any state k, let $X^{(k)}$ denote the set of all men i such that $\mu^{(k)}(i) = 0$, and let $V^{(k)}$ denote the set of all men in $X^{(k)}$ such that $p_i^{(k)} \neq w(i, c(G^{(k)}, i))$. (The men in $V^{(k)}$ violate predicate (P4) in state k.) We need to prove that $V^{(3)}$ is empty. Since (P4) holds in state 0, it is easy to see that $V^{(1)} \subseteq \{i_0\}$. We consider two cases.

Case 1: $\mu^{(1)}(j_0) = 0$. Since $V^{(1)} \subseteq \{i_0\}$, $X^{(3)} = X^{(1)} - i_0$, and the priorities and weights of the men are the same in states 1 and 3, we deduce that $V^{(3)}$ is empty.

Case 2: $\mu^{(1)}(j_0) \neq 0$. Let X' denote the set of men $X^{(1)} - i_0$. It is easy to check that $X^{(2)} = X' + i_1$, $X^{(3)} = X' + i_2$, and $I' \cap X'$ is empty. It follows that for each man i in X', the priority and weight of i is the same in states 1, 2, and 3. Hence $V^{(2)} \subseteq \{i_1\}$ and $V^{(3)} \subseteq \{i_2\}$. By Claim 2, (P3) holds in state 2, and hence $p_{i_2}^{(2)} \leq w(G^{(2)}, i_2)$. It follows that $p_{i_2}^{(3)} = w(G^{(2)}, i_2)$, which in turn is equal to $w(G^{(3)}, i_2)$ since $G^{(2)} = G^{(3)}$. Thus i_2 does not belong to $V^{(3)}$, and we conclude that $V^{(3)}$ is empty. □

Lemma 19. *Consider an iteration of the loop in Algorithm 1. Suppose (P1), (P3), and (P5) hold at the start of the iteration. Then (P5) hold at the end of the iteration.*

Proof. Let state 0 refer to the initial state before the execution of line 5, let state 1 refer to the intermediate state before the execution of line 19, let state 2 refer to the intermediate state before the execution of line 20, and let state 3 refer to the final state after the execution of line 21. For any state k, we write $G^{(k)}$ and $\mu^{(k)}$ to refer to the values of G and μ in state k. Since (P1) holds in state 0, Lemma 16 implies that $\mu^{(k)}$ is a matching of $G^{(k)}$ in any state k. For any man i, we write $p_i^{(k)}$ to denote the priority of man i in state k. (If $\mu^{(0)}(j_0) = 0$, then line 19 is not executed; in that case, we leave states 1 and 2 and the associated symbols undefined.) Our task is to show that (P5) holds in state 3.

Lemma 16 implies that for any state k and any edge (i,j) in $G^{(k)}$, we have $\mu^{(k)}(j) \neq 0$. For any state k, we define $V^{(k)}$ as the set of all edges (i,j) in $G^{(k)}$ such that $p_i^{(k)} > p_{\mu^{(k)}(j)}^{(k)}$. (The edges in $V^{(k)}$ violate (P5) in state k.) Thus(P5) holds in state k if and only if $V^{(k)}$ is empty.

We begin by considering the case $\mu^{(0)}(j_0) = 0$. Let (i,j) be an arbitrary edge in $G^{(3)}$. We claim that (i,j) does not belong to $V^{(3)}$. To prove the claim, first assume that $j \neq j_0$. Then (i,j) also belongs to $G^{(0)}$, and since (P5) holds in state 0, we conclude that (i,j) does not belong to $V^{(0)}$. Since $\mu^{(3)} = \mu^{(0)} + (i_0, j_0)$ and the priorities of the men are the same in states 0 and 3, we deduce that (i,j) does not belong to $V^{(3)}$. Now assume that $j = j_0$. Since $\mu^{(0)}(j_0) = 0$ and (P1) holds in state 0, we conclude that j_0 has degree zero in $G^{(0)}$. Since the edge set of $G^{(3)}$ is equal to that of $G^{(0)}$ plus (i_0, j_0), we deduce that $i = i_0$. Since $\mu^{(3)}(j_0) = i_0$, we deduce that (i,j) does not belong to $V^{(3)}$, completing the proof of the claim. The claim implies that (P5) holds in state 3, as required. (Note that states 1 and 2 are undefined since $\mu^{(0)}(j_0) = 0$.)

For the remainder of the proof, we assume that $\mu^{(0)}(j_0) \neq 0$.

Claim 1: $V^{(1)} \subseteq \{(i_1, j_0)\}$. Let (i,j) be an edge in $G^{(1)}$. We need to prove that either $(i,j) = (i_1, j_0)$ or (i,j) does not belong to $V^{(1)}$.

Case 1: $(i,j) = (i_0, j_0)$. If $\mu^{(1)}(j_0) = i_0$, then (i_0, j_0) does not belong to $V^{(1)}$. If $\mu^{(1)}(j_0) \neq i_0$, then $i_1 = i_0$ and hence $(i,j) = (i_1, j_0)$.

Case 2: $(i,j) \neq (i_0, j_0)$. It follows that edge (i,j) belongs to $G^{(0)}$. Since (P5) holds in state 0, we know that (i,j) does not belong to $V^{(0)}$. Since the priorities of the men are the same in states 0 and 1, either (i,j) does not belong to $V^{(1)}$ or $\mu^{(1)}(j) \neq \mu^{(0)}(j)$. In the latter case, $j = j_0$ and $i_0 >_{j_0} i_1$. Moreover, since (i_0, j_0) is the only edge incident on j_0 in $G^{(1)}$, we deduce that $(i,j) = (i_0, j_0)$, a contradiction.

Claim 2: For any man i on P, either (1) $i = i_1$ and edge (i_1, j_0) is on P, or (2) $p_i^{(1)} \leq w(G^{(1)}, i_2)$. Let i be a man on P such that either (1) $i = i_1$ and edge (i_1, j_0) is not on P, or (2) $i \neq i_1$. By Claim 1, no edge on the suffix of P from i to i_2 belongs to $V^{(1)}$; it follows that $p_i^{(1)} \leq p_{i_2}^{(1)}$. Since (P3) holds in state 0, the priorities of the men are the same in states 0 and 1, and the weights of the men do not decrease, we deduce that $p_{i_2}^{(1)} = p_{i_2}^{(0)} \leq w(G^{(0)}, i_2) \leq w(G^{(1)}, i_2)$. Thus $p_i^{(1)} \leq w(G^{(1)}, i_2)$, as required.

Claim 3: For any man i on P, we have $p_i^{(2)} \geq w(G^{(1)}, i_2)$; furthermore, this inequality is tight if $i \neq i_1$. Let i be a man on P. The claimed inequality holds since $p_i^{(2)} = \max(p_i^{(1)}, w(G^{(1)}, i_2))$. Claim 2 implies that the claimed inequality is tight for $i \neq i_1$.

Claim 4: $V^{(2)} \subseteq \{(i_1, j_0)\}$ and if (i_1, j_0) is not an edge on P then $V^{(2)}$ is empty. Let (i,j) be an arbitrary edge in $G^{(2)}$. Since $G^{(2)} = G^{(1)}$, edge (i,j) belongs to $G^{(1)}$. Observe that $\mu^{(2)} = \mu^{(1)}$. Lemma 16 implies that (P1) holds in state 2, and hence that $\mu^{(2)}(j) \neq 0$. Let i' denote $\mu^{(2)}(j)$. We consider two cases.

Case 1: (i,j) does not belong to $V^{(1)}$. Thus $p_i^{(1)} \leq p_{i'}^{(1)} \leq p_{i'}^{(2)}$. We consider two subcases.

Case 1.1: $p_i^{(2)} = p_i^{(1)}$. Thus $p_i^{(2)} \leq p_{i'}^{(2)}$, and hence (i,j) does not belong to $V^{(2)}$.

Case 1.2: $p_i^{(2)} > p_i^{(1)}$. Thus i belongs to I' and $p_i^{(2)} = w(G^{(1)}, i_2)$. Moreover, i' belongs to I' and hence $p_{i'}^{(2)} \geq w(G^{(1)}, i_2)$. Thus (i,j) does not belong to $V^{(2)}$.

Case 2: (i,j) belongs to $V^{(1)}$. Claim 1 implies $(i,j) = (i_1, j_0)$. We need to prove that if (i_1, j_0) is not an edge on P, then (i_1, j_0) does not belong to $V^{(2)}$. Assume that (i_1, j_0) is not an edge on P. By Claim 2, $p_{i_1}^{(1)} \leq w(G^{(1)}, i_2)$. Thus $p_{i_1}^{(2)} = w(G^{(1)}, i_2)$. It is easy to see that $\mu^{(1)}(j_0) \neq 0$ because $\mu^{(1)}(j_0)$ is either $\mu^{(0)}(j_0)$ or i_0; let i'' denote $\mu^{(1)}(j_0)$. Since i'' belongs to I', we have $p_{i''}^{(2)} \geq w(G^{(1)}, i_2)$ and hence $p_{i_1}^{(2)} \leq p_{i''}^{(2)}$. Thus (i_1, j_0) does not belong to $V^{(2)}$.

Claim 5: $V^{(3)}$ is empty. Observe that $G^{(3)} = G^{(2)}$. Let (i,j) be an edge in $G^{(2)}$. Lemma 16 implies that (P1) holds in states 2 and 3, so $\mu^{(2)}(j) \neq 0$ and $\mu^{(3)}(j) \neq 0$. Let men i' and i'' denote $\mu^{(2)}(j)$ and $\mu^{(3)}(j)$, respectively. Since $\mu^{(3)} = \mu^{(2)} \oplus P$, Claim 3 implies that $p_{i'}^{(2)} = w(G^{(1)}, i_2) \leq p_{i''}^{(2)}$. We consider two cases.

Case 1: (i,j) does not belong to $V^{(2)}$. Then $p_i^{(3)} = p_i^{(2)} \leq p_{i'}^{(2)} \leq p_{i''}^{(2)} = p_{i''}^{(3)}$, and hence (i,j) does not belong to $V^{(3)}$.

Case 2: (i,j) belongs to $V^{(2)}$. By Claim 4, (i,j) is equal to (i_1, j_0) and is an edge on P. Since $\mu^{(3)} = \mu^{(2)} \oplus P$, we have $\mu^{(3)}(j_0) = i_1$. Hence (i_1, j_0) does not belong to $V^{(3)}$. □

Proof of Lemma 3. It is easy to see that conditions (P1), (P2), (P3), (P4), and (P5) hold before the loop. So by induction, Lemmas 17, 16, 18, and 19 imply that conditions (P1), (P2), (P3), (P4), and (P5) hold when the algorithm terminates.

To show condition (O1), let $i \in I$ be a man such that $\mu(i) \neq 0$. Since $(i, \mu(i))$ is an edge in G, condition (P2) implies that $\mu(i) \in J_{i,c(G,i)}$. Since $\mu(i) \in J_{i,c(G,i)}$, we have $J_{i,c(G,i)}$ is non-empty, and hence $c(G, i) > 0$. Since $c(G, i) > 0$, condition (P3) implies $w(i, c(G, i) - 1) \leq p_i \leq w(i, c(G, i))$. Hence

$$\sum_{\substack{j \in J \\ j' >_i \mu(i)}} x_{i,j} \leq w(i, c(G, i) - 1) \leq p_i \leq w(i, c(G, i)) \leq 1.$$

To show condition (O2), let $(i, j) \in \mu$. Since $(i, j) \in \mu$, condition (P1) implies that (i, j) is an edge in G. Since (i, j) is an edge in G, condition (P2) implies that $j \in J_{i,c(G,i)}$. Hence $j \geq_i 0$ and $i \geq_j 0$.

To show condition (O3), let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i \mu(i)$ and $i \geq_j 0$. Then $j \in J_{i,c(G,i)}$. Let I'' be the set of all men i'' such that $j \in J_{i'',c(G,i'')}$. Since $i \in I''$, the set I'' is non-empty. Let i' be a most preferred man in I'' under the preference relation $\geq_j$. Condition (P2) implies that (i', j) is an edge in G. Since j has non-zero degree in G, condition (P1) implies that $\mu(j) \neq 0$. Since $(\mu(j), j)$ is an edge in G, condition (P2) implies that $\mu(j) \geq_j i$.

To show condition (O4), let $i \in I$ be a man such that $\mu(i) \neq 0$. Let $j \in J$ be a woman such that $j \geq_i 0$ and $i \geq_j 0$. Suppose $p_i > \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$. Since $(i, \mu(i))$ is an edge in G, condition (P2) implies $\mu(i) \in J_{i,c(G,i)}$. Since $\mu(i) \in J_{i,c(G,i)}$, we have $J_{i,c(G,i)}$ is non-empty, and hence $c(G, i) > 0$. Since $c(G, i) > 0$, condition (P3) implies $w(i, c(G, i)) \geq p_i$. Hence

$$w(i, c(G, i)) \geq p_i > \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}.$$

Thus $j \in J_{i,c(G,i)}$. Let I'' be the set of all men i'' such that $j \in J_{i'',c(G,i'')}$. Since $i \in I''$, the set I'' is non-empty. Let i' be a most preferred man in I'' under the preference relation $\geq_j$. Condition (P2) implies that (i', j) is an edge in G. Since j has non-zero degree in G, condition (P1) implies that $\mu(j) \neq 0$. Since $(\mu(j), j)$ is an edge in G, condition (P2) implies $\mu(j) \geq_j i$.

To show condition (O5), let $i \in I$ be a man such that $\mu(i) \neq 0$. Let $j \in J$ be a woman such that $j \geq_i 0$ and $\mu(j) =_j i$. Suppose $p_i > \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$. Since $(i, \mu(i))$ is an edge in G, condition (P2) implies that $\mu(i) \in J_{i,c(G,i)}$. Since $\mu(i) \in J_{i,c(G,i)}$, we have $J_{i,c(G,i)}$ is non-empty, and hence $c(G, i) > 0$. Since $c(G, i) > 0$, condition (P3) implies $w(i, c(G, i)) \geq p_i$. Hence

$$w(i, c(G, i)) \geq p_i > \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}.$$

Thus $j \in J_{i,c(G,i)}$. Let I'' be the set of all men i'' such that $j \in J_{i'',c(G,i'')}$. Since $i \in I''$, the set I'' is non-empty. Let i' be a most preferred man in I'' under the preference relation $\geq_j$. Condition (P2) implies that (i', j) is an edge in G. Since j has non-zero degree in G, condition (P1) implies $\mu(j) \neq 0$. Since $j \in J_{i,c(G,i)}$ and $\mu(j) =_j i$, condition (P2) implies that (i, j) is an edge in G. Since (i, j) is an edge in G, condition (P5) implies $p_{\mu(j)} \geq p_i$.

To show condition (O6), let $i \in I$ be a man such that $\mu(i) = 0$. Let $j \in J$ be a woman such that $j \geq_i 0$ and $\mu(j) =_j i$. Since $\mu(i) = 0$, we have $c(G, i) = |J_i|$. Thus $j \in J_{i,c(G,i)}$. Let I'' be the set of all men i'' such that $j \in J_{i'',c(G,i'')}$. Since $i \in I''$, the set I'' is non-empty. Let i' be a most preferred man in I'' under the preference relation $\geq_j$. Condition (P2) implies that (i', j) is an edge in G. Since j has non-zero degree in G, condition (P1) implies that $\mu(j) \neq 0$. Since $j \in J_{i,c(G,i)}$ and $\mu(j) =_j i$, condition (P2) implies that (i, j) is an edge in G. Since (i, j) is an edge in G, condition (P5) implies $p_{\mu(j)} \geq p_i$. Since $\mu(i) = 0$, condition (P4) implies $p_i = w(i, c(G, i)) = 1$. Hence $p_{\mu(j)} \geq p_i = 1$. □

A.3 Auxiliary Charges

In this subsection, we first prove Lemma 5 and Lemma 6. Then we prove Lemma 7 after we present Lemmas 20 and 21.

Proof of Lemma 5.

(1) Let $j \in J$ be a woman. Suppose $x_{i,j} = 0$ or $p_i \leq \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$. We consider two cases.

Case 1: $x_{i,j} = 0$. Then

$$0 \leq \widetilde{x}_{i,j} = \min\Big(x_{i,j}, \max\Big(0, p_i - \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}\Big)\Big) \leq x_{i,j} = 0.$$

Case 2: $p_i \leq \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$. Then

$$0 \leq \widetilde{x}_{i,j} = \min\Big(x_{i,j}, \max\Big(0, p_i - \sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}\Big)\Big) \leq \min(x_{i,j}, \max(0, 0)) \leq 0.$$

(2) Let women $j, j' \in J$ satisfy $\widetilde{x}_{i,j} > 0$ and $j' >_i j$. Since $\widetilde{x}_{i,j} > 0$, part (1) implies

$$p_i > \sum_{\substack{j'' \in J \\ j'' >_i j}} x_{i,j''} \geq x_{i,j'} + \sum_{\substack{j'' \in J \\ j'' >_i j'}} x_{i,j''}.$$

Hence

$$x_{i,j'} \geq \widetilde{x}_{i,j'} = \min\Big(x_{i,j'}, \max\Big(0, p_i - \sum_{\substack{j'' \in J \\ j'' >_i j'}} x_{i,j''}\Big)\Big) \geq \min(x_{i,j'}, \max(0, x_{i,j'})) = x_{i,j'}.$$

(3) Let $J'_i = \{j \in J \colon \widetilde{x}_{i,j} > 0\}$. It suffices to show that $\sum_{j \in J'_i} \widetilde{x}_{i,j} \leq p_i$. For the sake of contradiction, suppose

$$\sum_{j \in J'_i} \widetilde{x}_{i,j} > p_i. \tag{15}$$

By condition (O1) of Lemma 3, we have $p_i \geq 0$. Since $p_i \geq 0$, inequality (15) implies J'_i is non-empty. Let $j' \in J'_i$ be the least preferred woman in J'_i under the preference relation $\geq_i$. Then

$$\sum_{j \in J'_i} \widetilde{x}_{i,j} \leq \sum_{\substack{j \in J \\ j \geq_i j'}} \widetilde{x}_{i,j} = \widetilde{x}_{i,j'} + \sum_{\substack{j \in J \\ j >_i j'}} \widetilde{x}_{i,j} = \widetilde{x}_{i,j'} + \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j}, \tag{16}$$

where the last equality follows from part (2). Combining (15) and (16) gives

$$\widetilde{x}_{i,j'} > p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j}. \tag{17}$$

By the definition of $\widetilde{x}_{i,j'}$, we have

$$\widetilde{x}_{i,j'} = \min\Big(x_{i,j'}, \max\Big(0, p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j}\Big)\Big) \leq \max\Big(0, p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j}\Big).$$

We consider two cases.

Case 1: $p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} \leq 0$. Then $\widetilde{x}_{i,j'} \leq 0$, which contradicts $j' \in J'_i$.

Case 2: $p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} > 0$. Then $\widetilde{x}_{i,j'} \leq p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j}$, which contradicts (17).

(4) Let $J'_i = \{j \in J \colon \widetilde{x}_{i,j} < x_{i,j}\}$. It suffices to show that $\sum_{j \in J'_i} (x_{i,j} - \widetilde{x}_{i,j}) \leq 1 - p_i$. For the sake of contradiction, suppose

$$\sum_{j \in J'_i} (x_{i,j} - \widetilde{x}_{i,j}) > 1 - p_i. \tag{18}$$

By condition (O1) of Lemma 3, we have $p_i \leq 1$. Since $p_i \leq 1$, inequality (18) implies that J'_i is non-empty. Let $j' \in J'_i$ be the most preferred woman in J'_i under the preference relation $\geq_i$. Then

$$\sum_{j \in J'_i} (x_{i,j} - \widetilde{x}_{i,j}) \leq \sum_{\substack{j \in J \\ j' \geq_i j}} (x_{i,j} - \widetilde{x}_{i,j}) \leq \Big(\sum_{\substack{j \in J \\ j' \geq_i j}} x_{i,j} \Big) - \widetilde{x}_{i,j'}. \tag{19}$$

Also, by constraint (1), we have

$$1 \geq \sum_{j \in J} x_{i,j} = \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} + \sum_{\substack{j \in J \\ j' \geq_i j}} x_{i,j}. \tag{20}$$

Combining (18), (19), and (20) gives

$$p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} > \widetilde{x}_{i,j'}. \tag{21}$$

By the definition of $\widetilde{x}_{i,j'}$, we have

$$\widetilde{x}_{i,j'} = \min \Big(x_{i,j'}, \max \Big(0, p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} \Big) \Big).$$

We consider two cases.
Case 1: $x_{i,j'} \leq \max \Big(0, p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} \Big)$. Then $\widetilde{x}_{i,j'} = x_{i,j}$, which contradicts $j' \in J'_i$.
Case 2: $x_{i,j'} > \max \Big(0, p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} \Big)$. Then

$$\widetilde{x}_{i,j'} = \max \Big(0, p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} \Big) \geq p_i - \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j},$$

which contradicts (21). □

Proof of Lemma 6.

(1) Let $i \in I^*$ be a man. Then

$$0 \leq q_i = \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} \leq \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} x_{i,j} \leq \sum_{j \in J} x_{i,j} \leq 1,$$

where the last inequality follows from constraint (1).

(2) Let $j \in J^*$ be a woman. Then

$$
\begin{aligned}
0 \le y_j = &\sum_{\substack{i \in I^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} \\
\le &\sum_{\substack{i \in I^* \\ \mu(j) =_j i}} x_{i,j} \\
\le &\sum_{i \in I} x_{i,j} - \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j} - \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} x_{i,j} \\
\le &\, 1 - \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j} - \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} x_{i,j} = 1 - (p_{\mu(j)} - z_j) - \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} x_{i,j},
\end{aligned} \tag{22}
$$

where the last inequality follows from constraint (2), and the last equality follows from the definition of z_j. We consider two cases.

Case 1: $q_{\mu(j)} \le z_j$. Then (22) implies

$$
\begin{aligned}
0 \le y_j \le 1 - p_{\mu(j)} + z_j - \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} x_{i,j} &= 1 - \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j} - \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} x_{i,j} \\
&\le 1 - 0 - 0 \\
&= \min(1 - q_{\mu(j)} + z_j, 1).
\end{aligned}
$$

Case 2: $q_{\mu(j)} > z_j$. Then (22) implies

$$
\begin{aligned}
0 \le y_j &\le 1 - p_{\mu(j)} + z_j - \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} x_{i,j} \\
&\le 1 - p_{\mu(j)} + z_j - 0 \\
&\le 1 - \Big(\sum_{j \in J} \widetilde{x}_{i,j} \Big) + z_j \\
&\le 1 - \Big(\sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} \Big) + z_j \\
&= 1 - q_{\mu(j)} + z_j \\
&= \min(1 - q_{\mu(j)} + z_j, 1),
\end{aligned}
$$

where the fourth inequality follows from part (3) of Lemma 5, and the first equality follows from the definition of $q_{\mu(j)}$. □

Lemma 20. *Let $i \in I^*$ be a man. Then the following conditions hold.*

(1) For every woman $j \in J$ such that $\widetilde{x}_{i,j} > 0$, we have $j \in J^*$ and $\mu(j) \ge_j i$.

(2) $\displaystyle\sum_{j \in J} x_{i,j} \le 1 - p_i + q_i + \sum_{\substack{j \in J^* \\ \mu(j) >_j i}} x_{i,j}$.

Proof.

(1) Let $j \in J$ be a woman such that $\widetilde{x}_{i,j} > 0$. Since $\widetilde{x}_{i,j} > 0$, part (1) of Lemma 5 implies $x_{i,j} > 0$ and $p_i > \displaystyle\sum_{\substack{j' \in J \\ j' >_i j}} x_{i,j'}$. Since $x_{i,j} > 0$, constraint (4) implies $j \ge_i 0$ and $i \ge_j 0$. So by (O4) of Lemma 3, we have $j \in J^*$ and $\mu(j) \ge_j i$.

(2) By part (4) of Lemma 5, we have

$$\sum_{j\in J}(x_{i,j}-\widetilde{x}_{i,j}) \le 1-p_i. \tag{23}$$

Also part (1) implies

$$\sum_{j\in J}\widetilde{x}_{i,j} \le \sum_{\substack{j\in J^*\\ \mu(j)=_j i}}\widetilde{x}_{i,j} + \sum_{\substack{j\in J^*\\ \mu(j)>_j i}}\widetilde{x}_{i,j} = q_i + \sum_{\substack{j\in J^*\\ \mu(j)>_j i}}\widetilde{x}_{i,j} \le q_i + \sum_{\substack{j\in J^*\\ \mu(j)>_j i}} x_{i,j}, \tag{24}$$

where the equality follows from the definition of q_i. Combining (23) and (24) gives the desired inequality. □

Lemma 21. *Let $i\in I\setminus I^*$ be a man. Let $j\in J$ be a woman such that $x_{i,j}>0$. Then $j\in J^*$ and $\mu(j)\ge_j i$. Furthermore, if $\mu(j)=_j i$ then $p_{\mu(j)}\ge 1$.*

Proof. Since $x_{i,j}>0$, constraint (4) implies $j\ge_i 0$ and $i\ge_j 0$. Since $j\ge_i 0=\mu(i)$ and $i\ge_j 0$, condition (O3) of Lemma 3 implies $j\in J^*$ and $\mu(j)\ge_j i$. Furthermore, if $\mu(j)=_j i$ then condition (O6) implies $p_{\mu(j)}\ge 1$. □

Proof of Lemma 7. Consider

$$\sum_{i\in I}\sum_{j\in J}x_{i,j} = \sum_{i\in I^*}\sum_{j\in J}x_{i,j} + \sum_{i\in I\setminus I^*}\sum_{j\in J}x_{i,j}. \tag{25}$$

Part (2) of Lemma 20 implies

$$\begin{aligned}\sum_{i\in I^*}\sum_{j\in J}x_{i,j} &\le \sum_{i\in I^*}(1-p_i+q_i) + \sum_{i\in I^*}\sum_{\substack{j\in J^*\\ \mu(j)>_j i}}x_{i,j}\\ &= |I^*| + \sum_{i\in I^*}(-p_i+q_i) + \sum_{i\in I^*}\sum_{\substack{j\in J^*\\ \mu(j)>_j i}}x_{i,j}. \end{aligned}\tag{26}$$

Lemma 21 implies

$$\begin{aligned}\sum_{i\in I\setminus I^*}\sum_{j\in J}x_{i,j} &= \sum_{i\in I\setminus I^*}\sum_{\substack{j\in J^*\\ \mu(j)>_j i}}x_{i,j} + \sum_{i\in I\setminus I^*}\sum_{\substack{j\in J^*\\ \mu(j)=_j i}}x_{i,j}\cdot \mathbb{1}_{[0,\infty)}(p_{\mu(j)}-1)\\ &= \sum_{i\in I\setminus I^*}\sum_{\substack{j\in J^*\\ \mu(j)>_j i}}x_{i,j} + \sum_{j\in J^*}\mathbb{1}_{[0,\infty)}(p_{\mu(j)}-1)\sum_{\substack{i\in I\setminus I^*\\ \mu(j)=_j i}}x_{i,j}. \end{aligned}\tag{27}$$

Part (2) of Lemma 6 implies

$$\begin{aligned}\sum_{j\in J^*}\mathbb{1}_{[0,\infty)}(p_{\mu(j)}-1)\sum_{\substack{i\in I\setminus I^*\\ \mu(j)=_j i}}x_{i,j} &\le \sum_{j\in J^*}\mathbb{1}_{[0,\infty)}(p_{\mu(j)}-1)\cdot(1-p_{\mu(j)}+z_j-y_j)\\ &\le \sum_{j\in J^*}\max(z_j-y_j,0)\\ &= \sum_{i\in I^*}\max(z_{\mu(i)}-y_{\mu(i)},0), \end{aligned}\tag{28}$$

where the equality follows from the change of variables $i=\mu(j)$. Also, we have

$$\sum_{i\in I^*}\sum_{\substack{j\in J^*\\ \mu(j)>_j i}}x_{i,j} + \sum_{i\in I\setminus I^*}\sum_{\substack{j\in J^*\\ \mu(j)>_j i}}x_{i,j} = \sum_{j\in J^*}\sum_{\substack{i\in I\\ \mu(j)>_j i}}x_{i,j} = \sum_{j\in J^*}(p_{\mu(j)}-z_j) = \sum_{i\in I^*}(p_i-z_{\mu(i)}), \tag{29}$$

where the second equality follows from the definitions of $\{z_j\}_{j \in J^*}$, and the third equality follows from the change of variables $i = \mu(j)$. Combining (25), (26), (27), (28), and (29) gives

$$\begin{aligned}\Big(\sum_{i\in I}\sum_{j\in J} x_{i,j}\Big) - |I^*| &\le \sum_{i\in I^*}(q_i - z_{\mu(i)} + \max(z_{\mu(i)} - y_{\mu(i)}, 0)) \\ &= \sum_{i\in I^*}(q_i - \min(y_{\mu(i)}, z_{\mu(i)})).\end{aligned}$$ □

A.4 The Stability Constraint and The Tie-Breaking Criterion

In this subsection, we first present Lemmas 22 and 23. Then we use them to prove Lemma 10.

Lemma 22. *Let $i \in I^*$ be a man and $0 \le \xi \le 1$. Let*

$$J_i^* = \Big\{ j \in J^* \colon \mu(j) =_j i \text{ and } \sum_{\substack{j'' \in J \\ \mu(j'') =_{j''} i \\ j \ge_i j''}} \widetilde{x}_{i,j''} \ge \xi \Big\}. \tag{30}$$

Then $\max(q_i - \xi, 0) \le \sum_{j \in J_i^*} \widetilde{x}_{i,j}$.

Proof. We may assume that $q_i > \xi$, for otherwise $\max(q_i - \xi, 0) \le 0 \le \sum_{j\in J_i^*} \widetilde{x}_{i,j}$. Let

$$J_i' = \Big\{ j \in J^* \colon \mu(j) =_j i \text{ and } \sum_{\substack{j'' \in J \\ \mu(j'') =_{j''} i \\ j \ge_i j''}} \widetilde{x}_{i,j''} < \xi \Big\}.$$

Then

$$\max(q_i - \xi, 0) = q_i - \xi = \Big(\sum_{\substack{j\in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j}\Big) - \xi = \Big(\sum_{j\in J_i^*} \widetilde{x}_{i,j} + \sum_{j \in J_i'} \widetilde{x}_{i,j}\Big) - \xi,$$

where the second equality follows from the definition of q_i. So, it suffices to show that $\sum_{j\in J_i'} \widetilde{x}_{i,j} \le \xi$.

For the sake of contradiction, suppose $\sum_{j \in J_i'} \widetilde{x}_{i,j} > \xi$. Since $\xi \ge 0$, the set J_i' is non-empty. Let $j' \in J_i'$ be the most preferred woman in J_i' under the preference relation $\ge_i$. Then

$$\sum_{\substack{j'' \in J \\ \mu(j'') =_{j''} i \\ j' \ge_i j''}} \widetilde{x}_{i,j''} \ge \sum_{j'' \in J_i'} \widetilde{x}_{i,j''} > \xi,$$

which contradicts $j' \in J_i'$. □

Lemma 23. *Let $i \in I^*$ be a man and $0 \le \xi \le 1$. Let J_i^* be the set defined by* (30)*. Then, for every woman $j \in J_i^*$ such that $\widetilde{x}_{i,j} > 0$, we have $z_j \ge \xi$.*

Proof. Since $j \in J_i^*$, we have $j \in J^*$ and $\mu(j) =_j i$. Since $\widetilde{x}_{i,j} > 0$ and $\mu(j) =_j i$, Lemma 9 implies

$$z_j \ge \sum_{\substack{j' \in J \\ \mu(j') =_{j'} i \\ j \ge_i j'}} \widetilde{x}_{i,j'} \ge \xi,$$

where the second inequality follows from the definition of J_i^*. □

Proof of Lemma 10. Suppose I^* is non-empty. Let $0 \le \xi \le 1$. For every man $i \in I^*$, let J_i^* be the set defined by (30). Then Lemma 22 implies

$$\begin{aligned}
\frac{1}{|I^*|}\sum_{i\in I^*}\max(q_i-\xi,0) &\le \frac{1}{|I^*|}\sum_{i\in I^*}\sum_{j\in J_i^*}\widetilde{x}_{i,j} \\
&\le \frac{1}{|I^*|}\sum_{i\in I^*}\sum_{j\in J_i^*}\widetilde{x}_{i,j}\cdot \mathbb{1}_{[0,\infty)}(z_j-\xi) \\
&\le \frac{1}{|I^*|}\sum_{i\in I^*}\sum_{\substack{j\in J^*\\ \mu(j)=_j i}}\widetilde{x}_{i,j}\cdot \mathbb{1}_{[0,\infty)}(z_j-\xi) \\
&= \frac{1}{|I^*|}\sum_{j\in J^*}\mathbb{1}_{[0,\infty)}(z_j-\xi)\sum_{\substack{i\in I^*\\ \mu(j)=_j i}}\widetilde{x}_{i,j} \\
&= \frac{1}{|I^*|}\sum_{j\in J^*}\mathbb{1}_{[0,\infty)}(z_j-\xi)\cdot y_j \\
&= \frac{1}{|I^*|}\sum_{i\in I^*}\mathbb{1}_{[0,\infty)}(z_{\mu(i)}-\xi)\cdot y_{\mu(i)},
\end{aligned}$$

where the second inequality follows from Lemma 23, the third inequality follows from the definition of J_i^*, the second equality follows from the definition of $\{y_j\}_{j\in J^*}$, and the third equality follows from the change of variables $i = \mu(j)$. □

A.5 The Associated Optimization Problem

In this subsection, we first present Lemmas 24 and 25. Then we use them to prove Lemma 15.

Lemma 24. *The function f defined by* (13) *is bijective and strictly increasing.*

Proof. It is easy to see that the function f is strictly increasing on $\{u\colon \frac{1}{2} \le u \le 1\}$. Since

$$\frac{d}{du}f(u) = 2 - \frac{2}{1+2u} > 2 - \frac{2}{1+0} = 0$$

for every $0 < u < \frac{1}{2}$, the function f is also strictly increasing on $\{u\colon 0 \le u \le \frac{1}{2}\}$. So, bijectivity follows from continuity and strict monotonicity. □

Lemma 25. *The following inequalities hold.*

(1) For every $1-\ln 2 \le z \le 1$, we have

$$e^{z-1} - z \le -2^{-1} + \ln 2.$$

(2) For every $0 \le u_1 \le \frac{1}{2}$, we have

$$(1+2u_1)^{-1} + \ln(1+2u_1) \le 2^{-1} + \ln 2.$$

(3) For every $0 \le u_1 \le \frac{1}{2}$ and $0 \le u_3 \le \frac{1}{2}$, we have

$$(1+2u_1)^{-1} + (1+2u_3)^{-1}(\ln(1+2u_3) + (1+4u_3)\ln(1+2u_1) - 4u_1u_3) \le \ln 4.$$

Proof.

(1) Let

$$g(z) = e^{z-1} - z$$

for every $1 - \ln 2 \le z \le 1$. It suffices to show that $g(z) \le g(1 - \ln 2)$ for every $1 - \ln 2 \le z \le 1$.
For every $1 - \ln 2 < z < 1$, we have

$$\frac{d}{dz} g(z) = e^{z-1} - 1 < e^{1-1} - 1 = 0.$$

Hence $g(z) \le g(1 - \ln 2)$ for every $1 - \ln 2 \le z \le 1$.

(2) Let

$$g(u_1) = (1 + 2u_1)^{-1} + \ln(1 + 2u_1)$$

for every $0 \le u_1 \le \frac{1}{2}$. It suffices to show that $g(u_1) \le g(\frac{1}{2})$ for every $0 \le u_1 \le \frac{1}{2}$.
For every $0 < u_1 < \frac{1}{2}$, we have

$$\frac{d}{du_1} g(u_1) = \frac{4u_1}{(1 + 2u_1)^2} > \frac{4(0)}{(1 + 2u_1)^2} = 0.$$

Hence $g(u_1) \le g(\frac{1}{2})$ for every $0 \le u_1 \le \frac{1}{2}$.

(3) Let

$$g(u_1, u_3) = (1 + 2u_1)^{-1} + (1 + 2u_3)^{-1}(\ln(1 + 2u_3) + (1 + 4u_3)\ln(1 + 2u_1) - 4u_1u_3)$$

for every $0 \le u_1 \le \frac{1}{2}$ and $0 \le u_3 \le \frac{1}{2}$. It suffices to show that $g(u_1, u_3) \le g(u_1, \frac{1}{2}) \le g(\frac{1}{2}, \frac{1}{2})$ for every $0 \le u_1 \le \frac{1}{2}$ and $0 \le u_3 \le \frac{1}{2}$.
For every $0 \le u_1 \le \frac{1}{2}$ and $0 < u_3 < \frac{1}{2}$, we have

$$\frac{\partial}{\partial u_3} g(u_1, u_3) = \frac{2 - 2(2u_1 - \ln(1 + 2u_1)) - 2\ln(1 + 2u_3)}{(1 + 2u_3)^2} > \frac{2 - 2(1 - \ln 2) - 2\ln(1 + 2(\frac{1}{2}))}{(1 + 2u_3)^2} = 0,$$

since Lemma 24 implies $2u_1 - \ln(1 + 2u_1) = f(u_1) < f(\frac{1}{2}) = 1 - \ln 2$. Hence $g(u_1, u_3) \le g(u_1, \frac{1}{2})$ for every $0 \le u_1 \le \frac{1}{2}$ and $0 \le u_3 \le \frac{1}{2}$.
For every $0 < u_1 < \frac{1}{2}$, we have

$$\frac{\partial}{\partial u_1} g(u_1, \tfrac{1}{2}) = \frac{2u_1(1 - 2u_1)}{(1 + 2u_1)^2} > \frac{2u_1(1 - 2(\frac{1}{2}))}{(1 + 2u_1)^2} = 0.$$

Hence $g(u_1, \frac{1}{2}) \le g(\frac{1}{2}, \frac{1}{2})$ for every $0 \le u_1 \le \frac{1}{2}$. □

Proof of Lemma 15. By Lemma 24, there exist $0 \le u_1 \le \frac{1}{2}$ and $\frac{1}{2} \le u_2 \le 1$ such that

$$\min(q, 1 - \ln 2) = f(u_1) = 2u_1 - \ln(1 + 2u_1) \qquad \text{and} \qquad \max(q, 1 - \ln 2) = f(u_2) = 1 - (1 - u_2)\ln 4.$$

So we have

$$\begin{aligned}
&\mathrm{E}\Big[\max(q - f(U), 0)\Big] \\
&= \int_0^{1/2} 2(1 + 2u)^{-2} \cdot \max(q - 2u + \ln(1 + 2u), 0)\, du + \int_{1/2}^{1} 4^{u-1} \ln 4 \cdot \max(q - 1 + (1 - u)\ln 4, 0)\, du \\
&= \int_0^{u_1} 2(1 + 2u)^{-2} \cdot (q - 2u + \ln(1 + 2u))\, du + \int_{1/2}^{u_2} 4^{u-1} \ln 4 \cdot (q - 1 + (1 - u)\ln 4)\, du \\
&= 2(1 + 2u_1)^{-1}\Big((2 + q)u_1 - (1 + u_1)\ln(1 + 2u_1)\Big) + \Big(4^{u_2 - 1}(q + (1 - u_2)\ln 4) - 2^{-1}q - 2^{-1}\ln 2\Big) \\
&= \begin{cases} 1 + q - (1 + 2u_1)^{-1} - \ln(1 + 2u_1) & \text{if } q < 1 - \ln 2 \\ 1 - \ln 4 + e^{q-1} & \text{if } q \ge 1 - \ln 2 \end{cases} \qquad (31)
\end{aligned}$$

Also, by Lemma 24, there exist $0 \le u_3 \le \frac{1}{2}$ and $\frac{1}{2} \le u_4 \le 1$ such that

$$\min(z, 1-\ln 2) = f(u_3) = 2u_3 - \ln(1+2u_3) \qquad \text{and} \qquad \max(z, 1-\ln 2) = f(u_4) = 1-(1-u_4)\ln 4.$$

So we have

$$\begin{aligned}
&\mathrm{E}\left[y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \\
&= \int_0^{1/2} 2(1+2u)^{-2} \cdot y \cdot \mathbb{1}_{[0,\infty)}(z - 2u + \ln(1+2u))\, du + \int_{1/2}^{1} 4^{u-1} \ln 4 \cdot y \cdot \mathbb{1}_{[0,\infty)}(z - 1 + (1-u)\ln 4)\, du \\
&= \int_0^{u_3} 2(1+2u)^{-2} \cdot y\, du + \int_{1/2}^{u_4} 4^{u-1} (\ln 4) y\, du \\
&= 2u_3(1+2u_3)^{-1} y + (4^{u_4 - 1} - 2^{-1}) y \\
&= \begin{cases} 2u_3(1+2u_3)^{-1} y & \text{if } z < 1 - \ln 2 \\ e^{z-1} y & \text{if } z \ge 1 - \ln 2 \end{cases}
\end{aligned} \tag{32}$$

We consider five cases.

Case 1: $y < z$. Then by (31), we have

$$\begin{aligned}
\mathrm{E}\left[\max(q - f(U), 0)\right] &= \begin{cases} 1 + q - (1+2u_1)^{-1} - \ln(1+2u_1) & \text{if } q < 1 - \ln 2 \\ 1 - \ln 4 + e^{q-1} & \text{if } q \ge 1 - \ln 2 \end{cases} \\
&\ge \begin{cases} 1 + q - 2^{-1} - \ln 2 & \text{if } q < 1 - \ln 2 \\ 1 - \ln 4 + q & \text{if } q \ge 1 - \ln 2 \end{cases} \\
&\ge 1 - \ln 4 + q,
\end{aligned} \tag{33}$$

where the first inequality follows from $e^{q-1} \ge q$ and part (2) of Lemma 25. Also by (32), we have

$$\begin{aligned}
&\mathrm{E}\left[y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \\
&= \begin{cases} 2u_3(1+2u_3)^{-1} y & \text{if } z < 1 - \ln 2 \\ e^{z-1} y & \text{if } z \ge 1 - \ln 2 \end{cases} \\
&= \begin{cases} (1 - (1+2u_3)^{-1}) y & \text{if } z < 1 - \ln 2 \\ e^{z-1} y & \text{if } z \ge 1 - \ln 2 \end{cases} \\
&\le \begin{cases} (1 - (1+2(\frac{1}{2}))^{-1}) y & \text{if } z < 1 - \ln 2 \\ e^{1-1} y & \text{if } z \ge 1 - \ln 2 \end{cases} \\
&\le y,
\end{aligned} \tag{34}$$

where the first inequality follows from $u_3 \le \frac{1}{2}$ and $z \le 1$. Hence (33) and (34) imply

$$\mathrm{E}\left[\max(q - f(U), 0) - y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \ge 1 - \ln 4 + q - y = q - \min(y, z) - \lambda.$$

Case 2: $z \le y$ and $\max(q, z) < 1 - \ln 2$. Then by (31) and (32), we have

$$\begin{aligned}
&\mathrm{E}\left[\max(q - f(U), 0) - y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \\
&= \Big(1 + q - (1+2u_1)^{-1} - \ln(1+2u_1)\Big) - \Big(2u_3(1+2u_3)^{-1} y\Big) \\
&\ge 1 + q - (1+2u_1)^{-1} - \ln(1+2u_1) - 2u_3(1+2u_3)^{-1}(1 - q + z) \\
&= 1 + q - z - \Big((1+2u_1)^{-1} + (1+2u_3)^{-1}(\ln(1+2u_3) + (1+4u_3)\ln(1+2u_1) - 4u_1u_3)\Big) \\
&\ge 1 + q - z - \ln 4 \\
&= q - \min(y, z) - \lambda,
\end{aligned}$$

where the first inequality follows from $y \leq 1 - q + z$, the second equality follows from $q = 2u_1 - \ln(1 + 2u_1)$ and $z = 2u_3 - \ln(1 + 2u_3)$, and the second inequality follows from part (3) of Lemma 25.

Case 3: $z \leq y$ and $q < 1 - \ln 2 \leq z$. Then by (31) and (32), we have

$$\begin{aligned}
&\mathrm{E}\left[\max(q - f(U), 0) - y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \\
&= \Big(1 + q - (1 + 2u_1)^{-1} - \ln(1 + 2u_1)\Big) - \Big(e^{z-1} y\Big) \\
&\geq 1 + q - (1 + 2u_1)^{-1} - \ln(1 + 2u_1) - e^{z-1} \\
&\geq 1 + q - (2^{-1} + \ln 2) - (z - 2^{-1} + \ln 2) \\
&= q - \min(y, z) - \lambda,
\end{aligned}$$

where the first inequality follows from $y \leq 1$, and the second inequality follows from parts (1) and (2) of Lemma 25.

Case 4: $z \leq y$ and $z < 1 - \ln 2 \leq q$. Then by (31) and (32), we have

$$\begin{aligned}
&\mathrm{E}\left[\max(q - f(U), 0) - y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \\
&= \Big(1 - \ln 4 + e^{q-1}\Big) - \Big(2u_3(1 + 2u_3)^{-1} \cdot y\Big) \\
&\geq 1 - \ln 4 + e^{q-1} - 2u_3(1 + 2u_3)^{-1}(1 - q + z) \\
&= 1 - \ln 4 + e^{q-z+2u_3-1}(1 + 2u_3)^{-1} - 2u_3(1 + 2u_3)^{-1}(1 - q + z) \\
&\geq 1 - \ln 4 + (q - z + 2u_3)(1 + 2u_3)^{-1} - 2u_3(1 + 2u_3)^{-1}(1 - q + z) \\
&= q - \min(y, z) - \lambda,
\end{aligned}$$

where the first inequality follows from $y \leq 1 - q + z$, the second equality follows from $z = 2u_3 - \ln(1 + 2u_3)$, and the second inequality follows from $e^{q-z+2u_3-1} \geq q - z + 2u_3$.

Case 5: $z \leq y$ and $1 - \ln 2 \leq \min(q, z)$. Then by (31) and (32), we have

$$\begin{aligned}
&\mathrm{E}\left[\max(q - f(U), 0) - y \cdot \mathbb{1}_{[0,\infty)}(z - f(U))\right] \\
&= \Big(1 - \ln 4 + e^{q-1}\Big) - \Big(e^{z-1} y\Big) \\
&\geq 1 - \ln 4 + e^{z-1}(1 + q - z) - e^{z-1} y \\
&\geq \begin{cases} 1 - \ln 4 + e^{z-1}(1 + q - z) - e^{z-1}(1 - q + z) & \text{if } z \leq q \\ 1 - \ln 4 + e^{z-1}(1 + q - z) - e^{z-1} & \text{if } q < z \end{cases} \\
&= \begin{cases} 1 - \ln 4 + 2e^{z-1}(q - z) & \text{if } z \leq q \\ 1 - \ln 4 - e^{z-1}(z - q) & \text{if } q < z \end{cases} \\
&\geq \begin{cases} 1 - \ln 4 + 2e^{(1-\ln 2)-1}(q - z) & \text{if } z \leq q \\ 1 - \ln 4 - e^{1-1}(z - q) & \text{if } q < z \end{cases} \\
&= q - \min(y, z) - \lambda,
\end{aligned}$$

where the first inequality follows from $e^{q-z} \geq 1 + q - z$, the second inequality follows from $y \leq \min(1 + z - q, 1)$, and the third inequality follows from $1 - \ln 2 \leq z \leq 1$. □