A $(1+1/e)$-Approximation Algorithm for Maximum Stable Matching with One-Sided Ties and Incomplete Lists*

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Abstract

We study the problem of finding large weakly stable matchings when preference lists are incomplete and contain one-sided ties. Computing maximum weakly stable matchings is known to be NP-hard. We present a polynomial-time algorithm that achieves an improved approximation ratio of $1+1/e$. Like a number of existing approximation algorithms for this problem, our algorithm is based on a proposal process in which numerical priorities are adjusted according to the solution of a linear program, and are used for tie-breaking purposes. Our main idea is to use an infinitesimally small step size for incrementing the priorities. Our analysis involves solving an infinite-dimensional factor-revealing linear program. We also show that the ratio $1+1/e$ is an upper bound for the integrality gap, which matches the known lower bound.

1 Introduction

The stable matching problem, introduced by Gale and Shapley [5], involves two disjoint sets of agents, typically called men and women in the literature. Each agent has ordinal preferences over the agents of the opposite sex. The objective is to find a set of disjoint man-woman pairs, called a *matching*, such that no man and woman prefer each other to their partners. Matchings satisfying this property are said to be stable and can be computed efficiently using the Gale-Shapley algorithm. Stable matchings have applications such as centralized schemes for recruiting medical residents to hospitals [25].

Ties and incomplete lists arise naturally in real-world problems. The preference list of an agent is said to contain a tie when the agent is indifferent between two or more agents of the opposite sex. The preference list of an agent is said to be incomplete when one or more agents of the opposite sex are unacceptable to the agent. For such variants, the notion of stability can be generalized to weak stability, strong stability, or super-stability [10]. In this paper, we focus on weak stability since weakly stable matchings, unlike strongly stable or super-stable matchings, always exist. A weakly stable matching can be obtained by breaking all of the ties arbitrarily and then invoking the Gale-Shapley algorithm. When either ties or incomplete lists are absent, all weakly stable matchings have the same size [6, 25]. However, when both ties and incomplete lists are present, weakly stable matchings can vary in size.

The problem of finding large weakly stable matchings with ties and incomplete lists has been intensively studied. Iwama et al. [12] were the first to prove that finding a maximum weakly stable matching with ties and incomplete lists is NP-hard. Results by Yanagisawa [28] imply that getting an approximation ratio of $\frac{33}{29}-\varepsilon$ (≈ 1.1379) is NP-hard, and getting a ratio of $\frac{4}{3}-\varepsilon$ (≈ 1.3333) is UG-hard. On the positive side, it is straightforward to see that any weakly stable matching is a 2-approximate solution [20]. Using a local search approach, Iwama et al. [13] gave an algorithm with an approximation ratio of $\frac{15}{8}$ ($= 1.875$). Király [16] improved the approximation ratio to $\frac{5}{3}$ (≈ 1.6667) by introducing the idea of promoting unmatched agents to higher priorities for tie-breaking. The current best approximation ratio for two-sided ties and incomplete lists is $\frac{3}{2}$ ($= 1.5$), which is achieved by the polynomial-time algorithm of McDermid [21], and the linear-time algorithms of Paluch [23] and Király [17]. This ratio coincides with a lower bound for the integrality gap of an associated linear programming (LP) formulation [14], indicating a potential barrier to further improvements.

Ties often appear only on one side of the market, especially in settings where institutions need to evaluate a large number of candidates. For example, in the Scottish Foundation Allocation Scheme, residents have strict preferences, while the preferences of the hospitals may contain ties [11]. With one-sided ties and incomplete lists, the problem of finding a maximum weakly stable matching remains NP-hard [20]. Results by Halldórsson et al. [7] imply that getting an approximation ratio of $\frac{21}{19}-\varepsilon$ (≈ 1.105) is NP-hard, and achieving a ratio of $\frac{5}{4}-\varepsilon$ (≈ 1.25) is UG-hard. Király [16], who showed an approximation ratio of $\frac{3}{2}$ ($= 1.5$) for an algorithm based on the idea of promotion, conjectured that a $(\frac{3}{2}-\varepsilon)$-

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approximation is UG-hard even for one-sided ties. However, Iwama et al. [14] later established an approximation ratio of $\frac{25}{17}$ (≈ 1.4706) using an LP-based approach. Dean and Jalasutram [3] improved on this approach to obtain an approximation ratio of $\frac{19}{13}$ (≈ 1.4615). Huang and Kavitha [9] established an approximation ratio of $\frac{22}{15}$ (≈ 1.4667) using an algorithm based on rounding half-integral stable matchings. Subsequently, a tight analysis [2, 24] of their algorithm established an approximation ratio of $\frac{13}{9}$ (≈ 1.4444). It is known that $1 + 1/e$ (≈ 1.3679) is a lower bound for the integrality gap of the LP formulation associated with one-sided ties and incomplete lists [14]. In a paper by Huang et al. [8], the integrality gap is claimed to be at least $\frac{3}{2}$, but their proof contains an error.[1]

Our Techniques and Contributions. In this paper, we focus on the maximum stable matching problem with one-sided ties and incomplete lists. Our main result is a polynomial-time algorithm that achieves an improved approximation ratio of $1 + 1/e$ (≈ 1.3679).

Our algorithm is motivated by a proposal process similar to that of Iwama et al. [14], and that of Dean and Jalasutram [3], in which numerical priorities are adjusted according to the LP solution, and are used for tie-breaking purposes. However, instead of using their priority manipulation schemes, we introduce a method of priority incrementation based on an adjustable step size parameter. In § 3.1, we present the description of our process along with some key properties. Both the description and the properties are expressed in terms of the step size parameter. In § 3.2, we consider the limit of this process as the step size becomes infinitesimally small, and we present a polynomial-time algorithm that satisfies the key properties with the step size parameter set to zero.

We analyze the approximation ratio of our algorithm in § 4 by directly comparing the size of our output matching with the optimal value of the LP. Although this is a standard approach to analyze approximation algorithms, it has not been used in prior work on this problem. Prior non-LP-based analyses [2, 9, 16, 24] are based on considering the symmetric difference of the output matching and an unknown optimal matching, and counting augmenting paths of various lengths. Such symmetric difference arguments are also used in prior LP-based analyses [3, 14], where the output matching is compared to both an unknown optimal matching and an optimal LP solution. Instead of focusing on the symmetric difference, we develop a charging scheme for the priority increments. This scheme is described in § 4.1. In § 4.2, the key ingredients underlying our analysis are derived by applying the stability constraint and the tie-breaking criterion to all of the tie-related charges. While none of the prior analyses directly implies an upper bound for the integrality gap, our approach enables us to obtain an upper bound of $1 + 1/e$ for the integrality gap, which matches the known lower bound [14].

As part of our analysis, we formulate an infinite-dimensional factor-revealing LP; this LP is presented in § 4.3. The finite-dimensional factor-revealing LP technique was introduced by Jain et al. [15], and since then a number of variants have been proposed [4, 19, 22]. However, it is often difficult to obtain a nice closed-form solution. For the maximum stable matching problem with one-sided ties and incomplete lists, Dean and Jalasutram [3] obtained an approximation ratio of $\frac{19}{13}$ by enumerating the combinatorial structures of augmenting paths and resorting to a computer-assisted proof for solving a large factor-revealing LP. In contrast, our infinite-dimensional factor-revealing LP is derived from the tie-related charges. Even though our infinite-dimensional factor-revealing LP appears to be more complex than the one studied by Archer and Blasiak [1] in relation to the minimum latency problem, we are able to obtain an analytical solution using numerical results as guidance. In § 4.3.1, we discuss informally how calculus and numerical methods suggest a nice closed-form solution. In § 4.3.2, we provide a formal analytical proof using LP duality.

2 Stable Matching with One-Sided Ties and Incomplete Lists

2.1 The Model. The stable matching problem with one-sided ties and incomplete lists (SMOTI) involves a set I of men and a set J of women. We assume that the sets I and J are disjoint and do not contain the element 0, which we use to denote being unmatched. Each man $i \in I$ has a preference relation $\geq_i$ over the set $J \cup \{0\}$ that satisfies antisymmetry, transitivity, and totality. Each woman $j \in J$ has a preference relation $\geq_j$ over the

[1] In the proof of this claim [8, Theorem 19], Huang et al. exhibit a family of instances with $2k$ men and $2k$ women such that the corresponding LP has a feasible fractional value of $(3/2 - o(1))k$. It is asserted that a certain weakly stable matching of size k is a maximum weakly stable matching, but this assertion is incorrect. For the case when $k = 2$, there exists a weakly stable matching of size 3. Similarly, when $k > 2$, it can be shown that the maximum size of weakly stable matching is greater than k.

There is also a flaw related to the main result of their paper, which asserts an approximation ratio of $\frac{5}{4}$ for the special case where ties are one-sided and are restricted to the end of the preference lists. In the derivation of inequalities (11) and (12) in their proof [8, Lemma 16], it is claimed that $\frac{\delta_{m,w}}{1+\nu_w} \leq \delta_{m,w}$. This claim depends on the unproven assumption that $\delta_{m,w}$ is non-negative. It is unclear whether this flaw can be fixed. Both flaws have been acknowledged by Huang et al. in a personal communication.

set $I \cup \{0\}$ that satisfies transitivity and totality. We denote this SMOTI instance as $(I, J, \{\geq_i\}_{i\in I}, \{\geq_j\}_{j\in J})$.

Notice that the preference relations $\{\geq_j\}_{j\in J}$ of the women are not required to be antisymmetric, while the preference relations $\{\geq_i\}_{i\in I}$ of the men are required to be antisymmetric. So ties are allowed in the preferences of the women, but not in the preferences of the men. For every woman $j \in J$, we write $>_j$ and $=_j$ to denote the asymmetric part and the symmetric part of $\geq_j$, respectively. Similarly, for every man $i \in I$, we write $>_i$ to denote the asymmetric part of $\geq_i$.

For every man $i \in I$ and woman $j \in J$, we say that man i is acceptable to woman j if $i \geq_j 0$. Similarly, for every man $i \in I$ and woman $j \in J$, we say that woman j is acceptable to man i if $j \geq_i 0$. Notice that preference lists are allowed to be incomplete. In other words, there may exist $i \in I$ and $j \in J$ such that $0 >_j i$ or $0 >_i j$.

A matching is a subset $\mu \subseteq I \times J$ such that for every $(i, j), (i', j') \in \mu$, we have $i = i'$ if and only if $j = j'$. For every man $i \in I$, if $(i, j) \in \mu$ for some woman $j \in J$, we say that man i is matched to woman j in matching μ, and we write $\mu(i) = j$. Otherwise, we say that man i is unmatched in matching μ, and we write $\mu(i) = 0$. Similarly, for every woman $j \in J$, if $(i, j) \in \mu$ for some man $i \in I$, we say that woman j is matched to man i in matching μ, and we write $\mu(j) = i$. Otherwise, we say that woman j is unmatched in matching μ, and we write $\mu(j) = 0$.

A matching μ is *individually rational* if for every $(i, j) \in \mu$, we have $j \geq_i 0$ and $i \geq_j 0$. An individually rational matching μ is *weakly stable* if for every man $i \in I$ and woman $j \in J$, either $\mu(i) \geq_i j$ or $\mu(j) \geq_j i$. Otherwise, (i, j) forms a *strongly blocking pair*.

The goal of the maximum stable matching problem with one-sided ties and incomplete lists (MAX-SMOTI) is to find a maximum-cardinality weakly stable matching for a given SMOTI instance.

2.2 The Linear Programming Formulation. The following LP formulation is based on that of Rothblum [26], which extends that of Vande Vate [27].

$$\text{maximize} \quad \sum_{(i,j)\in I\times J} x_{i,j}$$

subject to

$$(2.1) \quad \sum_{j\in J} x_{i,j} \leq 1 \qquad \forall i \in I$$

$$(2.2) \quad \sum_{i\in I} x_{i,j} \leq 1 \qquad \forall j \in J$$

$$(2.3) \quad \sum_{\substack{j'\in J \\ j' >_i j}} x_{i,j'} + \sum_{\substack{i'\in I \\ i' \geq_j i}} x_{i',j} \geq 1 \qquad \forall (i, j) \in I \times J \text{ such that } j >_i 0 \text{ and } i >_j 0$$

$$(2.4) \quad x_{i,j} = 0 \qquad \forall (i, j) \in I \times J \text{ such that } 0 >_i j \text{ or } 0 >_j i$$

$$(2.5) \quad x_{i,j} \geq 0 \qquad \forall (i, j) \in I \times J$$

In Lemmas 2.1 and 2.2, we present two straightforward properties of the LP formulation. Vande Vate [27] used constraint (2.6) in the statement of Lemma 2.2 below, together with constraint (2.5) and tight versions of constraints (2.1) and (2.2), to characterize stable matchings in the special case where all preference lists are complete and the number of men is equal to the number of women. Rothblum [26] extended the result of Vande Vate and used constraints (2.1), (2.2), (2.3), (2.4), and (2.5) to characterize stable matchings for the model with strict preferences and incomplete lists, and where the number of men is not necessarily equal to the number of women. This formulation was adapted to study maximum weakly stable matching with one-sided ties and incomplete lists by Iwama et al. [14], and by Dean and Jalasutram [3]. Our model also allows a woman to be indifferent between being unmatched and being matched with some of the men. Accordingly, we provide proofs of Lemmas 2.1 and 2.2 for the sake of completeness.

LEMMA 2.1. *An integral solution $\{x_{i,j}\}_{(i,j)\in I\times J}$ corresponds to the indicator variables of a weakly stable matching if and only if it satisfies constraints* (2.1)*,* (2.2)*,* (2.3)*,* (2.4)*, and* (2.5)*.*

Proof. Suppose $\{x_{i,j}\}_{(i,j)\in I\times J}$ satisfies constraints (2.1), (2.2), (2.3), (2.4), and (2.5). Constraints (2.1), (2.2), and (2.5) imply that $\{x_{i,j}\}_{(i,j)\in I\times J}$ corresponds to a valid matching μ. Constraint (2.4) implies that μ is individually rational. To show the weak stability of μ, consider man $i \in I$ and woman $j \in J$. It suffices to show that (i, j) is not a strongly blocking pair. We may assume that $j >_i 0$ and $i >_j 0$, for otherwise individual rationality implies $\mu(i) \geq_i 0 \geq_i j$ or $\mu(j) \geq_j 0 \geq_j i$. Consider constraint (2.3) associated with (i, j). At least one of the two summations is equal to 1. If the first summation is equal to 1, then $\mu(i) >_i j$. If the second summation is equal to 1, then $\mu(j) \geq_j i$. Thus, μ is a weakly stable matching.

Conversely, suppose $\{x_{i,j}\}_{(i,j)\in I\times J}$ corresponds to a weakly stable matching μ. Since μ is a valid matching, constraints (2.1), (2.2), and (2.5) are satisfied. Also, the individual rationality of μ implies that constraint (2.4) is satisfied. To show that constraint (2.3) is satisfied, consider $(i, j) \in I \times J$ such that $j >_i 0$ and $i >_j 0$. It suffices to show that at least one of the two summations in constraint (2.3) associated with (i, j) is equal to 1. By the weak stability of μ, we have either $\mu(i) \geq_i j$ or $\mu(j) \geq_j i$. We consider two cases.

Case 1: $\mu(j) \geq_j i$. Since $\mu(j) \geq_j i >_j 0$, the second summation is equal to 1.

Case 2: $i >_j \mu(j)$ and $\mu(i) \geq_i j$. Since $i >_j \mu(j)$, we have $(i,j) \notin \mu$. Since $\mu(i) \geq_i j$ and $(i,j) \notin \mu$, we have $\mu(i) >_i j$. Since $\mu(i) >_i j >_i 0$, the first summation is equal to 1.

LEMMA 2.2. *Let $\{x_{i,j}\}_{(i,j)\in I\times J}$ be a fractional solution that satisfies constraints* (2.2), (2.3), (2.4), *and* (2.5). *Then the following constraint is also satisfied.*

$$(2.6) \quad \sum_{\substack{j'\in J\\ j'>_i j}} x_{i,j'} \geq \sum_{\substack{i'\in I\\ i>_j i'}} x_{i',j} \quad \forall (i,j)\in I\times J \text{ such that } j \geq_i 0 \text{ and } i \geq_j 0$$

Proof. Let $(i,j) \in I \times J$ with $j \geq_i 0$ and $i \geq_j 0$. We consider two cases.

Case 1: $i =_j 0$. Then

$$\sum_{\substack{i'\in I\\ i>_j i'}} x_{i',j} = \sum_{\substack{i'\in I\\ 0>_j i'}} x_{i',j} = 0 \leq \sum_{\substack{j'\in J\\ j'>_i j}} x_{i,j'},$$

where the second equality follows from constraint (2.4) and the inequality follows from constraint (2.5).

Case 2: $i >_j 0$. Since $j \in J$ and $j \geq_i 0$, we have $j >_i 0$. Since $j >_i 0$ and $i >_j 0$, constraint (2.3) implies

$$\sum_{\substack{j'\in J\\ j'>_i j}} x_{i,j'} \geq 1 - \sum_{\substack{i'\in I\\ i'\geq_j i}} x_{i',j} \geq \sum_{i'\in I} x_{i',j} - \sum_{\substack{i'\in I\\ i'\geq_j i}} x_{i',j} = \sum_{\substack{i'\in I\\ i>_j i'}} x_{i',j},$$

where the second inequality is a consequence of constraint (2.2).

3 The Algorithm

3.1 A Proposal Process with Priorities. In this subsection, we describe a proposal process with priorities which is similar to that of Iwama et al. [14], and that of Dean and Jalasutram [3]. Our algorithm is motivated by this process, and is presented in § 3.2.

Our proposal process with priorities takes an SMOTI instance and a step size parameter $\eta > 0$ as input, and produces a weakly stable matching μ as output. In the preprocessing phase, we compute an optimal fractional solution $\{x_{i,j}\}_{(i,j)\in I\times J}$ to the associated LP. We also define

$$(3.7) \qquad w(i,j) = \begin{cases} 1 & \text{if } j = 0 \\ \displaystyle\sum_{\substack{j'\in J\\ j'>_i j}} x_{i,j'} & \text{if } j \neq 0 \end{cases}$$

for every man $i \in I$ and woman $j \in J \cup \{0\}$. Then, in the initialization phase, we assign the empty matching to μ and each man i is assigned a priority p_i equal to 0. For each man i, we also maintain a set L_i of women which is initialized to the empty set. We use the set L_i to store the women to whom man i must propose before his priority p_i is increased by η. After that, the process enters the proposal phase and proceeds iteratively.

In each iteration, we pick an unmatched man i with priority $p_i < 1 + \eta$. If the set L_i is empty, we increment his priority p_i by η and then update L_i to the set $\{j \in J : j \geq_i 0 \text{ and } w(i,j) \leq p_i\}$. Otherwise, the man i that we pick has a non-empty set L_i of women. Let j denote the most preferred woman of man i in L_i. We remove j from L_i and man i proposes to woman j. When woman j receives the proposal from man i, she tentatively accepts him if she is currently unmatched and he is acceptable to her. Otherwise, if woman j is currently matched to another man i', she tentatively accepts her preferred choice between men i and i', and rejects the other. In the event of a tie, she compares the current priorities p_i and $p_{i'}$ of the men and accepts the one with higher priority. (If the priorities of i and i' are equal, she breaks the tie arbitrarily.) If man i is tentatively accepted by woman j, the matching μ is updated accordingly.

When every unmatched man i has priority $p_i \geq 1 + \eta$, the process terminates and outputs the final matching μ.

Our process is similar to that of Iwama et al. [14], and that of Dean and Jalasutram [3], which also use a proposal scheme with priorities. In particular, the way that we populate the set L_i with a subset of women by referring to the LP solution is based on their methods. The major difference is that, in our process, priorities only increase by a small step size η, whereas in their algorithms, the priorities may increase by a possibly larger amount, essentially to ensure that a new woman is added to L_i. As in their algorithms, for every woman j, the sequence of tentative partners $\mu(j)$ of woman j satisfies a natural monotonicity property. Woman j is initially unmatched, and becomes matched the first time she receives a proposal from a man who is acceptable to her. In each subsequent iteration, she either keeps her current partner or gets a weakly preferred partner. Furthermore, if she is indifferent between her new partner and her old partner, then the new partner has a weakly larger priority. When the process terminates, the following predicates hold, which are analogous to predicates satisfied by the algorithms of Iwama et al. [14] and Dean and Jalasutram [3].

$\mathcal{P}_1(\mu)$: for every $(i,j) \in \mu$, we have $j \geq_i 0$ and $i \geq_j 0$.

$\mathcal{P}_2(\mu)$: for every man $i \in I$ and woman $j \in J$ such that $j \geq_i \mu(i)$ and $i \geq_j 0$, we have $\mu(j) \neq 0$ and $\mu(j) \geq_j i$.

$\mathcal{P}_3(\mu, p, \eta)$: for every man $i \in I$, we have $w(i, \mu(i)) \leq p_i \leq 1 + 2\eta$.

$\mathcal{P}_4(\mu, p, \eta)$: for every man $i \in I$ and woman $j \in J$ such that $j \geq_i 0$, $i \geq_j 0$, and $p_i - \eta > w(i, j)$, we have $\mu(j) \neq 0$, $\mu(j) \geq_j i$, and if $\mu(j) =_j i$ then $p_{\mu(j)} \geq p_i - \eta$.

To establish that $\mathcal{P}_1(\mu)$ holds, it is easy to see that man i proposes to woman j only if she is acceptable to him, and woman j accepts a proposal from man i only if he is acceptable to her. For $\mathcal{P}_2(\mu)$, if man i weakly prefers woman j to $\mu(i)$ and is acceptable to woman j, then man i has proposed to woman j. Thus the monotonicity property implies that $\mu(j) \neq 0$ and $\mu(j) \geq_j i$. For $\mathcal{P}_3(\mu, p, \eta)$, it is easy to see that the priority p_i of man i lies within the specified range when he proposes to woman $\mu(i)$. For $\mathcal{P}_4(\mu, p, \eta)$, if man i and woman j satisfy the stated assumptions, then man i proposed to woman j when his priority was equal to $p_i - \eta$, and this proposal was eventually rejected. Immediately after this proposal was rejected, woman j was matched with a man i' such that $i' \neq i$ and $i' \geq_j i$. The monotonicity property implies that $\mu(j) \neq 0$ and $\mu(j) \geq_j i' \geq_j i$. Furthermore, if $\mu(j) =_j i$, then $\mu(j) =_j i' =_j i$. Since $i' =_j i$, the priority of man i' was at least $p_i - \eta$ when the aforementioned proposal was rejected. Since $\mu(j) =_j i'$, the monotonicity property implies that $p_{\mu(j)} \geq p_i - \eta$.

3.2 The Implementation. The proposal process with priorities of § 3.1 depends on a step size parameter $\eta > 0$. To obtain a good approximation ratio, we would like the step size parameter η to be small. However, the running time of a naive implementation grows in proportion to η^{-1}. We can imagine that if we take an infinitesimally small step size, then $\mathcal{P}_3(\mu, p, \eta)$ and $\mathcal{P}_4(\mu, p, \eta)$ can be satisfied with $\eta = 0$.

Algorithm 1 is motivated by the idea of simulating the process of § 3.1 with an infinitesimally small step size. We maintain for every man $i \in I$ a priority p_i and a pointer $\ell_i \in J \cup \{0\}$ into the preference list of man i. For every man $i \in I$ and woman $j \in J$, we think of man i as having proposed to woman j if and only if $j >_i \ell_i$. Given $\ell = \{\ell_i\}_{i \in I}$ and $j \in J$, we define

$$I_j(\ell) = \{i \in I : j >_i \ell_i\}$$

as the set of all men i who have proposed to woman j. Given $\ell = \{\ell_i\}_{i \in I}$, we define $G(\ell)$ as the bipartite graph with vertex set $I \cup J$ and edge set

$$E(\ell) = \{(i, j) \in I \times J : i \in I_j(\ell) \text{ and } i \geq_j i' \text{ for every } i' \in I_j(\ell) \cup \{0\}\}.$$

For any ℓ and any matching μ, we define a *μ-alternating path in $G(\ell)$* as a (possibly zero-length) simple path in $G(\ell)$ that alternates between edges not in μ and edges in μ.

The following lemma is proven in Appendix A. This result is used in § 4 to establish that Algorithm 1 achieves a $1 + 1/e$ approximation guarantee.

Lemma 3.1. *When Algorithm 1 terminates, $\mathcal{P}_1(\mu)$, $\mathcal{P}_2(\mu)$, $\mathcal{P}_3(\mu, p, 0)$, and $\mathcal{P}_4(\mu, p, 0)$ hold.*

Let us define the weight of any edge (i, j) in $E(\ell)$ as $w(i, \ell_i)$. We use the abbreviations MCM and MWMCM to denote the terms maximum-cardinality matching and maximum-weight MCM, respectively. The following lemma is proven in Appendix B.

Lemma 3.2. *An invariant of the Algorithm 1 loop is that μ is an MWMCM of $G(\ell)$.*

Lemma 3.2 leads us to consider Algorithm 2, which has a more succinct description than Algorithm 1, and does not maintain a priority vector p. In Appendix B, we use our analysis of Algorithm 1 to prove Lemma 3.3 below. This result is used in § 4 to establish that Algorithm 2 also achieves an approximation ratio of $1 + 1/e$.

Lemma 3.3. *When Algorithm 2 terminates, $\mathcal{P}_1(\mu)$ and $\mathcal{P}_2(\mu)$ hold. Furthermore, there exist priorities $\{p_i\}_{i \in I}$ such that $\mathcal{P}_3(\mu, p, 0)$ and $\mathcal{P}_4(\mu, p, 0)$ hold.*

Lemma 3.4. *Algorithms 1 and 2 each produce a weakly stable matching in polynomial time.*

Proof. Letting μ denote the output matching produced by Algorithm 1 (resp., Algorithm 2), Lemma 3.1 (resp., Lemma 3.3) implies that $\mathcal{P}_1(\mu)$ and $\mathcal{P}_2(\mu)$ hold. Since $\mathcal{P}_1(\mu)$ holds, μ is individually rational. To establish weak stability of μ, consider $(i, j) \in I \times J$. It suffices to show that (i, j) is not a strongly blocking pair. For the sake of contradiction, suppose $j >_i \mu(i)$ and $i >_j \mu(j)$. If $0 >_j i$, then $0 >_j i >_j \mu(j)$, contradicting the individual rationality of μ. If $i \geq_j 0$, then since $j >_i \mu(i)$, $i \geq_j 0$, and $\mathcal{P}_2(\mu)$ holds, we deduce that $\mu(j) \geq_j i$, contradicting the assumption that $i >_j \mu(j)$.

Algorithms 1 and 2 run in polynomial time since linear programming is polynomial-time solvable, the number of iterations of the loop is at most $|I| \times |J|$, and each iteration can be performed in polynomial time. (Using alternating breadth-first search from i_1, each iteration of Algorithm 1 can be performed in time linear in the size of $G(\ell)$.)

Algorithm 1

compute an optimal fractional solution $\{x_{i,j}\}_{(i,j)\in I\times J}$ to the associated LP
let $\{w(i,j)\}_{(i,j)\in I\times(J\cup\{0\})}$ be defined by (3.7)
initialize μ to the empty matching
for every man $i \in I$, initialize ℓ_i to the most preferred woman $j \in J \cup \{0\}$ with respect to $\geq_i$
for every man $i \in I$, initialize p_i to $w(i, \ell_i)$
while there exists a man $i \in I$ such that $\mu(i) = 0$ and $\ell_i >_i 0$ **do**
  let i_0 be such a man, and let j_0 denote the woman ℓ_{i_0}
  update ℓ_{i_0} to the most preferred woman $j \in \{j' \in J : j_0 >_{i_0} j'\} \cup \{0\}$ with respect to $\geq_{i_0}$
  if $\mu(j_0) = 0$ and $i_0 \geq_{j_0} 0$ **then**
    update μ to $\mu \cup \{(i_0, j_0)\}$
  else
    let $i_1 = \begin{cases} \mu(j_0) & \text{if } i_0 >_{j_0} \mu(j_0) \text{ or } (i_0 =_{j_0} \mu(j_0) \text{ and } p_{i_0} > p_{\mu(j_0)}) \\ i_0 & \text{otherwise} \end{cases}$
    let $\mu_0 = (\mu \cup \{(i_0, j_0)\}) \setminus \{(i_1, j_0)\}$
    let I_0 denote $\{i \in I : i \text{ is reachable from } i_1 \text{ via a } \mu_0\text{-alternating path in } G(\ell)\}$
    let i_2 be a man in $\arg\min_{i\in I_0} w(i, \ell_i)$
    let π_0 be a μ_0-alternating path from i_1 to i_2 in $G(\ell)$
    update p_i to $\max(p_i, w(i_2, \ell_{i_2}))$ for each man i in I_0
    update μ to $\mu_0 \oplus \pi_0$
  end if
end while
return matching μ

Algorithm 2

compute an optimal fractional solution $\{x_{i,j}\}_{(i,j)\in I\times J}$ to the associated LP
let $\{w(i,j)\}_{(i,j)\in I\times(J\cup\{0\})}$ be defined by (3.7)
initialize μ to the empty matching
for every man $i \in I$, initialize ℓ_i to the most preferred woman $j \in J \cup \{0\}$ with respect to $\geq_i$
while there exists a man $i \in I$ such that $\mu(i) = 0$ and $\ell_i >_i 0$ **do**
  let i_0 be such a man, and let j_0 denote the woman ℓ_{i_0}
  update ℓ_{i_0} to the most preferred woman $j \in \{j' \in J : j_0 >_{i_0} j'\} \cup \{0\}$ with respect to $\geq_{i_0}$
  update μ to an arbitrary MWMCM of $G(\ell)$
end while
return matching μ

4 Analysis of the Approximation Ratio

In this section, we analyze the approximation ratio and the integrality gap. Our analysis applies to both Algorithms 1 and 2. Throughout this section, whenever we mention $\{x_{i,j}\}_{(i,j)\in I\times J}$, $\{w(i,j)\}_{(i,j)\in I\times(J\cup\{0\})}$, and μ, we are referring to their values when the algorithm terminates. By Lemmas 3.1 and 3.3, conditions $\mathcal{P}_1(\mu)$, $\mathcal{P}_2(\mu)$, $\mathcal{P}_3(\mu, p, 0)$, and $\mathcal{P}_4(\mu, p, 0)$ hold for some $p = \{p_i\}_{i\in I}$. We fix such priority values p throughout this section. We also denote

$$I^* = \{i \in I : \mu(i) \neq 0\}$$

as the set of matched men and

$$J^* = \{j \in J : \mu(j) \neq 0\}$$

as the set of matched women in the output matching μ.

4.1 Auxiliary Charges.

Define the auxiliary charge

$$\widetilde{x}_{i,j} = \min(x_{i,j}, \max(0, p_i - w(i,j)))$$

for every man $i \in I$ and woman $j \in J$. So, for every man $i \in I$, the auxiliary charges $\{\widetilde{x}_{i,j}\}_{j\in J}$ correspond to the following charging process: Go through the list of all women j, from most preferred to least preferred under preference relation $\geq_i$, and charge an amount of at most $x_{i,j}$ until a total charge of p_i is reached, or the list of women is exhausted. The quantities $\{\widetilde{x}_{i,j}\}_{j\in J}$ correspond to the charged amounts.

In Lemma 4.1, we present some simple properties of the auxiliary charges. In Lemma 4.2, we present some simple necessary conditions for an auxiliary charge to be positive.

Lemma 4.1. *Let $i \in I$ be a man. Then the following properties hold.*

1. *For all women $j, j' \in J$ such that $\widetilde{x}_{i,j} > 0$ and $j' >_i j$, we have $\widetilde{x}_{i,j'} = x_{i,j'}$.*
2. $\sum_{j \in J} \widetilde{x}_{i,j} \le p_i$.
3. $\sum_{j \in J} (x_{i,j} - \widetilde{x}_{i,j}) \le 1 - p_i$.

Proof.

(1) Let women $j, j' \in J$ satisfy $\widetilde{x}_{i,j} > 0$ and $j' >_i j$. Since $\widetilde{x}_{i,j} > 0$, the definition of $\widetilde{x}_{i,j}$ implies $p_i > w(i,j)$. Hence

$$p_i > w(i,j) = \sum_{\substack{j'' \in J \\ j'' >_i j}} x_{i,j''} \ge x_{i,j'} + \sum_{\substack{j'' \in J \\ j'' >_i j'}} x_{i,j''} = x_{i,j'} + w(i,j'),$$

where the equalities follow from the definitions of $w(i,j)$ and $w(i,j')$. Since $p_i - w(i,j') \ge x_{i,j'}$, the definition of $\widetilde{x}_{i,j'}$ implies $\widetilde{x}_{i,j'} = x_{i,j'}$.

(2) Let $J'_i = \{j \in J : \widetilde{x}_{i,j} > 0\}$. It suffices to show that

$$\sum_{j \in J'_i} \widetilde{x}_{i,j} \le p_i.$$

For the sake of contradiction, suppose

$$\sum_{j \in J'_i} \widetilde{x}_{i,j} > p_i. \tag{4.8}$$

Since $p_i \ge 0$, inequality (4.8) implies J'_i is non-empty. Let $j' \in J'_i$ be the least preferred woman in J'_i under the preference relation $\ge_i$. Then

$$\begin{aligned}\sum_{j \in J'_i} \widetilde{x}_{i,j} \le \sum_{\substack{j \in J \\ j \ge_i j'}} \widetilde{x}_{i,j} &= \widetilde{x}_{i,j'} + \sum_{\substack{j \in J \\ j >_i j'}} \widetilde{x}_{i,j} \\ &= \widetilde{x}_{i,j'} + \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} \\ &= \widetilde{x}_{i,j'} + w(i,j'), \end{aligned} \tag{4.9}$$

where the second equality follows from part (1), and the third equality follows from the definition of $w(i,j')$. Combining (4.8) and (4.9) gives $\widetilde{x}_{i,j'} > p_i - w(i,j')$. Also $\widetilde{x}_{i,j'} > 0$ since $j' \in J'_i$. Hence $\widetilde{x}_{i,j'} > \max(0, p_i - w(i,j'))$, which contradicts the definition of $\widetilde{x}_{i,j'}$.

(3) Let $J'_i = \{j \in J : \widetilde{x}_{i,j} < x_{i,j}\}$. It suffices to show that

$$\sum_{j \in J'_i} (x_{i,j} - \widetilde{x}_{i,j}) \le 1 - p_i.$$

For the sake of contradiction, suppose

$$\sum_{j \in J'_i} (x_{i,j} - \widetilde{x}_{i,j}) > 1 - p_i. \tag{4.10}$$

$\mathcal{P}_3(\mu, p, 0)$ implies $p_i \le 1$, and so inequality (4.10) implies that J'_i is non-empty. Let $j' \in J'_i$ be the most preferred woman in J'_i under the preference relation $\ge_i$. Then

$$\begin{aligned}\sum_{j \in J'_i} (x_{i,j} - \widetilde{x}_{i,j}) &\le \sum_{\substack{j \in J \\ j' \ge_i j}} (x_{i,j} - \widetilde{x}_{i,j}) \\ &\le \Bigg(\sum_{\substack{j \in J \\ j' \ge_i j}} x_{i,j} \Bigg) - \widetilde{x}_{i,j'}. \end{aligned} \tag{4.11}$$

Also, by constraint (2.1), we have

$$\begin{aligned} 1 \ge \sum_{j \in J} x_{i,j} &= \sum_{\substack{j \in J \\ j >_i j'}} x_{i,j} + \sum_{\substack{j \in J \\ j' \ge_i j}} x_{i,j} \\ &= w(i,j') + \sum_{\substack{j \in J \\ j' \ge_i j}} x_{i,j}, \end{aligned} \tag{4.12}$$

where the second equality follows from the definition of $w(i,j')$. By combining (4.10), (4.11), and (4.12), we obtain

$$\widetilde{x}_{i,j'} < p_i - w(i,j') \le \max(0, p_i - w(i,j')).$$

Also $\widetilde{x}_{i,j'} < x_{i,j'}$ since $j' \in J'_i$. Hence $\widetilde{x}_{i,j'} < \min(x_{i,j'}, \max(0, p_i - w(i,j')))$, which contradicts the definition of $\widetilde{x}_{i,j'}$.

Lemma 4.2. *Let $i \in I$ be a man and $j \in J$ be a woman such that $\widetilde{x}_{i,j} > 0$. Then $j \in J^*$ and $\mu(j) \ge_j i$. Furthermore, if $i \in I \setminus I^*$ and $\mu(j) =_j i$, then $p_{\mu(j)} = 1$.*

Proof. Since $\widetilde{x}_{i,j} > 0$, the definition of $\widetilde{x}_{i,j}$ implies $x_{i,j} > 0$ and $p_i > w(i,j)$. Since $x_{i,j} > 0$, constraint (2.4) implies $j \ge_i 0$ and $i \ge_j 0$. Since $\mathcal{P}_4(\mu, p, 0)$ holds, we deduce that $j \in J^*$ and $\mu(j) \ge_j i$.

Furthermore, suppose $i \in I \setminus I^*$ and $\mu(j) =_j i$. Since $\mu(j) =_j i$, conditions $\mathcal{P}_3(\mu, p, 0)$ and $\mathcal{P}_4(\mu, p, 0)$ imply

$$1 \ge p_{\mu(j)} \ge p_i \ge w(i,0) = 1,$$

where the equality follows from the definition of $w(i,0)$.

For every man $i \in I^*$, we define the auxiliary quantity

$$q_i = \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j}.$$

For every woman $j \in J^*$, we define the auxiliary quantities

$$y_j = y_{\mu(j)} = \sum_{\substack{i \in I \setminus I^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j}$$

and

$$z_j = z_{\mu(j)} = \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j}.$$

We also denote

$$\mathbb{1}_{[0,\infty)}(\xi) = \begin{cases} 1 & \text{if } \xi \geq 0 \\ 0 & \text{if } \xi < 0 \end{cases}$$

as the Heaviside step function.

Intuitively, for every man $i \in I$ and woman $j \in J$, we think of the auxiliary charge $\widetilde{x}_{i,j}$ as *tie-related* if woman j is indifferent between i and $\mu(j)$. Then the quantity q_i corresponds to the total tie-related charge associated with man i and all matched women. Similarly, the quantity y_j corresponds to the total tie-related charge associated with woman j and all unmatched men. Finally, the quantity z_j corresponds to the right hand side of constraint (2.6) for $(i,j) = (\mu(j), j)$.

For every man $i \in I$ and woman $j \in J$, it is easy to see that $0 \leq \widetilde{x}_{i,j} \leq x_{i,j}$. Hence it follows from constraints (2.1) and (2.2) that $q_i, y_i, z_i \in [0,1]$ for every man $i \in I^*$. In Lemma 4.3, we present some simple properties of the auxiliary quantities. In Lemma 4.4, we show that the difference between the size of the matching produced by the algorithm and the optimal fractional value of the LP is bounded by an expression involving the auxiliary quantities.

LEMMA 4.3. *Let $i \in I^*$ be a man. Then $(1-p_i)y_i = 0$ and $q_i \leq p_i$.*

Proof. Let $j = \mu(i)$. By the definition of y_i, we have

$$\begin{aligned}(1-p_i)y_i = (1-p_{\mu(j)})y_j &= \sum_{\substack{i' \in I \setminus I^* \\ \mu(j) =_j i'}} (1-p_{\mu(j)})\widetilde{x}_{i',j} \\ &= \sum_{\substack{i' \in I \setminus I^* \\ \mu(j) =_j i'}} (1-1)\widetilde{x}_{i',j} = 0,\end{aligned}$$

where the third equality follows from Lemma 4.2. By the definition of q_i, we have

$$q_i = \sum_{\substack{j' \in J^* \\ \mu(j') =_{j'} i}} \widetilde{x}_{i,j'} \leq \sum_{j' \in J} \widetilde{x}_{i,j'} \leq p_i,$$

where the second inequality follows from part (2) of Lemma 4.1.

LEMMA 4.4. $\left(\sum_{i \in I} \sum_{j \in J} x_{i,j}\right) - |\mu| \leq \sum_{i \in I^*} (q_i - p_i + y_i + z_i).$

Proof. Consider

$$\sum_{i \in I} \sum_{j \in J} x_{i,j} = \sum_{i \in I} \sum_{j \in J} (x_{i,j} - \widetilde{x}_{i,j}) + \sum_{i \in I} \sum_{j \in J} \widetilde{x}_{i,j}. \tag{4.13}$$

Part (3) of Lemma 4.1 implies

$$\begin{aligned}\sum_{i \in I} \sum_{j \in J} (x_{i,j} - \widetilde{x}_{i,j}) &\leq \sum_{i \in I} (1 - p_i) \\ &= |\mu| - \sum_{i \in I^*} p_i + \sum_{i \in I \setminus I^*} (1 - p_i) \\ &\leq |\mu| - \sum_{i \in I^*} p_i + \sum_{i \in I \setminus I^*} (1 - w(i,0)) \\ &= |\mu| - \sum_{i \in I^*} p_i, \end{aligned} \tag{4.14}$$

where the second inequality follows from $\mathcal{P}_3(\mu, p, 0)$, and the second equality follows from the definitions of $\{w(i,0)\}_{i \in I \setminus I^*}$. Lemma 4.2 implies

$$\begin{aligned}&\sum_{i \in I} \sum_{j \in J} \widetilde{x}_{i,j} \\ &= \sum_{i \in I} \sum_{\substack{j \in J^* \\ \mu(j) >_j i}} \widetilde{x}_{i,j} + \sum_{i \in I} \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} \\ &= \sum_{i \in I} \sum_{\substack{j \in J^* \\ \mu(j) >_j i}} \widetilde{x}_{i,j} + \sum_{i \in I \setminus I^*} \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} \\ &\quad + \sum_{i \in I^*} \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} \\ &= \sum_{i \in I} \sum_{\substack{j \in J^* \\ \mu(j) >_j i}} \widetilde{x}_{i,j} + \sum_{i \in I \setminus I^*} \sum_{\substack{j \in J^* \\ \mu(j) =_j i}} \widetilde{x}_{i,j} + \sum_{i \in I^*} q_i, \end{aligned} \tag{4.15}$$

where the last equality follows from the definitions of $\{q_i\}_{i \in I^*}$. The definitions of $\{z_i\}_{i \in I^*}$ imply

$$\begin{aligned}\sum_{i \in I^*} z_i = \sum_{j \in J^*} z_j = \sum_{j \in J^*} \sum_{\substack{i \in I \\ \mu(j) >_j i}} x_{i,j} &= \sum_{i \in I} \sum_{\substack{j \in J^* \\ \mu(j) >_j i}} x_{i,j} \\ &\geq \sum_{i \in I} \sum_{\substack{j \in J^* \\ \mu(j) >_j i}} \widetilde{x}_{i,j}, \end{aligned} \tag{4.16}$$

where the inequality follows from the definitions of $\{\widetilde{x}_{i,j}\}_{(i,j)\in I\times J}$. The definitions of $\{y_i\}_{i\in I^*}$ imply

$$\sum_{i\in I^*} y_i = \sum_{j\in J^*} y_j = \sum_{j\in J^*} \sum_{\substack{i\in I\setminus I^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j}$$

$$= \sum_{i\in I\setminus I^*} \sum_{\substack{j\in J^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j}. \tag{4.17}$$

Combining (4.13), (4.14), (4.15), (4.16), and (4.17) gives the desired inequality.

4.2 The Stability Constraint and the Tie-Breaking Criterion. In this subsection, we develop the key ingredients underlying our analysis by making careful use of the stability constraint for matched women and the tie-breaking criterion for matched men. The stability constraint, which says that no matched woman can participate in a strongly blocking pair, corresponds to Lemma 2.2. The tie-breaking criterion corresponds to predicate $\mathcal{P}_4(\mu, p, 0)$.

In Lemma 4.5, we consider the stability constraint associated with a matched pair $(i, j) \in \mu$ and show that z_i is at most p_i.

LEMMA 4.5. *Let $i \in I^*$ be a man. Then $z_i \le p_i$.*

Proof. Let $j = \mu(i)$. Since $\mathcal{P}_1(\mu)$ holds, we have $j \ge_i 0$ and $i \ge_j 0$. By the definition of z_i, we have

$$z_i = z_j = \sum_{\substack{i'\in I \\ i>_j i'}} x_{i',j} \le \sum_{\substack{j'\in J \\ j'>_i j}} x_{i,j'} = w(i,j) \le p_i,$$

where the first inequality follows from Lemma 2.2, the third equality follows from the definition of $w(i, j)$, and the second inequality follows from $\mathcal{P}_3(\mu, p, 0)$.

In Lemma 4.6, we consider pairs $(i, j) \in I^* \times J^*$ where the associated auxiliary charge $\widetilde{x}_{i,j}$ is positive and woman j is indifferent between man i and the man $\mu(j)$ to whom she is matched. These are the pairs that contribute to the tie-related charges $\{q_i\}_{i\in I^*}$. By applying the tie-breaking criterion and the stability constraint to these pairs (i, j), we establish two useful inequalities.

LEMMA 4.6. *Let $i \in I^*$ be a man and $j \in J^*$ be a woman such that $\widetilde{x}_{i,j} > 0$ and $\mu(j) =_j i$. Then*

$$p_i \le p_{\mu(j)} \qquad \text{and} \qquad \sum_{\substack{j'\in J \\ \mu(j')=_{j'} i \\ j\ge_i j'}} \widetilde{x}_{i,j'} \le p_{\mu(j)} - z_j.$$

Proof. Since $\widetilde{x}_{i,j} > 0$, the definition of $\widetilde{x}_{i,j}$ implies $x_{i,j} > 0$ and $p_i > w(i, j)$. Since $x_{i,j} > 0$, constraint (2.4) implies $j \ge_i 0$ and $i \ge_j 0$. Thus $\mathcal{P}_4(\mu, p, 0)$ implies

$$p_i \le p_{\mu(j)}, \tag{4.18}$$

which establishes the first desired inequality.

Since $\widetilde{x}_{i,j} > 0$, parts (2) and (1) of Lemma 4.1 imply

$$p_i - \sum_{\substack{j'\in J \\ j'>_i j}} x_{i,j'} \ge \sum_{j'\in J} \widetilde{x}_{i,j'} - \sum_{\substack{j'\in J \\ j'>_i j}} \widetilde{x}_{i,j'} = \sum_{\substack{j'\in J \\ j\ge_i j'}} \widetilde{x}_{i,j'}$$

$$\ge \sum_{\substack{j'\in J \\ \mu(j')=_{j'} i \\ j\ge_i j'}} \widetilde{x}_{i,j'}. \tag{4.19}$$

By the definition of z_j, we have

$$z_j = \sum_{\substack{i'\in I \\ \mu(j)>_j i'}} x_{i',j} = \sum_{\substack{i'\in I \\ i>_j i'}} x_{i',j} \le \sum_{\substack{j'\in J \\ j'>_i j}} x_{i,j'}, \tag{4.20}$$

where the second equality follows from $\mu(j) =_j i$, and the inequality follows from Lemma 2.2. Combining (4.18), (4.19), and (4.20) gives the second desired inequality.

Let $\mathcal{F}$ be the set of all non-increasing functions $f: [0, 1] \to [0, +\infty)$. Lemma 4.7 below aggregates the inequalities of Lemma 4.6 with respect to functions $f \in \mathcal{F}$. The proof of Lemma 4.7 is presented after we establish a few technical results in Lemmas 4.8, 4.9, and 4.10.

LEMMA 4.7. *If I^* is non-empty, then for every $f \in \mathcal{F}$,*

$$\frac{1}{|I^*|}\sum_{i\in I^*} \max(q_i - f(p_i), 0)$$
$$\le \frac{1}{|I^*|}\sum_{i\in I^*} (1 - y_i - z_i)\cdot \mathbb{1}_{[0,\infty)}(p_i - z_i - f(p_i)).$$

LEMMA 4.8. *Let $i \in I^*$ be a man and $f \in \mathcal{F}$. Let*

$$J_i^* = \Big\{ j \in J^* : \mu(j) =_j i \text{ and } \sum_{\substack{j''\in J \\ \mu(j'')=_{j''} i \\ j\ge_i j''}} \widetilde{x}_{i,j''} \ge f(p_i) \Big\}. \tag{4.21}$$

Then $\max(q_i - f(p_i), 0) \le \sum_{j\in J_i^*} \widetilde{x}_{i,j}$.

Proof. We may assume that $q_i > f(p_i)$, for otherwise $\max(q_i - f(p_i), 0) = 0 \le \sum_{j\in J_i^*} \widetilde{x}_{i,j}$. Let

$$J_i' = \Big\{ j \in J^* \colon \mu(j) =_j i \text{ and } \sum_{\substack{j''\in J \\ \mu(j'')=_{j''} i \\ j \ge_i j''}} \widetilde{x}_{i,j''} < f(p_i) \Big\}.$$

Then

$$\begin{aligned}\max(q_i - f(p_i), 0) &= q_i - f(p_i) \\ &= \Big(\sum_{\substack{j\in J^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j} \Big) - f(p_i) \\ &= \Big(\sum_{j\in J_i^*} \widetilde{x}_{i,j} + \sum_{j\in J_i'} \widetilde{x}_{i,j} \Big) - f(p_i),\end{aligned}$$

where the second equality follows from the definition of q_i. So, it suffices to show that $\sum_{j\in J_i'} \widetilde{x}_{i,j} \le f(p_i)$.

For the sake of contradiction, suppose

$$\sum_{j\in J_i'} \widetilde{x}_{i,j} > f(p_i).$$

Since $f \in \mathcal{F}$, we have

$$0 \le f(p_i) < \sum_{j\in J_i'} \widetilde{x}_{i,j}.$$

So the set J_i' is non-empty. Let $j' \in J_i'$ be the most preferred woman in J_i' under the preference relation $\ge_i$. Then

$$\sum_{\substack{j''\in J \\ \mu(j'')=_{j''} i \\ j' \ge_i j''}} \widetilde{x}_{i,j''} \ge \sum_{j''\in J_i'} \widetilde{x}_{i,j''} > f(p_i),$$

which contradicts $j' \in J_i'$.

Lemma 4.9. *Let $i \in I^*$ be a man and $f \in \mathcal{F}$. Let J_i^* be the set defined by* (4.21)*. Then, for every woman $j \in J_i^*$ such that $\widetilde{x}_{i,j} > 0$, we have $p_{\mu(j)} - z_j \ge f(p_{\mu(j)})$.*

Proof. Let $j \in J_i^*$ be a woman such that $\widetilde{x}_{i,j} > 0$. Since $j \in J_i^*$, we have $j \in J^*$ and $\mu(j) =_j i$. Since $\widetilde{x}_{i,j} > 0$ and $\mu(j) =_j i$, Lemma 4.6 implies

$$p_i \le p_{\mu(j)} \tag{4.22}$$

and

$$\sum_{\substack{j'\in J \\ \mu(j')=_{j'} i \\ j \ge_i j'}} \widetilde{x}_{i,j'} \le p_{\mu(j)} - z_j. \tag{4.23}$$

Since $f \in \mathcal{F}$, inequality (4.22) implies

$$f(p_{\mu(j)}) \le f(p_i) \le \sum_{\substack{j'\in J \\ \mu(j')=_{j'} i \\ j \ge_i j'}} \widetilde{x}_{i,j'} \le p_{\mu(j)} - z_j,$$

where the second inequality follows from $j \in J_i^*$, and the third inequality follows from (4.23).

Lemma 4.10. *Let $j \in J^*$ be a woman. Then $\sum_{\substack{i\in I^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j} \le 1 - y_j - z_j$.*

Proof. By constraint (2.2), we have

$$\begin{aligned} 1 &\ge \sum_{i\in I} x_{i,j} \\ &\ge \sum_{\substack{i\in I \\ \mu(j)>_j i}} x_{i,j} + \sum_{\substack{i\in I\setminus I^* \\ \mu(j)=_j i}} x_{i,j} + \sum_{\substack{i\in I^* \\ \mu(j)=_j i}} x_{i,j} \\ &\ge \sum_{\substack{i\in I \\ \mu(j)>_j i}} x_{i,j} + \sum_{\substack{i\in I\setminus I^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j} + \sum_{\substack{i\in I^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j} \\ &= z_j + y_j + \sum_{\substack{i\in I^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j}, \end{aligned}$$

where the third inequality follows from the definitions of $\{\widetilde{x}_{i,j}\}_{i\in I}$, and the equality follows from the definitions of z_j and y_j.

Proof of Lemma 4.7. Suppose I^* is non-empty. Let $f \in \mathcal{F}$. For every man $i \in I^*$, let J_i^* be the set defined by (4.21). Then Lemma 4.8 implies

$$\begin{aligned} &\frac{1}{|I^*|} \sum_{i\in I^*} \max(q_i - f(p_i), 0) \\ &\le \frac{1}{|I^*|} \sum_{i\in I^*} \sum_{j\in J_i^*} \widetilde{x}_{i,j} \\ &= \frac{1}{|I^*|} \sum_{i\in I^*} \sum_{j\in J_i^*} \widetilde{x}_{i,j} \cdot \mathbb{1}_{[0,\infty)}(p_{\mu(j)} - z_j - f(p_{\mu(j)})) \\ &\le \frac{1}{|I^*|} \sum_{i\in I^*} \sum_{\substack{j\in J^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j} \cdot \mathbb{1}_{[0,\infty)}(p_{\mu(j)} - z_j - f(p_{\mu(j)})) \\ &= \frac{1}{|I^*|} \sum_{j\in J^*} \mathbb{1}_{[0,\infty)}(p_{\mu(j)} - z_j - f(p_{\mu(j)})) \sum_{\substack{i\in I^* \\ \mu(j)=_j i}} \widetilde{x}_{i,j} \\ &\le \frac{1}{|I^*|} \sum_{j\in J^*} \mathbb{1}_{[0,\infty)}(p_{\mu(j)} - z_j - f(p_{\mu(j)})) \cdot (1 - y_j - z_j) \\ &= \frac{1}{|I^*|} \sum_{i\in I^*} \mathbb{1}_{[0,\infty)}(p_i - z_i - f(p_i)) \cdot (1 - y_i - z_i), \end{aligned}$$

where the first equality follows from Lemma 4.9, the second inequality follows from the definition of J_i^*, the third inequality follows from Lemma 4.10, and the last equality follows from the change of variables $i = \mu(j)$.

4.3 Factor-Revealing Optimization. In order to obtain the approximation ratio, we use Lemmas 4.3, 4.5, and 4.7 to derive a bound for the right-hand side of Lemma 4.4. We can formulate this as the following factor-revealing optimization problem over $\{p_i, q_i, y_i, z_i\}_{i \in I^*}$.

$$
\begin{aligned}
&\text{maximize} \quad \frac{1}{|I^*|}\sum_{i\in I^*}(q_i - p_i + y_i + z_i)\\
&\text{subject to}\\
&\qquad (1-p_i)y_i = 0 \qquad \forall i \in I^*\\
&\qquad q_i \le p_i \qquad \forall i \in I^*\\
&\qquad z_i \le p_i \qquad \forall i \in I^*\\
&\frac{1}{|I^*|}\sum_{i\in I^*}\Big(\max(q_i - f(p_i), 0)\\
&\qquad - (1 - y_i - z_i)\\
&\qquad \cdot \mathbb{1}_{[0,\infty)}(p_i - z_i - f(p_i))\Big) \le 0 \qquad \forall f \in \mathcal{F}\\
&\qquad p_i, q_i, y_i, z_i \in [0,1] \qquad \forall i \in I^*
\end{aligned}
$$

The above factor-revealing optimization problem is non-convex in $\{p_i, q_i, y_i, z_i\}_{i \in I^*}$. Moreover, the formulation depends on the size of I^*. To overcome these issues, we reformulate it as the following infinite-dimensional factor-revealing LP over random variables $P, Q, Y, Z \in [0, 1]$.

$$
\begin{aligned}
&\text{maximize} \quad \mathrm{E}[Q - P + Y + Z]\\
&\text{subject to}\\
&(4.24) \qquad \Pr[(1-P)Y = 0\\
&\qquad\qquad \text{and } \max(Q,Z) \le P] = 1\\
&(4.25) \qquad \mathrm{E}[\max(Q - f(P), 0)\\
&\qquad\qquad - (1 - Y - Z)\\
&\qquad\qquad \cdot \mathbb{1}_{[0,\infty)}(P - Z - f(P))] \le 0 \qquad \forall f \in \mathcal{F}
\end{aligned}
$$

Notice that the above infinite-dimensional factor-revealing LP is linear in the joint distribution of (P, Q, Y, Z).

4.3.1 An Informal Investigation. We first present an informal investigation of the infinite-dimensional factor-revealing LP. A formal analytical solution is presented in § 4.3.2.

One approach to the infinite-dimensional factor-revealing LP is to obtaining a numerical solution via a suitable discretization. More precisely, we can bucket the support of (P, Q, Y, Z) into disjoint hypercubes, and assign to each such hypercube a variable that corresponds to the probability mass inside the hypercube. We can also restrict our attention to a finite subfamily of $\mathcal{F}$. This discretizes the infinite-dimensional factor-revealing LP into a finite LP which can be solved numerically. However, none of the numerical optimal solutions of the finite LP can equal $1/e$, which is irrational.

Our numerical results suggest that an optimal solution occurs when $P = Q = p_0 \approx 0.6$ and $Y = 0$. Suppose $P = Q = p_0$ and $Y = 0$. Then it is easy to see that constraint (4.25) depends only on $u = f(p_0)$. So the infinite-dimensional factor-revealing LP becomes

$$
\begin{aligned}
&\text{maximize} \quad \int_0^{p_0} z \cdot \rho_Z(z)\,dz\\
&\text{subject to}\\
&\qquad \int_0^{p_0} \rho_Z(z)\,dz = 1\\
&(p_0 - u) - \int_0^{p_0-u}(1-z)\cdot\rho_Z(z)\,dz \le 0 \qquad \forall u \in [0, p_0]
\end{aligned}
$$

the dual of which is

$$
\begin{aligned}
&\text{minimize} \quad \lambda - \int_0^{p_0}(p_0 - u)\cdot\rho_U(u)\,du\\
&\text{subject to}\\
&\qquad \lambda - \int_0^{p_0 - z}(1-z)\cdot\rho_U(u)\,du \ge z \qquad \forall z \in [0, p_0]
\end{aligned}
$$

Suppose the constraints in the dual are tight for all $z \in (0, p_0)$. Differentiating twice with respect to z gives

$$(1-z)\cdot\rho_U'(p_0 - z) - 2\rho_U(p_0 - z) = 0.$$

This ordinary differential equation has the solution $\rho_U(u) \propto (1 - p_0 + u)^{-2}$. Furthermore, assuming that the constraints in the dual are tight for all $z \in [0, p_0]$, we get $\rho_U(u) = (1 - p_0)(1 - p_0 + u)^{-2}$ and $\lambda = p_0$, with an objective value of $(p_0 - 1)\ln(1 - p_0)$. In fact, if $p_0 \le 1 - 1/e$, this objective value is attained by the primal solution

$$\rho_Z(z) = (1-z)^{-1} + (1 + \ln(1 - p_0))\cdot\delta(p_0 - z)$$

where δ is the Dirac delta function. The maximum objective value of $(p_0 - 1)\ln(1 - p_0)$ is $1/e$, which occurs when $p_0 = 1 - 1/e$.

4.3.2 A Formal Analytical Solution. To obtain a formal analytical solution, we consider functions in $\mathcal{F}$ of the form $f_u(p) = (1 - p)u$ where $u \in [0, \infty)$. (In our earlier technical report [18], we established an approximation ratio of $\ln 4$ by considering constant functions.) Then constraint (4.25) implies

$$\begin{aligned}(4.26)\quad &\mathrm{E}[\max(Q-(1-P)u,0)\\ &-(1-Y-Z)\\ &\cdot \mathbb{1}_{[0,\infty)}(P-Z-(1-P)u)]\le 0 \quad \forall u\in[0,\infty)\end{aligned}$$

Lemma 4.11 below presents a simple inequality. Lemma 4.12 can be regarded as a dual feasibility result. The probability density function ρ_U plays the role of the dual variables corresponding to constraint (4.26), while the dual variable λ corresponds to constraint (4.24). Lemma 4.13 uses Lemma 4.12 to bound the objective value of the infinite dimensional factor-revealing LP using Lemma 4.12. Our proof is analogous to that of the weak duality theorem for finite LP, in which any feasible primal solution is bounded by any feasible dual solution.

LEMMA 4.11. *For every $p\in[0,1)$, we have $(1-p)\ln(1-p)\ge -e^{-1}$.*

Proof. Let $g(p)=(1-p)\ln(1-p)$. It suffices to show that $g(p)\ge g(1-e^{-1})$ for every $p\in[0,1)$.

For every $p\in(0,1-e^{-1})$, we have

$$\frac{dg(p)}{dp}=-\ln(1-p)-1<-\ln(1-(1-e^{-1}))-1=0.$$

Hence $g(p)\ge g(1-e^{-1})$ for every $p\in[0,1-e^{-1}]$.

For every $p\in(1-e^{-1},1)$, we have

$$\frac{dg(p)}{dp}=-\ln(1-p)-1>-\ln(1-(1-e^{-1}))-1=0.$$

Hence $g(p)\ge g(1-e^{-1})$ for every $p\in[1-e^{-1},1)$.

LEMMA 4.12. *Let $p,q,y,z\in[0,1]$ such that $(1-p)y=0$ and $\max(q,z)\le p$. Let U be a continuous random variable with probability density function*

$$(4.27)\qquad \rho_U(u)=\begin{cases}(1+u)^{-2} & \text{if } u\ge 0\\ 0 & \text{otherwise}\end{cases}$$

and $\lambda=e^{-1}$. Then

$$\begin{aligned}&\mathrm{E}[\max(q-(1-p)U,0)\\ &-(1-y-z)\cdot\mathbb{1}_{[0,\infty)}(p-z-(1-p)U)]\\ &\ge q-p+y+z-\lambda.\end{aligned}$$

Proof. Since $0\le p\le 1$, we consider two cases.

Case 1: $p=1$. Then

$$\begin{aligned}&\mathrm{E}[\max(q-(1-p)U,0)\\ &\quad-(1-y-z)\cdot\mathbb{1}_{[0,\infty)}(p-z-(1-p)U)]\\ &=\mathrm{E}[q-(1-y-z)]\\ &=q-1+y+z\\ &=q-p+y+z\\ &>q-p+y+z-\lambda.\end{aligned}$$

Case 2: $0\le p<1$. Then $0\le z\le p<1$. Hence

$$\begin{aligned}&\mathrm{E}[\max(q-(1-p)U,0)]\\ &-\mathrm{E}[(1-y-z)\cdot\mathbb{1}_{[0,\infty)}(p-z-(1-p)U)]\\ &=\int_0^{q/(1-p)}(q-(1-p)u)\cdot\rho_U(u)\,du\\ &\quad-\int_0^{(p-z)/(1-p)}(1-y-z)\cdot\rho_U(u)\,du\\ &=\int_0^{q/(1-p)}\frac{q-(1-p)u}{(1+u)^2}\,du\\ &\quad-\int_0^{(p-z)/(1-p)}\frac{1-y-z}{(1+u)^2}\,du\\ &=\Big[\frac{-(1-p+q)}{1+u}-(1-p)\ln(1+u)\Big]_0^{q/(1-p)}\\ &\quad-\Big[\frac{-(1-y-z)}{1+u}\Big]_0^{(p-z)/(1-p)}\\ &=\Big(q-(1-p)\ln\frac{1-p+q}{1-p}\Big)-(1-y-z)\Big(\frac{p-z}{1-z}\Big)\\ &\ge\Big(q-(1-p)\ln\frac{1}{1-p}\Big)-(1-y-z)\Big(\frac{p-z}{1-z}\Big)\\ &\quad+y\Big(\frac{1-p}{1-z}\Big)\\ &\ge(q-e^{-1})-(1-y-z)\Big(\frac{p-z}{1-z}\Big)+y\Big(\frac{1-p}{1-z}\Big)\\ &=q-\lambda-p+y+z,\end{aligned}$$

where the first inequality follows from $q\le p$ and $(1-p)y=0$, and the second inequality follows from Lemma 4.11.

LEMMA 4.13. *Let $P,Q,Y,Z\in[0,1]$ be random variables satisfying constraints (4.24) and (4.25). Then $\mathrm{E}[Q-P+Y+Z]\le e^{-1}$.*

Proof. Pick U independent of (P,Q,Y,Z) according to the probability density function ρ_U defined by (4.27). Since constraint (4.24) is satisfied by (P,Q,Y,Z), Lemma 4.12 implies

$$\begin{aligned}&\mathrm{E}[\max(Q-(1-P)U,0)\\ &\quad-(1-Y-Z)\cdot\mathbb{1}_{[0,\infty)}(P-Z-(1-P)U)]\\ &\ge\mathrm{E}[Q-P+Y+Z]-e^{-1}.\end{aligned}$$

Also since constraint (4.25) is satisfied by (P, Q, Y, Z), constraint (4.26) is satisfied by (P, Q, Y, Z). Hence

$$\begin{aligned} 0 \geq \mathrm{E}[&\max(Q-(1-P)U, 0) \\ &-(1-Y-Z)\cdot \mathbb{1}_{[0,\infty)}(P-Z-(1-P)U)] \end{aligned}$$

Thus $\mathrm{E}[Q - P + Y + Z] \leq e^{-1}$.

4.4 The Approximation Ratio. Lemma 4.14 below is obtained by putting together the results in the preceding subsections. Our main results are presented in Theorems 4.1 and 4.2 and proved using Lemma 4.14.

LEMMA 4.14. $\sum_{i\in I}\sum_{j\in J} x_{i,j} \leq (1+e^{-1})\cdot|\mu|$.

Proof. We may assume that I^* is non-empty, for otherwise Lemma 4.4 implies

$$\sum_{i\in I}\sum_{j\in J} x_{i,j} = |\mu| = 0.$$

Let $(P, Q, Y, Z) = (p_{i'}, q_{i'}, y_{i'}, z_{i'})$, where i' is a man picked uniformly at random from I^*. Then Lemmas 4.3 and 4.5 imply that (P, Q, Y, Z) satisfies (4.24). Also, Lemma 4.7 implies that (P, Q, Y, Z) satisfies (4.25). Hence Lemma 4.13 implies

$$e^{-1} \geq \mathrm{E}[Q-P+Y+Z] = \frac{1}{|I^*|}\sum_{i\in I^*}(q_i - p_i + y_i + z_i).$$

Thus Lemma 4.4 implies

$$\begin{aligned} \sum_{i\in I}\sum_{j\in J} x_{i,j} &\leq |I^*| + \sum_{i\in I^*}(q_i - p_i + y_i + z_i) \\ &\leq |I^*| + e^{-1}\cdot|I^*| \\ &= (1+e^{-1})\cdot|\mu|. \end{aligned}$$

THEOREM 4.1. *Algorithms 1 and 2 are polynomial-time $(1+e^{-1})$-approximation algorithms for* MAX-SMOTI.

Proof. By Lemma 3.4, Algorithms 1 and 2 each run in polynomial time and produce a weakly stable matching μ. Let μ' be a maximum weakly stable matching, and $\{x'_{i,j}\}_{(i,j)\in I\times J}$ be the indicator variables of μ'. Since μ' is weakly stable, Lemma 2.1 implies that $\{x'_{i,j}\}_{(i,j)\in I\times J}$ satisfies constraints (2.1), (2.2), (2.3), (2.4), and (2.5). Hence Lemma 4.14 implies

$$(1+e^{-1})\cdot|\mu| \geq \sum_{i\in I}\sum_{j\in J} x_{i,j} \geq \sum_{i\in I}\sum_{j\in J} x'_{i,j} = |\mu'|,$$

where the second inequality follows from the optimality of $\{x_{i,j}\}_{(i,j)\in I\times J}$.

THEOREM 4.2. *The integrality gap of the LP formulation in § 2.2 is $1+e^{-1}$.*

Proof. By Lemmas 3.4 and 4.14, there exists a weakly stable matching μ such that

$$(1+e^{-1})\cdot|\mu| \geq \sum_{i\in I}\sum_{j\in J} x_{i,j}.$$

Let $\{x'_{i,j}\}_{(i,j)\in I\times J}$ be the indicator variables of μ. Since μ is weakly stable, Lemma 2.1 implies that $\{x'_{i,j}\}_{(i,j)\in I\times J}$ is an integral solution satisfying constraints (2.1), (2.2), (2.3), (2.4), and (2.5). Since

$$(1+e^{-1})\sum_{i\in I}\sum_{j\in J} x'_{i,j} = (1+e^{-1})\cdot|\mu| \geq \sum_{i\in I}\sum_{j\in J} x_{i,j},$$

the integrality gap is at most $1+e^{-1}$. This upper bound matches the known lower bound for the integrality gap [14, § 5.1].

We remark that our analysis of the infinite-dimensional factor-revealing LP in Lemma 4.13 is tight, since our upper bound for the integrality gap is tight. Similarly, our reformulation of the finite-dimensional factor-revealing optimization problem into the infinite-dimensional factor-revealing LP is tight in the limit as $|I^*|$ tends to infinity.

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A Proof of Lemma 3.1

The purpose of this section is to prove Lemma 3.1. We begin by establishing additional properties of Algorithm 1. It is straightforward to prove that throughout any execution of Algorithm 1, the program variable μ corresponds to a matching in the sense defined in § 2.1. Likewise, where it is defined, the program variable μ_0 corresponds to a matching. Accordingly, throughout our analysis, we assume that μ and μ_0 are matchings. It is convenient to define the following predicates.

$\mathcal{Q}_1(\ell)$: $\ell_i \geq_i 0$ for all $i \in I$.

$\mathcal{Q}_2(\ell, \mu)$: μ is a matching of $G(\ell)$ such that for every $i \in I$ and $j \in J$, if $i \in I_j(\ell)$ and $i \geq_j 0$, then $\mu(j) \neq 0$.

$\mathcal{Q}_3(\ell, p)$: for every $i \in I$, we have $p_i \leq w(i, \ell_i)$.

$\mathcal{Q}_4(\ell, \mu, p)$: for every $i \in I$ such that $\mu(i) = 0$, we have $p_i = w(i, \ell_i)$.

$\mathcal{Q}_5(\ell, p)$: for every $i \in I$ and $j \in J$ such that $j >_i \ell_i$, we have $w(i, j) \leq p_i$.

$\mathcal{Q}_6(\ell, \mu, p)$: for every $i, i' \in I$ and $j \in J$ such that $(i, j) \in E(\ell)$ and $\mu(i') = j$, we have $p_i \leq p_{i'}$.

In addition we, define $\mathcal{Q}(\ell, \mu, p)$ as the conjunction of predicates $\mathcal{Q}_1(\ell)$, $\mathcal{Q}_2(\ell, \mu)$, $\mathcal{Q}_3(\ell, p)$, $\mathcal{Q}_4(\ell, \mu, p)$, $\mathcal{Q}_5(\ell, p)$, and $\mathcal{Q}_6(\ell, \mu, p)$.

LEMMA A.1. *Let ℓ, μ, and p be such that $\mathcal{Q}(\ell, \mu, p)$ holds. Then $\mathcal{P}_1(\mu)$, $\mathcal{P}_2(\mu)$, $\mathcal{P}_3(\mu, p, 0)$, and $\mathcal{P}_4(\mu, p, 0)$ hold.*

Proof. We begin by proving that $\mathcal{P}_1(\mu)$ holds. Let $(i, j) \in \mu$. Since $\mathcal{Q}_2(\ell, \mu)$ holds, μ is matching of $G(\ell)$.

Since $(i,j) \in \mu$ and μ is a matching of $G(\ell)$, we have $(i,j) \in E(\ell)$. Since $(i,j) \in E(\ell)$, we have $i \in I_j(\ell)$ and $i \geq_j 0$. Since $i \in I_j(\ell)$, we have $j >_i \ell_i$. Since $\mathcal{Q}_1(\ell)$ holds, we have $\ell_i \geq_i 0$. Since $j >_i \ell_i$ and $\ell_i \geq_i 0$, we have $j >_i 0$.

We now prove that $\mathcal{P}_2(\mu)$ holds. holds. Let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i \mu(i)$ and $i \geq_j 0$. We prove that $i \in I_j(\ell)$ by considering two cases.

Case 1: $\mu(i) = 0$. The loop termination condition implies $0 \geq_i \ell_i$. Since $j \in J$ and $j \geq_i \mu(i) = 0$, we have $j >_i 0$. Since $j >_i 0 \geq_i \ell_i$, we have $i \in I_j(\ell)$.

Case 2: $\mu(i) \neq 0$. Since $(i, \mu(i)) \in \mu \subseteq E(\ell)$, we have $i \in I_j(\ell)$.

Having established that $i \in I_j(\ell)$, we now complete the proof that $\mathcal{P}_2(\mu)$ holds. Since $i \in I_j(\ell)$ and $i \geq_j 0$, condition $\mathcal{Q}_2(\ell,\mu)$ implies $\mu(j) \neq 0$. Since $(\mu(j), j) \in E(\ell)$ and $i \in I_j(\ell)$, the definition of $E(\ell)$ implies $\mu(j) \geq_j i$.

We now prove that $\mathcal{P}_3(\mu, p, 0)$ holds. Let $i \in I$ be a man. We consider two cases.

Case 1: $\mu(i) = 0$. Since $\mathcal{Q}_1(\ell)$ holds, we have $\ell_i \geq_i 0$. Since $\mu(i) = 0$ and $\ell_i \geq_i 0$, the loop termination condition implies that $\ell_i = 0$. Since $\ell_i = 0$, we have $w(i, \ell_i) = 1$. Since $\mu(i) = 0$, $w(i, \ell_i) = 1$, and $\mathcal{Q}_4(\ell, \mu, p)$ holds, we have $p_i = 1$.

Case 2: $\mu(i) \neq 0$. Let j denote $\mu(i)$. Since $\mathcal{Q}_2(\ell,\mu)$ holds, μ is a matching of $G(\ell)$. Since $\mu(i) = j$ and μ is a matching of $G(\ell)$, we have $(i,j) \in E(\ell)$. Since $(i,j) \in E(\ell)$, we have $i \in I_j(\ell)$ and hence $j >_i \ell_i$. Since $j >_i \ell_i$ and $\mathcal{Q}_5(\ell, p)$ holds, we have $w(i,j) \leq p_i$. It remains to argue that $p_i \leq 1$. Since constraint (2.1) holds, we have $w(i, \ell_i) \leq 1$. Since $w(i, \ell_i) \leq 1$ and $\mathcal{Q}_3(\ell, p)$ holds, we have $p_i \leq 1$.

It remains to prove that $\mathcal{P}_4(\mu, p, 0)$ holds. Let $i \in I$ be a man and $j \in J$ be a woman such that $j \geq_i 0$, $i \geq_j 0$, and $p_i > w(i,j)$. Since $p_i > w(i,j)$ and $\mathcal{Q}_3(\ell, p)$ holds, we have $j >_i \ell_i$ and hence $i \in I_j(\ell)$. Since $i \in I_j(\ell)$, $i \geq_j 0$, and $\mathcal{Q}_2(\ell,\mu)$ holds, we know that μ is a matching of $G(\ell)$ with $\mu(j) \neq 0$. Let $i' \in I$ denote $\mu(j)$. Since μ is a matching of $G(\ell)$ and (i', j) belongs to μ, we have $(i', j) \in E(\ell)$. Since $(i', j) \in E(\ell)$ and $i \in I_j(\ell)$, the definition of $E(\ell)$ implies that $i' \geq_j i$. It remains to prove that if $i' =_j i$ then $p_i \leq p_{i'}$. Assume $i' =_j i$. Since $(i', j) \in E(\ell)$, $i \in I_j(\ell)$, and $i' =_j i$, the definition of $E(\ell)$ implies that $(i,j) \in E(\ell)$. Since $(i,j) \in E(\ell)$, $\mu(i') = j$, and $\mathcal{Q}_6(\ell, \mu, p)$ holds, we have $p_i \leq p_{i'}$.

The following lemma is proven in § A.1.

Lemma A.2. *Consider an iteration of the Algorithm 1 loop. Let ℓ^-, μ^-, and p^- denote the values of ℓ, μ, and p at the start of the iteration. Assume that the loop condition is satisfied, and that $\mathcal{Q}(\ell^-, \mu^-, p^-)$ holds. Let ℓ^+, μ^+, and p^+ denote the values of ℓ, μ, and p at the end of the iteration. Then $\mathcal{Q}(\ell^+, \mu^+, p^+)$ holds.*

Proof of Lemma 3.1. Fix an execution of Algorithm 1. It is straightforward to verify that $\mathcal{Q}(\ell, \mu, p)$ holds the first time the loop is reached. Lemma A.2 implies that $\mathcal{Q}(\ell, \mu, p)$ holds upon termination of the loop. Thus the claim of the lemma follows by Lemma A.1.

A.1 Proof of Lemma A.2. The purpose of this section is to prove Lemma A.2. Throughout this section, we fix an iteration of the Algorithm 1 loop. Let ℓ^-, μ^-, and p^- denote the values of ℓ, μ, and p before the iteration, and let ℓ^+, μ^+, p^+ denote the values of ℓ, μ, and p after the iteration. Assume that ℓ^- and μ^- are such that the loop condition holds, so that the loop body is executed.

Lemma A.3. *Assume that $\mathcal{Q}_1(\ell^-)$ holds. Then $\mathcal{Q}_1(\ell^+)$ holds.*

Proof. The only line in the loop body that modifies ℓ is line 8, which updates ℓ_{i_0}. The definition of i_0 implies that $\ell^-_{i_0} >_{i_0} 0$. It follows that $\ell^+_{i_0} \geq_{i_0} 0$ holds.

The following lemma characterizes how $E(\ell)$ changes in a single iteration of the loop of Algorithm 1. We omit the proof, which is straightforward but tedious.

Lemma A.4. *Assume that $\mathcal{Q}_2(\ell^-, \mu^-)$ holds. Then the following claims hold: (1) $\mu^-(j) \geq_j 0$ for all $j \in J$; (2) if $i_0 <_{j_0} \mu^-(j_0)$, then $E(\ell^+) = E(\ell^-)$; (3) if $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$, then $E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$; (4) if $\mu^-(j_0) \neq 0$ and $i_0 =_{j_0} \mu^-(j_0)$, then $E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$; (5) if $\mu^-(j_0) \neq 0$ and $i_0 >_{j_0} \mu^-(j_0)$, then $E(\ell^+) = \{(i,j) \in E(\ell^-) \colon j \neq j_0\} \cup \{(i_0, j_0)\}$.*

Lemma A.5. *Assume that $\mathcal{Q}_2(\ell^-, \mu^-)$ holds. Then $\mathcal{Q}_2(\ell^+, \mu^+)$ holds. Furthermore, if $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$, then $\mathcal{Q}_2(\ell^+, \mu_0)$ holds.*

Proof. Since $\mathcal{Q}_2(\ell^-, \mu^-)$ holds, we know that μ^- is a matching of $G(\ell^-)$. Let $i \in I$ and $j \in J$ be such that $i \in I_j(\ell^+)$ and $i \geq_j 0$.

Case 1: $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$. Then $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$. Since $\mu^-(j_0) = 0$, $i_0 \geq_{j_0} 0$, and $\mathcal{Q}_2(\ell^-, \mu^-)$ holds, part (3) of Lemma A.4 implies that $E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$. Since μ^- is a matching of $G(\ell^-)$, $\mu^-(i_0) = 0$, $\mu^-(j_0) = 0$, $E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0\}$, and $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, we find that μ^+ is a matching of $G(\ell^+)$. To establish that $\mathcal{Q}_2(\ell^+, \mu_0)$ holds, it remains to prove that $\mu^+(j) \neq 0$.

Case 1.1: $j \neq j_0$. Then $I_j(\ell^+) = I_j(\ell^-)$, and hence $i \in I_j(\ell^-)$. Since $i \in I_j(\ell^-)$, $i \geq_j 0$, and $\mathcal{Q}_2(\ell^-, \mu^-)$ holds, we have $\mu^-(j) \neq 0$. Since $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$

and $j \neq j_0$, we have $\mu^+(j) = \mu^-(j)$. Since $\mu^+(j) = \mu^-(j)$ and $\mu^-(j) \neq 0$, we have $\mu^+(j) \neq 0$.

Case 1.2: $j = j_0$. Since $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, μ^+ is a matching of $G(\ell^+)$, and $j = j_0$, we deduce that $\mu^+(j) = i_0 \neq 0$.

Case 2: $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$. We need to prove that $\mathcal{Q}_2(\ell^+, \mu_0)$ and $\mathcal{Q}_2(\ell^+, \mu^+)$ hold. We begin by establishing two useful claims.

The first claim is that μ_0 is a matching of $G(\ell^+)$ that matches the same set of women as μ^-. To prove this claim, we consider three cases.

(a) $i_0 <_{j_0} \mu^-(j_0)$. Then $i_1 = i_0$ and part (2) of Lemma A.4 implies $E(\ell^+) = E(\ell^-)$. Since $i_1 = i_0$, we have $\mu_0 = \mu^-$. Since $\mu_0 = \mu^-$, $E(\ell^+) = E(\ell^-)$, and μ^- is a matching of $G(\ell^-)$, the claim follows.

(b) $i_0 =_{j_0} \mu^-(j_0)$. Then part (4) of Lemma A.4 implies $E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$. Since $\mu_0 = (\mu^- \cup \{(i_0, j_0)\}) \setminus \{(i_1, j_0)\}$, $E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$, and μ^- is a matching of $G(\ell^-)$, the claim follows.

(c) $i_0 >_{j_0} \mu^-(j_0)$. Then $i_1 \neq i_0$ and part (5) of Lemma A.4 implies $E(\ell^+) = \{(i,j) \in E(\ell^-) \colon j \neq j_0\} \cup \{(i_0, j_0)\}$. Since $\mu_0 = (\mu^- \cup \{(i_0, j_0)\}) \setminus \{(i_1, j_0)\}$, $E(\ell^+) = \{(i,j) \in E(\ell^-) \colon j \neq j_0\} \cup \{(i_0, j_0)\}$, and μ^- is a matching of $G(\ell^-)$, the claim follows.

The second claim is that μ^+ is a matching of $G(\ell^+)$ that matches the same set of women as μ_0. Since μ_0 is a matching of $G(\ell^+)$ and μ^+ is the symmetric difference between μ_0 and an oriented μ_0-alternating path in $G(\ell^+)$ from i_1 to i_2, the second claim follows.

Given the two preceding claims, we can establish that $\mathcal{Q}_2(\ell^+, \mu_0)$ and $\mathcal{Q}_2(\ell^+, \mu^+)$ hold by proving that $\mu^-(j) \neq 0$. If $j = j_0$, the latter inequality follows from the Case 2 condition. Now suppose that $j \neq j_0$. Then $I_j(\ell^+) = I_j(\ell^-)$, and hence $i \in I_j(\ell^-)$. Since $i \in I_j(\ell^-)$, $i \geq_j 0$, and $\mathcal{Q}_2(\ell^-, \mu^-)$ holds, we have $\mu^-(j) \neq 0$.

Lemma A.6. *Assume that $\mathcal{Q}_3(\ell^-, p^-)$ holds. Then $\mathcal{Q}_3(\ell^+, p^+)$ holds.*

Proof. Let $i \in I$. We consider two cases.

Case 1: $p_i^+ = p_i^-$. Since $\mathcal{Q}_3(\ell^-, p^-)$ holds, we have $p_i^- \leq w(i, \ell_i^-)$. Line 8 of Algorithm 1 implies $w(i, \ell_i^-) \leq w(i, \ell_i^+)$. Thus $p_i^+ = p_i^- \leq w(i, \ell_i^-) \leq w(i, \ell_i^+)$.

Case 2: $p_i^+ \neq p_i^-$. Then line 17 of Algorithm 1 implies $i \in I_0$ and $p_i^+ = w(i_2, \ell_{i_2}^+)$. Since $i \in I_0$, line 15 of Algorithm 1 implies $w(i_2, \ell_{i_2}^+) \leq w(i, \ell_i^+)$. Thus $p_i^+ = w(i_2, \ell_{i_2}^+) \leq w(i, \ell_i^+)$.

Lemma A.7. *Assume $\mathcal{Q}_3(\ell^-, p^-)$ and $\mathcal{Q}_4(\ell^-, \mu^-, p^-)$ hold. Then $\mathcal{Q}_4(\ell^+, \mu^+, p^+)$ holds.*

Proof. Let $i \in I$ be such that $\mu^+(i) = 0$. Since $\mathcal{Q}_3(\ell^-, p^-)$ holds, Lemma A.6 implies that $p_i^+ \leq w(i, \ell_i^+)$. Thus it is sufficient to prove that $p_i^+ \geq w(i, \ell_i^+)$. We consider two cases.

Case 1: $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$. In this case, we have $p_i^+ = p_i^-$. Since $\mu^+(i) = 0$, line 10 of Algorithm 1 implies $i \neq i_0$ and $\mu^-(i) = 0$. Since $\mu^-(i) = 0$, condition $\mathcal{Q}_4(\ell^-, \mu^-, p^-)$ implies $p_i^- = w(i, \ell_i^-)$. Since $i \neq i_0$, line 8 of Algorithm 1 implies $w(i, \ell_i^-) = w(i, \ell_i^+)$. Thus $p_i^+ = p_i^- = w(i, \ell_i^-) = w(i, \ell_i^+)$.

Case 2: $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$. We consider two subcases.

Case 2.1: $i = i_2$. Then line 15 of Algorithm 1 implies $i_2 \in I_0$. Since $i_2 \in I_0$, line 17 of Algorithm 1 implies $p_{i_2}^+ \geq w(i_2, \ell_{i_2}^+)$.

Case 2.2: $i \neq i_2$. Since $\mu^+(i) = 0$, $i \neq i_2$, and $\{i' \in I \colon \mu^+(i') \neq 0\} = (\{i' \in I \colon \mu^-(i') \neq 0\} \cup \{i_0\}) \setminus \{i_2\}$, we deduce that $i \neq i_0$ and $\mu^-(i) = 0$. Line 17 of Algorithm 1 implies $p_i^+ \geq p_i^-$. Since $\mu^-(i) = 0$ and $\mathcal{Q}_4(\ell^-, \mu^-, p^-)$ holds, we have $p_i^- = w(i, \ell_i^-)$. Since $i \neq i_0$, line 8 of Algorithm 1 implies $w(i, \ell_i^-) = w(i, \ell_i^+)$. Thus $p_i^+ \geq p_i^- = w(i, \ell_i^-) = w(i, \ell_i^+)$.

Lemma A.8. *Assume $\mathcal{Q}_4(\ell^-, \mu^-, p^-)$ and $\mathcal{Q}_5(\ell^-, p^-)$ hold. Then $\mathcal{Q}_5(\ell^+, p^+)$ holds.*

Proof. Let $i \in I$ and $j \in J$ be such that $j >_i \ell_i^+$. Line 17 of Algorithm 1 implies $p_i^+ \geq p_i^-$. We consider two cases.

Case 1: $j >_i \ell_i^-$. Then $\mathcal{Q}_5(\ell^-, p^-)$ implies $p_i^- \geq w(i,j)$. Thus $p_i^+ \geq p_i^- \geq w(i,j)$.

Case 2: $\ell_i^- \geq_i j$. Since $\ell_i^- \geq_i j >_i \ell_i^+$, line 8 of Algorithm 1 implies $i = i_0$ and $j = \ell_i^-$. Since $i = i_0$ and $\mu^-(i_0) = 0$, condition $\mathcal{Q}_4(\ell^-, \mu^-, p^-)$ implies $p_i^- = w(i, \ell_i^-)$. Thus $p_i^+ \geq p_i^- = w(i, \ell_i^-) = w(i,j)$.

Lemma A.9. *Assume that $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$. Further assume that $\mathcal{Q}_3(\ell^-, p^-)$ and $\mathcal{Q}_6(\ell^-, \mu^-, p^-)$ hold. Then the following conditions hold: (1) $\mathcal{Q}_6(\ell^+, \mu_0, p^-)$; (2) $\mathcal{Q}_6(\ell^+, \mu_0, p^+)$; (3) $p_i^+ = w(i_2, \ell_{i_2}^+)$ for every man i on path π_0; (4) $\mathcal{Q}_6(\ell^+, \mu^+, p^+)$.*

Proof.

(1) Let $i, i' \in I$ and $j \in J$ be such that $(i,j) \in E(\ell^+)$ and $\mu_0(i') = j$. We need to prove that $p_i^- \leq p_{i'}^-$. We consider two cases.

Case 1: $j \neq j_0$. Since $j \neq j_0$, we have $\mu^-(j) = \mu_0(j) = i'$. In addition, Lemma A.4 implies that $(i,j) \in E(\ell^-)$. Since $\mu^-(j) = i'$, $(i,j) \in E(\ell^-)$, and $\mathcal{Q}_6(\ell^-, \mu^-, p^-)$ holds, we conclude that $p_i^- \leq p_{i'}^-$.

Case 2: $j = j_0$. Thus $i' = \mu_0(j_0)$. Let $i'' \in I$ denote $\mu^-(j_0)$. We consider two subcases.

Case 2.1: $i \neq i_0$. Since $i \neq i_0$, Lemma A.4 implies that $(i, j_0) \in E(\ell^-)$. Since $(i, j_0) \in E(\ell^-)$ and $\mathcal{Q}_6(\ell^-, \mu^-, p^-)$ holds, we have $p_i^- \leq p_{i''}^-$. Since $i \neq i_0$ and $(i, j_0) \in E(\ell^+)$, Lemma A.4 implies that $i \geq_{j_0} i_0$. Since $i \geq_{j_0} i_0$, lines 12 and 13 of Algorithm 1 imply that $p_{i''}^- \leq p_{i'}^-$. Since $p_i^- \leq p_{i''}^-$ and $p_{i''}^- \leq p_{i'}^-$, we have $p_i^- \leq p_{i'}^-$.

Case 2.2: $i = i_0$. Since $(i_0, j_0) \in E(\ell^+)$, we have $i_0 \geq_{j_0} i''$. Since $i_0 \geq_{j_0} i''$, lines 12 and 13 of Algorithm 1 imply that $p_{i_0}^- \leq p_{i'}^-$.

(2) Let $i, i' \in I$ and $j \in J$ be such that $(i, j) \in E(\ell^+)$ and $\mu_0(i') = j$. By part (1), we have $p_i^- \leq p_{i'}^-$. We need to prove that $p_i^+ \leq p_{i'}^+$. If $i = i'$, this inequality is immediate, so we can assume that $i \neq i'$. We consider two cases.

Case 1: $p_i^+ = p_i^-$. We have $p_i^+ = p_i^- \leq p_{i'}^- \leq p_{i'}^+$.

Case 2: $p_i^+ \neq p_i^-$. In this case, i belongs to I_0 and $p_i^+ = w(i_2, \ell_{i_2}^+)$. Moreover, i' also belongs to I_0 and hence $p_{i'}^+ \geq w(i_2, \ell_{i_2}^+)$. Thus $p_i^+ \leq p_{i'}^+$.

(3) Let $i_1 = i'_1, \ldots, i'_s = i_2$ denote the sequence of men on path π_0. By part (1), we have $p_{i'_t}^- \leq p_{i'_{t+1}}^-$ for $1 \leq t < s$. It follows that $p_i^- \leq p_{i_2}^-$ for every man i on path π_0. Since $\mathcal{Q}_3(\ell^-, p^-)$ holds, we have $p_{i_2}^- \leq w(i_2, \ell_{i_2}^-) \leq w(i_2, \ell_{i_2}^+)$. Thus $p_i^- \leq w(i_2, \ell_{i_2}^+)$ for every man i on path π_0. Since every man on path π_0 belongs to I_0, line 17 of Algorithm 1 implies that $p_i^+ = w(i_2, \ell_{i_2}^+)$ for every man i on path π_0.

(4) Let J' denote the set of women who are matched in μ_0. Line 18 of Algorithm 1 ensures that the set of women who are matched in μ^+ is also J'. Moreover, by part (3), $p_{\mu^+(j)}^+ = p_{\mu_0(j)}^+$ for every woman j in J'. Consequently, part (2) implies that $\mathcal{Q}_6(\ell^+, \mu^+, p^+)$ holds.

Lemma A.10. *Assume that $\mathcal{Q}_2(\ell^-, \mu^-)$, $\mathcal{Q}_3(\ell^-, p^-)$, and $\mathcal{Q}_6(\ell^-, \mu^-, p^-)$ hold. Then $\mathcal{Q}_6(\ell^+, \mu^+, p^+)$ holds.*

Proof. If $\mu^-(j_0) \neq 0$ or $0 >_{j_0} i_0$, then part (4) of Lemma A.9 implies that $\mathcal{Q}_6(\ell^+, \mu^+, p^+)$ holds. For the remainder of the proof, assume that $\mu^-(j_0) = 0$ and $i_0 \geq_{j_0} 0$. Thus $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, $p^+ = p^-$, and part (3) of Lemma A.4 implies $E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$.

Let $i, i' \in I$ and $j \in J$ be such that $(i, j) \in E(\ell^+)$ and $\mu^+(i') = j$. We need to prove that $p_i^+ \leq p_{i'}^+$. We consider two cases.

Case 1: $j \neq j_0$. Since $j \neq j_0$ and μ^+ is equal to $\mu^- \cup \{(i_0, j_0)\}$, we have $\mu^-(j) = \mu^+(j) = i'$. Since $(i, j) \in E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$ and $j \neq j_0$, we have $(i, j) \in E(\ell^-)$. Since $(i, j) \in E(\ell^-)$, $\mu^-(i') = j$, and $\mathcal{Q}_6(\ell^-, \mu^-, p^-)$ holds, we have $p_i^- \leq p_{i'}^-$. Since $p_i^- \leq p_{i'}^-$ and $p^+ = p^-$, we have $p_i^+ \leq p_{i'}^+$.

Case 2: $j = j_0$. Since $\mu^-(j_0) = 0$ and $\mathcal{Q}_2(\ell^-, \mu^-)$ holds, we deduce that none of the edges in $E(\ell^-)$ are incident on j_0. Since $j = j_0$ and none of the edges in $E(\ell^-)$ are incident on j_0, we have $(i, j) \notin E(\ell^-)$. Since $(i, j) \notin E(\ell^-)$ and $(i, j) \in E(\ell^+) = E(\ell^-) \cup \{(i_0, j_0)\}$, we have $(i, j) = (i_0, j_0)$. Since $\mu^+ = \mu^- \cup \{(i_0, j_0)\}$, we have $\mu^+(j_0) = i_0$. Since $(i, j) = (i_0, j_0)$ and $\mu^+(j_0) = i_0$, we have $i' = \mu^+(j) = \mu^+(j_0) = i_0 = i$. Since $i = i'$ we have $p_i^+ = p_{i'}^+$.

Proof of Lemma A.2. Immediate from Lemmas A.3, A.5, A.6, A.7, A.8, and A.10.

B Proofs of Lemmas 3.2 and 3.3

The purpose of this section is to prove Lemmas 3.2 and 3.3. For any ℓ, any matching μ, and any men $i, i' \in I$, we define an *oriented μ-alternating path in $G(\ell)$ from i to i'* as a μ-alternating path π in $G(\ell)$ such that no edge in $\pi \cap \mu$ is incident on i.

Lemma B.1. *Let ℓ, μ, and p satisfy $\mathcal{Q}_6(\ell, \mu, p)$, let $i, i' \in I$, and let π be an oriented μ-alternating path in $G(\ell)$ from i to i'. Then $p_i \leq p_{i'}$.*

Proof. If $i = i'$ then $p_i = p_{i'}$, so we can assume that $i \neq i'$. Let $i = i_1, i_2, \ldots, i_k = i'$ denote the sequence of $k > 1$ men appearing on path π. Since $\mathcal{Q}_6(\ell, \mu, p)$ holds and π is an oriented μ-alternating path in $G(\ell)$ from i to i', we deduce that $p_{i_j} \leq p_{i_{j+1}}$ for all j such that $1 \leq j < k$. Hence $p_i = p_{i_1} \leq p_{i_k} = p_{i'}$.

Lemma B.2. *Let ℓ, μ, and p satisfy $\mathcal{Q}_2(\ell, \mu)$, $\mathcal{Q}_3(\ell, p)$, $\mathcal{Q}_4(\ell, \mu, p)$, and $\mathcal{Q}_6(\ell, \mu, p)$. Then μ is an MWMCM of $G(\ell)$.*

Proof. Since $\mathcal{Q}_2(\ell, \mu)$ holds, μ is an MCM of $G(\ell)$. Let μ' be an MWMCM of $G(\ell)$. Since μ' is an MCM of $G(\ell)$, $\mathcal{Q}_2(\ell, \mu)$ implies that μ and μ' match the same set of women. Thus $\mu \oplus \mu'$ corresponds to a collection $\mathcal{X}$ of cycles (of positive even length) and man-to-man paths (of positive even length). For any cycle γ in $\mathcal{X}$, the edges of μ on γ match the same set of men as the edges of μ' on γ. Thus the total weight (in $G(\ell)$) of the edges of μ on γ is equal to the total weight of the edges of μ' on γ.

Now consider a man-to-man path π in $\mathcal{X}$. Let the endpoints of π be i and i', where i is matched in μ and not in μ', and i' is matched in μ' and not in μ. Since μ' is an MWMCM of $G(\ell)$, and since $\mu' \oplus \pi$ is an MCM of $G(\ell)$, we deduce that $w(i, \ell_i) \leq w(i', \ell_{i'})$. Since $\mu(i') = 0$ and $\mathcal{Q}_4(\ell, \mu, p)$ holds, we have $p_{i'} = w(i', \ell_{i'})$. Since $\mathcal{Q}_6(\ell, \mu, p)$ holds and π is an oriented μ-alternating path in $G(\ell)$ from i to i', Lemma B.1 implies that $p_i \geq p_{i'}$. Since $\mathcal{Q}_3(\ell, p)$ holds, we have $p_i \leq w(i, \ell_i)$. Since $p_i \geq p_{i'}$ and $p_i \leq w(i, \ell_i) \leq w(i', \ell_{i'}) = p_{i'}$, we deduce that $p_i = w(i, \ell_i) = w(i', \ell_{i'}) = p_{i'}$. Thus the total weight (in $G(\ell)$) of the edges of μ on π is equal to the total weight of the edges of μ' on π.

The foregoing analysis of the cycles and paths in $\mathcal{X}$ implies that the weight of μ is equal to that of μ', and hence that μ is an MWMCM of $G(\ell)$.

Proof of Lemma 3.2. It is easy to check that μ is an MWMCM of $G(\ell)$ when the Algorithm 1 loop is first encountered. Hence the claim of the lemma follows by Lemmas A.2 and B.2.

Lemma B.3. *Let ℓ, μ, and p satisfy $\mathcal{Q}_2(\ell, \mu)$, $\mathcal{Q}_3(\ell, p)$, $\mathcal{Q}_4(\ell, \mu, p)$, and $\mathcal{Q}_6(\ell, \mu, p)$, and let μ' be an MWMCM of $G(\ell)$. Then $\mathcal{Q}_2(\ell, \mu')$, $\mathcal{Q}_4(\ell, \mu', p)$, and $\mathcal{Q}_6(\ell, \mu', p)$ hold.*

Proof. Lemma B.2 implies that μ is an MWMCM of $G(\ell)$. Let J' denote the set of women with nonzero degree in $G(\ell)$. Since $\mathcal{Q}_2(\ell, \mu)$ holds, the set of women matched by μ is J'. Since μ' is an MCM, we deduce that the set of women matched by μ' is also J', and hence that $\mathcal{Q}_2(\ell, \mu')$ holds. Thus $\mu \oplus \mu'$ corresponds to a collection $\mathcal{X}$ of cycles (of positive even length) and man-to-man paths (of positive even length).

Consider a cycle γ in $\mathcal{X}$. Since $\mathcal{Q}_6(\ell, \mu, p)$ holds and there is an oriented μ-alternating path in $G(\ell)$ from i to i' for every pair of men i and i' on γ, Lemma B.1 implies that $p_i = p_{i'}$ for all men i and i' on γ.

Consider a path π in $\mathcal{X}$. Let the endpoints of π be i and i', where i is matched in μ and not in μ', and i' is matched in μ' and not in μ. Since $\mathcal{Q}_6(\ell, \mu, p)$ holds and π is an oriented μ-alternating path in $G(\ell)$ from i' to i, there are oriented μ-alternating paths in $G(\ell)$ from i' to i'' and from i'' to i for every man i'' on π. Thus Lemma B.1 implies that $p_{i'} \leq p_{i''} \leq p_i$ for every man i'' on π. Since μ and μ' are each MWMCMs, and $\mu \oplus \pi$ and $\mu' \oplus \pi$ are MCMs of $G(\ell)$, we deduce that $w(i, \ell_i) = w(i', \ell_{i'})$. Since $\mathcal{Q}_4(\ell, \mu, p)$ holds, we have $p_{i'} = w(i', \ell_{i'})$. Since $\mathcal{Q}_3(\ell, p)$ holds, we have $p_i \leq w(i, \ell_i)$. Since $p_i \leq w(i, \ell_i) = w(i', \ell_{i'}) = p_{i'} \leq p_i$, we deduce that $p_i = w(i, \ell_i) = p_{i'}$. Since $p_i = w(i, \ell_i)$, we conclude that $\mathcal{Q}_4(\ell, \mu', p)$ holds. Since $p_i = p_{i'}$ and $p_{i'} \leq p_{i''} \leq p_i$ for every man i'' on π, we deduce that $p_i = p_{i''}$ for every man i'' on π.

The foregoing analysis of the cycles and paths in $\mathcal{X}$ implies that $p_{\mu(j)} = p_{\mu'(j)}$ for every woman j in J'. Since $\mathcal{Q}_6(\ell, \mu, p)$ holds, we deduce that $\mathcal{Q}_6(\ell, \mu', p)$ holds.

We now use our results concerning Algorithm 1 to reason about Algorithm 2. To do this, it is convenient to introduce an intermediate algorithm, which we define by modifying Algorithm 1 as follows: At the end of each iteration of the while loop, update the matching μ to to an arbitrary MWMCM of $G(\ell)$. We refer to this intermediate algorithm as Algorithm 3.

Lemma B.4. *Consider an iteration of the Algorithm 3 loop. Let ℓ^-, μ^-, and p^- denote the values of ℓ, μ, and p at the start of the iteration. Assume that the loop condition is satisfied, and that $\mathcal{Q}(\ell^-, \mu^-, p^-)$ holds. Let ℓ^+, μ^+, and p^+ denote the values of ℓ, μ, and p at the end of the iteration. Then $\mathcal{Q}(\ell^+, \mu^+, p^+)$ holds.*

Proof. Lemma A.2 implies that $\mathcal{Q}(\ell, \mu, p)$ holds just before μ is updated to an arbitrary MWMCM of $G(\ell)$. Lemma B.2 implies that μ is an MWMCM of $G(\ell)$ at this point in the execution. Thus Lemma B.3 implies that $\mathcal{Q}_2(\ell^+, \mu^+)$, $\mathcal{Q}_4(\ell^+, \mu^+, p^+)$, and $\mathcal{Q}_6(\ell^+, \mu^+, p^+)$ hold. Since $\mathcal{Q}(\ell, \mu, p)$ holds just before μ is updated to an arbitrary MWMCM of $G(\ell)$, we conclude that $\mathcal{Q}_1(\ell^+)$, $\mathcal{Q}_3(\ell^+, p^+)$ and $\mathcal{Q}_5(\ell^+, p^+)$ hold. Hence $\mathcal{Q}(\ell^+, \mu^+, p^+)$ holds, as required.

The converse of the following lemma also holds, but we only need the stated direction.

Lemma B.5. *Fix an execution of Algorithm 2, and let n denote the number of times the body of the loop is executed. For $0 \leq i \leq n$, let $\ell^{(i)}$ and $\mu^{(i)}$ denote the values of the corresponding program variables after i iterations of the loop. Then there is an n-iteration execution of Algorithm 3 such that, for $0 \leq i \leq n$, the program variables ℓ and μ are equal to $\ell^{(i)}$ and $\mu^{(i)}$, respectively, after i iterations of the loop.*

Proof. Observe that Algorithms 2 and 3 are equivalent in terms of their initialization of ℓ and μ, and also in terms of the set of possible updates to ℓ and μ associated with any given iteration. (While Algorithm 3 also maintains a priority vector p, this priority vector has no influence on the overall update applied to ℓ and μ in a given iteration.) Given this observation, the claim of the lemma is straightforward to prove by induction on i.

Proof of Lemma 3.3. Fix an execution of Algorithm 2, and let ℓ^* and μ^* denote the final values of ℓ and μ. Lemma B.5 implies that there exists an execution of Algorithm 3 with the same final values of ℓ and μ. Fix such an execution of Algorithm 3, and let p^* denote the final value of p. It is straightforward to verify that $\mathcal{Q}(\ell, \mu, p)$ holds the first time the loop is reached in this execution of Algorithm 3. Thus, Lemma B.4 implies that $\mathcal{Q}(\ell^*, \mu^*, p^*)$ holds. Hence the claim of the lemma follows by Lemma A.1.