Object Allocation Over a Network of Objects: Mobile Agents with Strict Preferences*

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Abstract

In recent work, Gourvès, Lesca, and Wilczynski propose a variant of the classic housing markets model where the matching between agents and objects evolves through Pareto-improving swaps between pairs of adjacent agents in a social network. To explore the swap dynamics of their model, they pose several basic questions concerning the set of reachable matchings. In their work and other follow-up works, these questions have been studied for various classes of graphs: stars, paths, generalized stars (i.e., trees where at most one vertex has degree greater than two), trees, and cliques. For generalized stars and trees, it remains open whether a Pareto-efficient reachable matching can be found in polynomial time.

In this paper, we pursue the same set of questions under a natural variant of their model. In our model, the social network is replaced by a network of objects, and a swap is allowed to take place between two agents if it is Pareto-improving and the associated objects are adjacent in the network. In those cases where the question of polynomial-time solvability versus NP-hardness has been resolved for the social network model, we are able to show that the same result holds for the network-of-objects model. In addition, for our model, we present a polynomial-time algorithm for computing a Pareto-efficient reachable matching in generalized star networks. Moreover, the object reachability algorithm that we present for path networks is significantly faster than the known polynomial-time algorithms for the same question in the social network model.

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1 Introduction

Problems related to resource allocation under preferences are widely studied in both computer science and economics. Research in this area seeks to gain mathematical insight into the structure of resource allocation problems, and to exploit this structure to design fast algorithms. In one important class of resource allocation problems, sometimes referred to as one-sided matching problems [13], we seek to allocate indivisible objects to a set of agents, where each agent has preferences over the objects and wants to receive at most one object (unit demand). The allocation should enjoy one or more strong game-theoretic properties, such as Pareto-efficiency.

In a seminal work, Shapley and Scarf [18] introduced the notion of a housing market, which corresponds to the special case of one-sided matching in which there are an equal number of agents and objects, each agent is initially endowed with a distinct object, and each agent is required to be matched to exactly one object. They present an elegant algorithm (attributed to David Gale) for housing markets called the top trading cycles (TTC) algorithm. The TTC algorithm enjoys a number of strong game-theoretic properties. For example, when agents have strict preferences, the output of the TTC algorithm is the unique matching in the core. The TTC algorithm has subsequently been generalized to handle more complex variants of the original housing market problem (e.g., [3, 8, 9, 15, 17]).

Like many one-sided matching algorithms, the TTC algorithm is centralized: it takes all of the agent preference information as input and computes the output matching. In some resource allocation scenarios of practical interest, it may be difficult or impossible to coordinate such a global recomputation of the matching. Accordingly, researchers have studied decentralized (or distributed) variants of one-sided matching problems in which the initial allocation gradually evolves as "local" trading opportunities arise. In this setting, restrictions are imposed on the sets of agents that are allowed to participate in a single trade. For example, we might only allow (certain) pairs of agents to trade. In addition, all trades are required to be Pareto-improving. Locality-based restrictions on trade are generally enforced through graph-theoretic constraints.

Of particular relevance to the present paper is the line of research initiated by Gourvès et al. [10] on decentralized allocation in housing markets. They propose a model in which agents have strict preferences and are embedded in an underlying social network. A pair of agents are allowed to swap objects with each other only if (1) they will be better off after the swap, and (2) they are directly connected (socially tied) via the network. The underlying social network is modeled as an undirected graph, and five different graph classes are considered: paths, stars, generalized stars, trees, and general graphs. The swap dynamics of the model are investigated by considering three computational questions. The first question, Reachable Object, asks whether there is a sequence of swaps that results in a given agent being matched to a given target object. The second question, Reachable Matching, asks whether there is a sequence of swaps that results in a given target matching. The third question, Pareto Efficiency, asks how to find a sequence of swaps that results in a Pareto-efficient matching with respect to the set of reachable matchings.

Gourvès et al. [10] studied each of the three questions in the context of the aforementioned graph classes, with the goal of either exhibiting a polynomial-time algorithm or establishing NP-hardness. For some of these problems, it is a relatively straightforward exercise to design a polynomial-time algorithm (even for the search version). In particular, this is the case for all three reachability questions on stars, for Pareto Efficiency on paths, and for Reachable Matching on trees (which subsumes Reachable Matching on generalized stars, and hence also on paths). Gourvès et al. present an elegant reduction from 2P1N-SAT [19] to establish the NP-completeness of Reachable Object on generalized stars (and hence also on trees and general graphs). They establish the NP-completeness of Reachable Matching on general graphs via a reduction from Reachable Object on trees. The latter reduction has the property that for any given instance of Reachable Object on trees, the target matching in the instance of Reachable Matching on general graphs produced by the transformation matches each agent to its most preferred object. Consequently, the same reduction establishes the NP-hardness of Pareto Efficiency on general graphs. The work of Gourvès et al. left three of these problems open: Reachable Object on paths and Pareto Efficiency on generalized stars and trees. Subsequently, two sets of authors independently presented polynomial-time algorithms for Reachable Object on paths [4, 11]. Both groups obtained an $O(n^4)$-time algorithm by carefully studying the structure of swap dynamics on paths and then reducing the problem to 2-SAT. The complexity of Pareto Efficiency remains open for generalized stars and for trees. Gourvès et al. noted, "It appears interesting to see if Pareto (Efficiency) is polynomial time solvable in a generalized star by a combination of the techniques used to solve the cases of paths and stars."

Bentert et al. [4] established that Reachable Object on cliques is NP-complete, and Müller and Bentert [14] established that Reachable Matching on cliques is NP-complete. It is easy to extend the latter result to show that Pareto Efficiency on cliques is NP-hard. These three hardness results for cliques subsume the corresponding results obtained previously for general graphs by Gourvès et al.

We study a natural variant of the decentralized housing markets model of Gourvès et al. [10]. Instead of enforcing locality constraints on trade via a network where the locations of the agents are fixed (since they correspond to the vertices of the network) and the objects move around (due to swaps), we consider a network where the locations of the objects are fixed and the agents move around. We refer to these two models as the object-moving model and the agent-moving model. Table 1 summarizes the current state of the art for the object-moving model.

To motivate the study of the agent-moving model, consider a cloud computing environment with a large number of servers (objects) connected by a network that are available to rent. A set of customers (agents) are each interested in renting one server. The servers vary in CPU capacity, storage capacity, physical security, and rental cost. Varying customer workloads and requirements result in varying customer preferences over the servers. Rather than attempting to globally optimize the entire matching of customers to servers, it might be preferable to allow local swaps between adjacent servers to gradually optimize the matching. Given that customer workloads are likely to vary significantly over time, an optimization strategy based on frequent local updates might outperform a strategy based on less frequent global updates. Alternatively, one can envision a system that performs occasional global updates to optimize the matching, and that relies on local updates to maintain a reasonable matching between successive global updates.

Our Results. We initiate the study of the agent-moving model by revisiting each of the questions associated with Table 1 in the context of the agent-moving model. We

	Reachable Object	Reachable Matching	Pareto Efficiency
Star	poly-time	(poly-time)	poly-time
Path	poly-time	(poly-time)	poly-time
Generalized Star	NP-complete	(poly-time)	open
Tree	(NP-complete)	poly-time	open
Clique	NP-complete	NP-complete	NP-hard

Table 1: This table presents known complexity results for various questions related to the object-moving model of Gourvès et al. [10]. The results in parentheses follow directly from other table entries. For the agent-moving model, we obtain the same results, except that we also give a polynomial-time algorithm for Pareto Efficiency on generalized stars.

emphasize that the sole difference between the agent-moving model and the object-moving model is that the locality constraint prevents an agent a currently matched to an object b from trading with an agent a' currently matched to an object b' unless objects b and b' (two vertices in a given network of objects) are adjacent, rather than requiring agents a and a' (two vertices in a given network of agents) to be adjacent. Both models also require swaps to be Pareto-improving. The two models have strong similarities. In fact, for all of the questions in Table 1 for which a polynomial-time algorithm or hardness result has been established in the object-moving model, we establish a corresponding result in the agent-moving model. Moreover, for Pareto Efficiency on generalized stars, which is open in the object-moving model, we provide a polynomial-time algorithm in the agent-moving model. In some cases, it is relatively straightforward to adapt known results for the object-moving model to the agent-moving model. Below we highlight our four main technical contributions, which address more challenging cases.

Our first main technical result is an $O(n^2)$ time algorithm for Reachable Object on paths in the agent-moving model, which is much faster than the known $O(n^4)$-time algorithms for Reachable Object on paths in the object-moving model. (Here n denotes the number of agents/objects; the size of the input is quadratic in n since the preference list of each agent is of length n.) The speedup is due to a simpler local characterization of the reachable matchings on a path in the agent-moving model.

In our second main technical result, we obtain the same $O(n^2)$ time bound for Pareto Efficiency on paths. Our algorithms for Reachable Object and Pareto Efficiency are based on an efficient subroutine for solving a certain constrained reachability problem. Roughly speaking, this subroutine determines all of the possible matches for a given agent when certain agent-object pairs are required to be matched to one another. Our implementation involves a trivial $O(n^2)$-time preprocessing phase followed by an $O(n)$-time greedy phase. The preferences of the agents are only examined during the preprocessing phase. The proof of correctness of the greedy phase is somewhat nontrivial. We solve Reachable Object on paths using a single application of the subroutine, yielding an $O(n^2)$ bound. Our polynomial-time algorithm for Pareto Efficiency on paths uses n applications of our algorithm for Reachable Object on paths. Since the preprocessing phase only needs to be performed once, the overall running time remains $O(n^2)$.

In our third main technical result, we present a polynomial-time algorithm for Pareto Efficiency on generalized stars, which remains open in the object-moving model. To tackle this problem, we use the serial dictatorship algorithm with the novel idea of dynamically choosing the dictator sequence. We also leverage our techniques for solving Pareto Efficiency on paths.

The faster time bounds discussed above for the case of paths suggest that the agent-moving model is simpler than the object-moving model, at least from an upper bound perspective. Accordingly, we can expect it to be a bit more challenging to establish the NP-completeness results stated in Table 1 for the agent-moving model than for the object-moving model. In our fourth main technical result, we adapt an NP-completeness proof developed by Bentert et al. [4] in the context of the object-moving model to the more challenging setting of the agent-moving model. Specifically, we modify their reduction from 2P1N-SAT to establish that Reachable Object on cliques remains NP-complete in the agent-moving model.

Related work. For the object-moving model, Huang and Xiao [11] study Reachable Object with weak preferences, i.e., where an agent can be indifferent between different objects. Bentert et al. [4] establish NP-hardness for Reachable Object on cliques, and consider the case where the preference lists have bounded length. Saffidine and Wilczynski [16] propose a variant of Reachable Object where we ask whether a given agent is guaranteed to achieve a specified level of satisfaction after any maximal sequence of rational exchanges. Müller and Bentert [14] study Reachable Matching on cliques and cycles. Aspects related to social connectivity are also addressed in recent work on envy-free allocations [5, 7] and on trade-offs between efficiency and fairness [12].

Our agent-moving model can be viewed as a game in which each agent seeks to be matched to an object that is as high as possible on its preference list. If the game reaches a state in which no further swaps can be performed, we say that an equilibrium matching has been reached. Agarwal et al. [2] study a similar game motivated by Schelling's well-known residential segregation model. As in our game, there are an equal number of agents and objects, the objects correspond to the nodes of a graph, a matching is maintained between the agents and the objects, and the matching evolves via Pareto-improving, agent-moving swaps. There are also some significant differences. In our model, each agent has static preferences over the set of objects, and swaps can only occur between adjacent agents (i.e, agents matched to adjacent objects). In the Agarwal et al. game, each agent has a type, the desirability of an object b to an agent a depends on the current fraction of agents in the "neighborhood" of b (i.e., the set of agents matched to an object adjacent to b) with the same type as a, and swaps can occur between any pair of agents. Agarwal et al. study the existence, computational complexity, and quality of equilibrium matchings in such games. Bilò et. al [6] further investigated the influence of the graph structure on the resulting strategic multi-agent system.

Organization of the paper. The remainder of the paper is organized as follows. Section 2 provides formal definitions. Section 3 presents our polynomial-time algorithms for Reachable Object and Pareto Efficiency on paths. Section 4 presents our polynomial-time algorithm for Pareto Efficiency on generalized stars. Section 5 presents our NP-completeness result for Reachable Object on cliques. Section 6 presents our other NP-completeness and NP-hardness results. Section 7 briefly discusses simple algorithms for justifying the other polynomial-time entries in Table 1. Section 8 offers concluding remarks.

2 Preliminaries

We define an object allocation framework (OAF) as a 4-tuple $F = (A, B, \succ, E)$ where A is a set of agents, B is a set of objects such that $|A| = |B|$, $\succ$ is a collection of strict linear orderings $\{\succ_a\}_{a \in A}$ over B such that $\succ_a$ specifies the preferences of agent a over B, and E is the edge set of some undirected graph (B, E).

We define a matching μ of given OAF $F = (A, B, \succ, E)$ as a subset of $A \times B$ such that no agent or object belongs to more than one pair in μ. (Put differently, μ is a matching in the complete bipartite graph of agents and objects.) We say that such a matching is perfect if $|\mu| = |A|$. For any matching μ, we define agents(μ) (resp., objects(μ)) as the set of all matched agents (resp., objects) with respect to μ. For any matching μ and any agent a that is matched in μ, we use the shorthand notation $\mu(a)$ to refer to the object matched to agent a. For any matching μ and any object b that is matched in μ, we use the notation $\mu^{-1}(b)$ to refer to the agent matched to object b.

For any OAF $F = (A, B, \succ, E)$, any perfect matching μ of F, and any edge $e = (b, b')$ in E such that $b' \succ_a b$ and $b \succ_{a'} b'$ where $a = \mu^{-1}(b)$ and $a' = \mu^{-1}(b')$, we say that a swap operation is applicable to μ across edge e, and we write $\mu \rightarrow_{F,e} \mu'$ where

$$\mu' = (\mu \setminus \{(a, b), (a', b')\}) \cup \{(a, b'), (a', b)\},$$

is the matching of F that results from applying this operation. We write $\mu \rightarrow_F \mu'$ to denote that $\mu \rightarrow_{F,e} \mu'$ for some edge e. We write $\mu \rightsquigarrow_F \mu'$ if there exists a sequence $\mu = \mu_0, \ldots, \mu_k = \mu'$ of matchings of F such that $\mu_{i-1} \rightarrow_F \mu_i$ for $1 \leq i \leq k$.

We define a configuration as a pair $\chi = (F, \mu)$ where F is an OAF and μ is a perfect matching of F.

For any configuration $\chi = (F, \mu)$ where $F = (A, B, \succ, E)$, any agent a in A, and any object b in B, we define $\chi(a)$ as a shorthand for the object $\mu(a)$, and we define $\chi^{-1}(b)$ as a shorthand for the agent $\mu^{-1}(b)$.

For any configuration $\chi = (F, \mu)$ where $F = (A, B, \succ, E)$, and any matching μ' of F such that $\mu \rightarrow_{F,e} \mu'$ for some edge e in E, we say that a swap is applicable to χ across edge e, and the result of applying this operation is the configuration (F, μ').

For any configuration $\chi = (F, \mu)$, we define reach(χ) as the set of all perfect matchings μ' of F such that $\mu \rightsquigarrow_F \mu'$. For any configuration $\chi = (F, \mu)$ and any matching μ' of F, we define reach(χ, μ') as the set of all matchings μ'' in reach(χ) such that μ'' contains μ'.

We now state the three reachability problems studied in this paper.

- The reachable matching problem: Given a configuration $\chi = (F, \mu)$ and a perfect matching μ' of F, determine whether μ' belongs to reach(χ).

- The reachable object problem: Given a configuration $\chi = (F, \mu)$ where $F = (A, B, \succ, E)$, an agent a in A, and an object b in B, determine whether there is a matching μ' in reach(χ) such that $\mu'(a) = b$.

- The Pareto-efficient matching problem: Given a configuration χ, find a matching in reach(χ) that is not Pareto-dominated by any other matching in reach(χ).

3 Reachability Over a Path Network

We begin by introducing some notation.

For any nonnegative integer n, we define $[n]$ as $\{1, \ldots, n\}$. Without loss of generality, in this section we restrict attention to OAFs of the form $([n], [n], \succ, \{(b, b+1) \mid 1 \leq b < n\})$ for some positive integer n. We use the notation $(n, \succ)$ to refer to such an OAF.

For any nonnegative integer n, we define $\Phi(n)$ as the set of all matchings μ such that $\text{agents}(\mu) = [|\mu|]$ and $\text{objects}(\mu) \subseteq [n]$.

For any matching μ in $\Phi(n)$, we define $\max(\mu)$ as the maximum matched object in $\text{objects}(\mu)$, or as 0 if $\mu = \emptyset$.

For any matching μ in $\Phi(n)$, we define $\text{hole}(\mu)$ as the minimum positive integer that does not belong to $\text{objects}(\mu)$.

For any matching μ in $\Phi(n)$ and any agent a in $\text{agents}(\mu)$, define $\text{span}(\mu, a)$ as $\{b \in [n] \mid \mu(a) \leq b \leq a\} \cup \{b \in [n] \mid a \leq b \leq \mu(a)\}$.

For any OAF $F = (n, \succ)$, we define μ_F as the matching $\{(i, i) \mid i \in [n]\}$, and we define χ_F as the configuration (F, μ_F).

For any OAF $F = (n, \succ)$ and any agent a in $[n]$, we define $\text{left}(\succ, a)$ as the minimum object b in $[n]$ such that $b \succ_a b+1 \succ_a \cdots \succ_a \mu_F(a) = a$, and we define $\text{right}(\succ, a)$ as the maximum object b in $[n]$ such that $b \succ_a b-1 \succ_a \cdots \succ_a a$. Thus if a matching μ belongs to $\text{reach}(\chi_F)$, then the match $\mu(a)$ of agent a is at least $\text{left}(\succ, a)$ and at most $\text{right}(\succ, a)$, regardless of the preferences of the remaining agents.

For any OAF $F = (n, \succ)$, any matching μ in $\Phi(n)$, and any agent a in $\text{agents}(\mu)$, we say that the predicate $\text{IR}(\succ, \mu, a)$ holds (where "IR" stands for "individually rational") if $\text{left}(\succ, a) \leq \mu(a) \leq \text{right}(\succ, a)$. We say that the predicate $\text{IR}(\succ, \mu)$ holds if $\text{IR}(\succ, \mu, a)$ holds for all agents a in $\text{agents}(\mu)$.

3.1 A Useful Subroutine

This section presents Algorithm 1, a greedy subroutine that we use in Sections 3.3 and 3.4 to solve reachability problems over a path network.

Recall that the reachable object problem with path configuration χ_F is to check whether an object b is reachable for an agent a. Algorithm 1 addresses a variant of this problem in which the agents less than a are all required to be matched to specific objects. The input matching μ_0 specifies the required match for each of these agents.

3.2 Proof of Correctness of Algorithm 1

In this section, we establish the correctness of Algorithm 1.

We begin by defining a specific subset $\Phi^*(n)$ of $\Phi(n)$. For any matching μ in $\Phi(n)$ and any integer i in $[|\mu|]$, let μ_i be the matching such that $\mu_i \subseteq \mu$ and $\text{agents}(\mu) = [i]$. Then $\Phi^*(n)$ is the set of all matchings μ such that μ belongs to $\Phi(n)$ and for each i in $[|\mu| - 1]$, either $\mu(i+1) = \text{hole}(\mu_i)$ or $\max(\mu_i) < \mu(i+1) \leq n$.

We now present a number of useful structural properties of matchings in $\Phi^*(n)$.

Algorithm 1: A greedy path reachability subroutine.

Input: An OAF $F = (n, \succ)$, a matching μ_0 in $\Phi(n)$ such that $|\mu_0| < n$ and reach$(\chi_F, \mu_0) \neq \emptyset$, and a matching $\mu_1 = \mu_0 + (|\mu_1|, b_0)$ where $\max(\mu_0) < b_0 \leq \text{right}(\succ, |\mu_1|)$

Output: A matching μ in reach(χ_F, μ_1), or $\emptyset$ if this set is empty

```
μ = μ_1;
while 0 < |μ| < n do
    if left(≻, |μ| + 1) ≤ hole(μ) then
        μ = μ + (|μ| + 1, hole(μ));
    else if max(μ) < right(≻, |μ| + 1) then
        μ = μ + (|μ| + 1, max(μ) + 1);
    else
        μ = ∅;
    end
end
return μ
```

3.2.1 Representing a Matching as a Pair of Binary Strings

For any binary string α, we let $|\alpha|$ denote the length of α, and we let $w(\alpha)$ denote the Hamming weight of α. For any binary string α and any integer i in $[|\alpha|]$, we let α_i denote bit i of α. For any binary string α, and any integers i and j in $[|\alpha|]$, we let $\alpha_{i,j}$ denote the substring $\alpha_i \cdots \alpha_j$ of α.

For any integers m and n such that $0 \leq m \leq n$, we let $\Psi(m, n)$ denote the set of all pairs of binary strings (α, β) such that $|\alpha| = m$, $|\beta| = n$, $w(\alpha_{1,i}) \geq w(\beta_{1,i})$ holds for all i in $[m]$, $w(\alpha) = w(\beta)$, and $m < n$ implies $\beta_n = 1$.

For any (α, β) in $\Psi(m, n)$, we define $[\alpha, \beta]$ as the cardinality-m matching μ in $\Phi(n)$ constructed as follows: for any agent a in $[m]$ such that α_a is the ith 0 (resp., 1) in α, we define $\mu(a)$ as the index of the ith 0 (resp., 1) in β.

Observation 3.1. *Let (α, β) belong to $\Psi(m, n)$, let μ denote $[\alpha, \beta]$, and let b belong to $[n]$. If b is unmatched in μ, then $\beta_b = 0$. Otherwise, the following conditions hold, where a denotes $\mu^{-1}(b)$: $a > b$ implies $\alpha_a = \beta_b = 0$, $a < b$ implies $\alpha_a = \beta_b = 1$, and $a = b$ implies $\alpha_a = \beta_a$.*

Observation 3.2. *Let (α, β) belong to $\Psi(m, n)$ and let μ denote $[\alpha, \beta]$. If $m < n$ then $(\alpha 0, \beta)$ belongs to $\Psi(m+1, n)$ and $[\alpha 0, \beta] = \mu + (m+1, \text{hole}(\mu))$. Furthermore, for any nonnegative integer k, $(\alpha 1, \beta 0^k 1)$ belongs to $\Psi(m+1, n+k+1)$ and $[\alpha 1, \beta 0^k 1] = \mu + (m+1, n+k+1)$.*

For any (α, β) in $\Psi(m, n)$, and any agent a in $[m]$ such that $\alpha_a = \beta_a$ and $w(\alpha_{1,a}) = w(\beta_{1,a})$, we say that a complement operation is applicable to (α, β) at agent a. The result of applying this operation is the pair of binary strings (α', β') that is the same as (α, β) except $\alpha'_a = \beta'_a = 1 - \alpha_a$. It is easy to see that (α', β') belongs to $\Psi(m, n)$.

For any (α, β) and (α', β') in $\Psi(m, n)$, we write $(\alpha, \beta) \simeq (\alpha', \beta')$ to denote that (α, β) can be transformed into (α', β') via a sequence of complement operations.

Observation 3.3. *Let (α, β) and (α', β') belong to $\Psi(m, n)$. Then $[\alpha, \beta] = [\alpha', \beta']$ if and only if $(\alpha, \beta) \simeq (\alpha', \beta')$.*

Observation 3.4. *Let (α, β) belong to $\Psi(m, n)$, and let (α', β') belong to $\Psi(m', n')$ where $m \leq m'$ and $n \leq n'$. Then $[\alpha, \beta] \subseteq [\alpha', \beta']$ if and only if $(\alpha, \beta) \simeq (\alpha'_{1,|\alpha|}, \beta'_{1,|\beta|})$.*

For any (α, β) in $\Psi(m, n)$, and any object b in $[n-1]$ such that $\beta_b = 1$ and $\beta_{b+1} = 0$, we say that a sort operation is applicable to (α, β) across objects b and $b+1$. The result of applying this operation is the pair of binary strings (α, β') that is the same as (α, β) except $\beta'_b = 0$ and $\beta'_{b+1} = 1$.

Observation 3.5. *Let (α, β) belong to $\Psi(m, n)$ and let (α, β') be the result of applying a sort operation to (α, β) across objects b and $b+1$. Then (α, β') belongs to $\Psi(m, n)$. Furthermore, if $m = n$ then*

$$[\alpha, \beta'] = (\mu \setminus \{(a, b), (a', b+1)\}) \cup \{(a', b), (a, b+1)\}$$

where μ denotes $[\alpha, \beta]$, a denotes $\mu^{-1}(b)$, and a' denotes $\mu^{-1}(b+1)$.

For any (α, β) and (α', β') in $\Psi(m, n)$, we write $(\alpha, \beta) \rightsquigarrow (\alpha', \beta')$ to denote that (α, β) can be transformed into (α', β') via a sequence of complement and sort operations.

Observation 3.6. *Let α be a binary string of length m, and let β be a binary string of length n such that $m \leq n$. Then (α, β) belongs to $\Psi(m, n)$ if and only if $(0^m, 0^n) \rightsquigarrow (\alpha, \beta)$.*

For any nonnegative integer n, we define $\Phi'(n)$ as the set of all matchings μ in $\Phi(n)$ such that $\mu = [\alpha, \beta]$ for some (α, β) in $\Psi(|\mu|, \max(\mu))$.

For any (α, β) in $\Psi(n, n)$, and any agent a in $[n-1]$ such that $w(\alpha_{1,a}) > w(\beta_{1,a})$ and $\alpha_a = 1$, we say that a pivot operation is applicable to (α, β) at agent a. The result of applying this operation is the pair of binary strings (α', β) that is the same as (α, β) except $\alpha'_a = 0$ and $\alpha'_{a'} = 1$, where a' denotes the minimum agent greater than a for which $w(\alpha_{1,a'}) = w(\beta_{1,a'})$. (The agent a' is well-defined since $w(\alpha) = w(\beta)$.)

Observation 3.7. *Let (α, β) belong to $\Psi(n, n)$, and let (α', β) be the result of applying a pivot operation to (α, β) at agent a. Then (α', β) belongs to $\Psi(n, n)$ and* $\mathrm{span}([\alpha', \beta], a')$ *is contained in* $\mathrm{span}([\alpha, \beta], a')$ *for all agents a' in $[n] - a$.*

For any (α, β) in $\Psi(n, n)$, and any object b in $[n-1]$ such that $w(\alpha_{1,b}) > w(\beta_{1,b})$, $\beta_b = 0$, and $\beta_{b+1} = 1$, we say that an unsort operation is applicable to (α, β) across objects b and $b+1$. The result of applying this operation is the pair of binary strings (α, β') that is the same as (α, β) except $\beta'_b = 1$ and $\beta'_{b+1} = 0$.

Observation 3.8. *Let (α, β) belong to $\Psi(n, n)$, and let (α, β') be the result of applying an unsort operation to (α, β) across objects b and $b+1$. Then (α, β') belongs to $\Psi(n, n)$ and* $\mathrm{span}([\alpha, \beta'], a)$ *is contained in* $\mathrm{span}([\alpha, \beta], a)$ *for all agents a in $[n]$.*

3.2.2 Structural Properties of Matchings in $\Phi^*(n)$

Claim 3.9 below gives a a simple characterization of matchings in $\Phi'(n)$, and hence implies that $\Phi'(n) = \Phi^*(n)$.

Claim 3.9. *Let μ be a matching in $\Phi'(n)$ such that $|\mu| < n$, let a denote $|\mu|$, let a' denote $a+1$, let b denote $\max(\mu)$, and let μ' denote $\mu + (a', b^*)$. Then μ' belongs to $\Phi'(n)$ if and only if $b^* = \text{hole}(\mu)$ or $b < b^* \leq n$.*

Proof. Since μ belongs to $\Phi'(n)$, there exists (α, β) in $\Psi(a, b)$ such that $\mu = [\alpha, \beta]$. Let b' denote $\max(\mu')$.

For the "if" direction, we need to prove that there exists (α', β') in $\Psi(a', b')$ such that $\mu' = [\alpha', \beta']$. We consider two cases.

Case 1: $b^* = \text{hole}(\mu) \leq b$. Observation 3.2 implies that $(\alpha 0, \beta)$ belongs to $\Psi(a', b')$ and $\mu' = [\alpha 0, \beta]$.

Case 2: $b < b^* \leq n$. Let k denote $b^* - b - 1$. Observation 3.2 implies that $(\alpha 1, \beta 0^k 1)$ belongs to $\Psi(a', b')$ and $\mu' = [\alpha 1, \beta 0^k 1]$.

We now address the "only if" direction. Assume that μ' belongs to $\Phi'(n)$. Since $\mu' = \mu + (a', b^*)$, we deduce that b^* belongs to $[n] \setminus \text{objects}(\mu)$. It remains to prove that $b^* = \text{hole}(\mu)$ or $b < b^*$. Let B denote the set of objects $[b] \setminus \text{objects}(\mu)$. We consider two cases.

Case 1: $a = b$. Thus $B = \emptyset$ and since b^* is unmatched in μ, we have $b < b^*$.

Case 2: $a < b$. Since μ' belongs to $\Phi'(n)$, there exists (α', β') in $\Psi(a', b')$ such that $\mu' = [\alpha', \beta']$. Observation 3.4 implies that $(\alpha'_{1,a}, \beta'_{1,b})$ belongs to $\Psi(a, b)$ and $(\alpha'_{1,a}, \beta'_{1,b}) \simeq (\alpha, \beta)$. Since
$(\alpha'_{1,a}, \beta'_{1,b}) \simeq (\alpha, \beta)$, Observation 3.3 implies that $[\alpha'_{1,a}, \beta'_{1,b}] = [\alpha, \beta] = \mu$. Let B_0 denote the set of all objects i in B such that $\beta'_i = 0$. Observation 3.1 implies that $B_0 = B$. We consider two cases.

Case 2.1: $\alpha'_{a'} = 1$. Since $\mu'(a') = b^*$, Observation 3.1 implies that $\beta'_{b^*} = 1$. Thus b^* does not belong to $B = B_0$. Since b^* is unmatched in μ, we conclude that $b < b^*$.

Case 2.2: $\alpha'_{a'} = 0$. Since $a < b$ and $(\alpha'_{1,a}, \beta'_{1,b})$ belongs to $\Psi(a, b)$, we have $\beta'_b = 1$ and $w(\alpha'_{1,a}) = w(\beta'_{1,b}) > w(\beta'_{1,a})$. Let k denote the number of 0's in $\alpha'_{1,a}$, and let ℓ denote the number of 0's in $\beta'_{1,a}$. Since $w(\alpha'_{1,a}) > w(\beta'_{1,a})$, we have $\ell > k$. Let B' denote the indices of the first k 0's in $\beta'_{1,a}$, and let B'' denote the indices of the remaining $\ell - k$ 0's in $\beta'_{1,a}$. Since $\mu = [\alpha'_{1,a}, \beta'_{1,b}]$, we deduce that the objects in B' are all matched in μ and the objects in B'' are all unmatched in μ. Thus $B \cap [a] = B_0 \cap [a] = B'' \neq \emptyset$, It follows that hole$(\mu)$ is the minimum object in B''. Since $\mu' = [\alpha', \beta']$, $\alpha'_{a'} = 0$, and there are $k+1$ 0's in α', we deduce that b^* is the minimum object in B''. Thus $b^* = \text{hole}(\mu)$. □

Lemma 3.10 below establishes a one-to-one correspondence between matchings that are reachable from χ_F and matchings in $\Phi^*(n)$.

Lemma 3.10. *Let $F = (n, \succ)$ be an OAF. Then μ belongs to* $\text{reach}(\chi_F)$ *if and only if μ belongs to $\Phi^*(n)$, $|\mu| = n$, and* $\text{IR}(\succ, \mu)$ *holds.*

Proof. First we prove the "only if" direction. Suppose that μ belongs to reach(χ_F). Then there is a sequence $\mu_F = \mu_0, \ldots, \mu_k = \mu$ of perfect matchings of F such that $\mu_{i-1} \rightarrow_F \mu_i$ for all i in $[k]$.

For any i such that $0 \leq i \leq k$, let $P(i)$ denote the predicate asserting that the following conditions hold: μ_i belongs to $\Phi^*(n)$; $|\mu_i| = n$; $\mathrm{IR}(\succ, \mu_i)$ holds. We prove by induction on i that $P(i)$ holds for all i in $\{0, \ldots, k\}$. Using the definition of μ_F, it is easy to see that $P(0)$ holds. Now consider the induction step. Fix i in $[k]$ and assume that $P(i-1)$ holds. We need to prove that $P(i)$ holds. Since $P(i-1)$ holds, we know that μ_{i-1} belongs to $\Phi^*(n)$, $|\mu_{i-1}| = n$, and $\mathrm{IR}(\succ, \mu_{i-1})$. Since μ_{i-1} belongs to $\Phi^*(n)$ and $|\mu_{i-1}| = n$, there exists (α, β) in $\Psi(n, n)$ such that $\mu_{i-1} = [\alpha, \beta]$. Let b denote the object in $[n-1]$ such that $\mu_{i-1} \to_{F,(b,b+1)} \mu_i$.

Let (α', β') and (α'', β'') be defined as follows. First, if a complement operation is applicable to (α, β) at agent b and $\beta_b = 0$, then (α', β') is the result of applying this operation to (α, β), and otherwise (α', β') is equal to (α, β). Second, if a complement operation is applicable to (α, β) at agent $b+1$ and $\beta_{b+1} = 1$, then (α'', β'') is the result of applying this operation to (α', β'), and otherwise (α'', β'') is equal to (α', β'). Observation 3.3 implies that (α'', β'') belongs to $\Psi(n, n)$ and $\mu_{i-1} = [\alpha'', \beta'']$.

Using Observation 3.1, it is straightforward to prove that $\beta''_b = 1$ and $\beta''_{b+1} = 0$. Thus a sort operation is applicable to (α'', β'') across objects b and $b+1$; let (α'', β''') denote the result of applying this sort operation. Observation 3.5 implies that (α'', β''') belongs to $\Psi(n, n)$ and $\mu_i = [\alpha'', \beta''']$. Thus μ_i belongs to $\Phi^*(n)$ and $|\mu_i| = n$. Since $\mathrm{IR}(\succ, \mu_{i-1})$ holds and $\mu_{i-1} \to_F \mu_i$, we deduce that $\mathrm{IR}(\succ, \mu_i)$ holds. We conclude that $P(i)$ holds, completing the proof by induction.

We now prove the "if" direction. Assume that μ belongs to $\Phi^*(n)$, $|\mu| = n$, and $\mathrm{IR}(\succ, \mu)$ holds. Observation 3.6 implies there exists (α, β) in $\Psi(n, n)$ such that $(0^n, 0^n) \rightsquigarrow (\alpha, \beta)$. It follows that there is a sequence $(0^n, 0^n) = (\alpha^{(0)}, \beta^{(0)}), \ldots, (\alpha^{(k)}, \beta^{(k)}) = (\alpha, \beta)$ of pairs in $\Psi(n, n)$ such that an applicable complement or sort operation transforms $(\alpha^{(i-1)}, \beta^{(i-1)})$ into $(\alpha^{(i)}, \beta^{(i)})$ for all i in $[k]$. Let μ_i denote $[\alpha^{(i)}, \beta^{(i)}]$ for all i such that $0 \leq i \leq k$. Thus $\mu_0 = \mu_F$ and $\mu_k = \mu$.

Observations 3.3 and 3.5 imply that for all i in $[k]$, either $\mu_i = \mu_{i-1}$ or μ_i is obtained from μ_{i-1} via an exchange across two adjacent objects. It remains to prove that any such exchanges are swaps, i.e., do not violate individual rationality. Below we accomplish this by proving that $\mathrm{IR}(\succ, \mu_i)$ holds for $0 \leq i \leq k$.

For any i in $[k]$, let $P(i)$ denote the predicate $\mathrm{span}(\mu_{i-1}, a) \subseteq \mathrm{span}(\mu_i, a)$ for all agents a in $[n]$. We claim that $P(i)$ holds for all i in $[k]$. To prove the claim, fix an integer i in $[k]$. We consider two cases.

Case 1: A complement operation transforms $(\alpha^{(i-1)}, \beta^{(i-1)})$ into $(\alpha^{(i)}, \beta^{(i)})$. In this case, Observation 3.3 implies that $\mu_i = \mu_{i-1}$. Thus $\mathrm{span}(\mu_{i-1}, a) = \mathrm{span}(\mu_i, a)$ for all agents a in $[n]$.

Case 2: A sort operation transforms $(\alpha^{(i-1)}, \beta^{(i-1)})$ into $(\alpha^{(i)}, \beta^{(i)})$. Assume that the sort operation is applied to $(\alpha^{(i-1)}, \beta^{(i-1)})$ across objects b and $b+1$. Then $\beta_b^{(i-1)} = 1$ and $\beta_{b+1}^{(i)} = 0$, and Observation 3.1 implies that $\mu_{i-1}^{-1}(b) \leq b$ and $\mu_{i-1}^{-1}(b+1) \geq b+1$. Thus Observation 3.5 implies $\mathrm{span}(\mu_{i-1}, a) \subseteq \mathrm{span}(\mu_i, a)$ for all agents a in $[n]$.

Since $P(i)$ holds for all i in $[k]$ and $\mathrm{IR}(\succ, \mu_k)$ holds, we deduce that $\mathrm{IR}(\succ, \mu_i)$ holds for all i such that $0 \leq i \leq k$, as required. □

The next three lemmas are concerned with enlarging a given matching μ in $\Phi^*(n)$ such that $|\mu| < n$ by introducing a suitable match for agent $|\mu| + 1$. Lemma 3.11 (resp., Lemma 3.12) addresses the case where agent $|\mu| + 1$ is matched to an object that is at most (resp., at least) $|\mu|+1$. By combining these two lemmas, we obtain Lemma 3.13, which shows that it is sufficient to consider matching agent $|\mu|+1$ with an object in $\{\text{hole}(\mu), \max(\mu)+1\}$.

Lemma 3.11. *Let $F = (n, \succ)$ be an OAF, let μ be a matching in $\Phi^*(n)$ such that $|\mu| < n$, let a denote $|\mu|$, let a' denote $a+1$, let b denote $\max(\mu)$, and let μ' denote $\mu + (a', \text{hole}(\mu))$. Assume that $a < b$, $\text{reach}(\chi_F, \mu) \neq \emptyset$, and $\text{left}(\succ, a') \leq \text{hole}(\mu)$. Then $\text{reach}(\chi_F, \mu') \neq \emptyset$.*

Proof. Let μ^* be a matching in $\text{reach}(\chi_F, \mu)$. Lemma 3.10 implies that $\text{IR}(\succ, \mu^*)$ holds and there exists (α^*, β^*) in $\Psi(n, n)$ such that $\mu^* = [\alpha^*, \beta^*]$. Using Observations 3.3 and 3.4, we deduce that $[\alpha^*_{1,a}, \beta^*_{1,b}]$ is equal to μ. We consider two cases.

Case 1: $\alpha^*_{a'} = 0$. Using Observation 3.2, we deduce that $[\alpha^*_{1,a'}, \beta^*_{1,b}]$ is equal to μ'. Hence Observation 3.4 implies that μ^* contains μ'. Thus $\text{reach}(\chi_F, \mu') \neq \emptyset$, as required.

Case 2: $\alpha^*_{a'} = 1$. In this case, it is straightforward to prove that a pivot operation is applicable to (α^*, β^*) at agent a'; let (α^{**}, β^*) denote the result of this operation. Observation 3.7 implies that (α^{**}, β^*) belongs to $\Psi(n, n)$. Let μ^{**} denote $[\alpha^{**}, \beta^*]$. Thus μ^{**} belongs to $\Phi^*(n)$. Observation 3.7 implies that $\text{IR}(\succ, \mu^{**}, a'')$ holds for all agents a'' in $[n] - a'$. Since the inequality $\text{left}(\succ, a') \leq \text{hole}(\mu)$ implies that $\text{IR}(\succ, \mu^{**}, a')$ holds, we deduce that $\text{IR}(\succ, \mu^{**})$ holds. Thus Lemma 3.10 implies that μ^{**} belongs to $\text{reach}(\chi_F)$. Since $\alpha^{**}_{1,a} = \alpha^*_{1,a}$, Observation 3.4 implies that μ^{**} contains μ. Using Observation 3.2, we deduce that $[\alpha^{**}_{1,a'}, \beta^*_{1,b}]$ is equal to μ'. Hence Observation 3.4 implies that μ^{**} contains μ'. Since μ^{**} belongs to $\text{reach}(\chi_F)$ and μ^{**} contains μ', we conclude that μ^{**} is contained in $\text{reach}(\chi_F, \mu')$. □

Lemma 3.12. *Let $F = (n, \succ)$ be an OAF, let μ be a matching in $\Phi^*(n)$ such that $|\mu| < n$, let a denote $|\mu|$, let a' denote $a+1$, let b denote $\max(\mu)$, let μ' denote $\mu + (a', b')$ where b' belongs to $[n] \setminus [b+1]$ and $\text{reach}(\chi_F, \mu') \neq \emptyset$, and let μ'' denote $\mu + (a', b+1)$. Then $\text{reach}(\chi_F, \mu'') \neq \emptyset$.*

Proof. Let μ^* be a matching in $\text{reach}(\chi_F, \mu')$. Lemma 3.10 implies that $\text{IR}(\succ, \mu^*)$ holds and there exists (α^*, β^*) in $\Psi(n, n)$ such that $\mu^* = [\alpha^*, \beta^*]$. Using Observations 3.1 and 3.4, we find that $\alpha^*_{a'} = \beta^*_{b'} = 1$ and $\beta^*_{b+k} = 0$ for all k in $[b' - b - 1]$. Using Observation 3.8, we deduce that a sequence of $b' - b - 1$ unsort operations can be used to transform (α^*, β^*) into a pair of binary strings (α^*, β^{**}) in $\Psi(n, n)$ such that $\beta^{**}_{1,b} = \beta^*_{1,b}$ and $\text{IR}(\succ, [\alpha^*, \beta^{**}])$ holds. Let μ^{**} denote $[\alpha^*, \beta^{**}]$; thus μ^{**} belongs to $\Phi^*(n)$. Lemma 3.10 implies that μ^{**} belongs to $\text{reach}(\chi_F)$. Since $\beta^{**}_{1,b} = \beta^*_{1,b}$, Observation 3.4 implies that μ^{**} contains μ. Using Observation 3.2, we deduce that $[\alpha^*_{1,a'}, \beta^{**}_{1,b+1}]$ is equal to μ''. Hence Observation 3.4 implies that μ^{**} contains μ''. Since μ^{**} belongs to $\text{reach}(\chi_F)$ and μ^{**} contains μ'', we conclude that μ^{**} belongs to $\text{reach}(\chi_F, \mu'')$. □

Lemma 3.13. *Let $F = (n, \succ)$ be an OAF, let μ be a matching in $\Phi^*(n)$ such that $|\mu| < n$, let a denote $|\mu|$, let a' denote $a+1$, let b denote $\max(\mu)$, and assume that $\text{reach}(\chi_F, \mu) \neq \emptyset$. Then there exists a matching in $\text{reach}(\chi_F, \mu)$ that matches agent a' to an object in $\{\text{hole}(\mu), b+1\}$.*

Proof. Let μ^* belong to $\text{reach}(\chi_F, \mu)$ and let b^* denote $\mu^*(a')$. Lemma 3.10 implies that μ^* belongs to $\Phi^*(n)$ and $\text{IR}(\succ, \mu^*)$ holds.

Since μ^* belongs to $\Phi^*(n)$, the definition of $\Phi^*(n)$ implies that $b^* = \text{hole}(\mu)$ or $b < b^*$. We consider two cases.

Case 1: $b^* = \mathrm{hole}(\mu)$. Since $\mathrm{IR}(\succ, \mu^*)$ holds, we deduce that $\mathrm{left}(\succ, a') \le b^*$. Hence the claim of the lemma follows from Lemma 3.11.

Case 2: $b < b^*$. Since $\mathrm{IR}(\succ, \mu^*)$ holds, we deduce that $b^* \le \mathrm{right}(\succ, a')$. If $b^* = b + 1$, the claim of the lemma is immediate. Otherwise, it follows from Lemma 3.12. □

We are now ready to prove the main technical lemma of this section, Lemma 3.14 below.

Lemma 3.14. *Consider an execution of Algorithm 1 with inputs $F = (n, \succ)$, μ_0, and μ_1. If the guard of the while loop is evaluated in a state where $\mu \neq \emptyset$, then the following conditions hold in that state: (1) μ belongs to $\Phi^*(n)$; (2) $\mathrm{IR}(\succ, \mu)$ holds; (3) $\mathrm{reach}(\chi_F, \mu_1) \neq \emptyset$ implies $\mathrm{reach}(\chi_F, \mu) \neq \emptyset$. Furthermore, if the guard of the while loop is evaluated in a state where $\mu = \emptyset$, then $\mathrm{reach}(\chi_F, \mu_1) = \emptyset$.*

Proof. We prove the claim by induction on the number of iterations of the loop. For the base case, we verify that the stated conditions hold the first time the loop is reached. The initialization of μ ensures that $\mu \neq \emptyset$, so we need to verify conditions (1) through (3). Since $\mathrm{reach}(\chi_F, \mu_0) \neq \emptyset$, Lemma 3.10 implies that $\mathrm{IR}(\succ, \mu_0)$ holds, and Lemma 3.10 and the definition of $\Phi^*(n)$together imply that μ_0 belongs to $\Phi^*(n)$. Since $b_0 \le \mathrm{right}(\succ, |\mu_1|) \le n$, the definition of $\Phi^*(n)$ implies that μ_1 belongs to $\Phi^*(n)$. Since $\mathrm{IR}(\succ, \mu_0)$ holds and the preconditions associated with Algorithm 1 ensure that $\mathrm{IR}(\succ, \mu_1, |\mu_1|)$ holds, we deduce that $\mathrm{IR}(\succ, \mu_1)$ holds. Since μ is initialized to μ_1, we conclude that conditions (1) through (3) hold the first time the loop is reached.

For the induction step, consider an arbitrary iteration of the loop. Such an iteration begins in a state where the guard of the while loop evaluates to true, so we can assume that $0 < |\mu| < n$ and that conditions (1) through (3) hold in this state. Let a denote $|\mu|$, let a' denote $a + 1$, let b denote $\max(\mu)$, and let μ' denote the value of the program variable μ immediately after this iteration of the loop body. We consider two cases.

Case 1: $\mu' = \emptyset$. In this case, we need to prove that $\mathrm{reach}(\chi_F, \mu_1) = \emptyset$. Assume for the sake of contradiction that $\mathrm{reach}(\chi_F, \mu_1) \neq \emptyset$. Condition (3) implies that $\mathrm{reach}(\chi_F, \mu) \neq \emptyset$. Thus Lemma 3.13 implies there exists a matching μ^* in $\mathrm{reach}(\chi_F, \mu)$ such that $\mathrm{left}(\succ, a') \le \mathrm{hole}(\mu)$ or $b + 1 \le \mathrm{right}(\succ, a')$. It follows by inspection of the code that $\mu' \neq \emptyset$, a contradiction.

Case 2: $\mu' \neq \emptyset$. In this case, we need to prove that conditions (1) through (3) hold with μ replaced by μ'; we refer to these conditions as postconditions (1) through (3). Since $\mu'(a') \le \mathrm{right}(\succ, a') \le n$ and condition (1) implies that μ belongs to $\Phi^*(n)$, the definition of $\Phi^*(n)$implies that μ' belongs to $\Phi^*(n)$. Thus postcondition (1) holds. Since condition (2) implies that $\mathrm{IR}(\succ, \mu)$ holds, and $\mathrm{IR}(\succ, \mu', a')$ holds by inspection of the code, we deduce that postcondition (2) holds. It remains to establish postcondition (3). In order to do so, we may assume that $\mathrm{reach}(\chi_F, \mu_1) \neq \emptyset$. Condition (3) implies that $\mathrm{reach}(\chi_F, \mu) \neq \emptyset$. To establish postcondition (3), we need to prove that $\mathrm{reach}(\chi_F, \mu') \neq \emptyset$. We consider two cases.

Case 2.1: $a < b$ and $\mathrm{left}(\succ, a') \le \mathrm{hole}(\mu)$. In this case, $\mu' = \mu + (a', \mathrm{hole}(\mu))$, and Lemma 3.11 implies that $\mathrm{reach}(\chi_F, \mu') \neq \emptyset$.

Case 2.2: $a = b$ or $\mathrm{hole}(\mu) < \mathrm{left}(\succ, a')$. In this case, $\mu' = \mu + (a', b+1)$, and Lemma 3.13 implies there exists a matching μ^* in $\mathrm{reach}(\chi_F, \mu)$ such that $\mu^*(a')$ is either $\mathrm{hole}(\mu)$ or $b + 1$. Since $\mathrm{IR}(\succ, \mu^*)$ holds, the Case 2.2 condition implies that if $\mu^*(a') = \mathrm{hole}(\mu)$, then $a = b$, in which case $\mathrm{objects}(\mu) = [b]$ and hence $\mathrm{hole}(\mu) = b + 1$. It follows that $\mu^*(a') = b + 1$, and hence that $\mathrm{reach}(\chi_F, \mu') \neq \emptyset$. □

Using Lemma 3.14, it is straightforward to establish the correctness of Algorithm 1.

Lemma 3.15. *Consider an execution of Algorithm 1 with inputs $F = (n, \succ)$, μ_0, and μ_1. The execution terminates correctly within $n - |\mu_1|$ iterations.*

Proof. Lemma 3.14 implies that each iteration of Algorithm 1 either increments the cardinality of matching μ or reduces it to zero. In the latter case, the algorithm terminates immediately. It follows that the algorithm terminates within $n - |\mu_1|$ iterations. Next, we argue that the algorithm terminates correctly. In what follows, let μ^* denote the final value of the program variable μ. We consider two cases.

Case 1: Algorithm 1 terminates with $\mu^* = \emptyset$. In this case, Lemma 3.14 implies that $\text{reach}(\chi_F, \mu_1) = \emptyset$, as required.

Case 2: Algorithm 1 terminates with $\mu^* \neq \emptyset$. Lemma 3.14 implies that conditions (1) through (3) in the statement of Lemma 3.14 hold with μ replaced by μ^*; we refer to these conditions as postconditions (1) through (3). Since $\mu^* \neq \emptyset$, no agent-object pairs are removed from μ during the execution of Algorithm 1. Since Algorithm 1 initializes μ to μ_1, we deduce that μ_1 is contained in μ^*. Since the guard of the loop evaluates to false when μ is equal to μ^* and postcondition (1) holds, we deduce that μ^* belongs to $\Phi^*(n)$ and $|\mu^*| = n$. Since postconditions (1) and (2) hold, Lemma 3.10 implies that μ^* belongs to $\text{reach}(\chi_F)$. Since μ_1 is contained in μ^*, we deduce that μ^* belongs to $\text{reach}(\chi_F, \mu_1)$, as required. □

3.3 Object Reachability

We now describe how to use Algorithm 1 to solve the reachable object problem on paths in $O(n^2)$ time. Let $F = (n, \succ)$ be a given OAF, let a^* be an agent in $[n]$, and let b^* be an object in $[n]$. Assume without loss of generality that $a^* < b^*$. We wish to determine whether there is a matching in $\text{reach}(\chi_F)$ that matches a^* to b^*, and if so, to compute such a matching. We begin by using a preprocessing phase to compute $\text{left}(\succ, a)$ and $\text{right}(\succ, a)$ for all agents a in $[n]$. We start with agent a^*, and check whether $\text{left}(\succ, a^*) \leq b^* \leq \text{right}(\succ, a^*)$. If this check fails, we halt and report failure. Otherwise, we proceed to the remaining agents. Barring failure, the overall cost of the preprocessing phase is $O(n^2)$. We now describe how to proceed in the special case where a^* is the leftmost agent on the path, i.e., where $a^* = 1$. Later we will see how to efficiently reduce the general case to this special case. In the special case $a^* = 1$, we call Algorithm 1 with $\mu_0 = \emptyset$ and $\mu_1 = \{(1, b^*)\}$. If there is a matching in $\text{reach}(\chi_F)$ that matches agent 1 to object b, then Algorithm 1 returns such a matching. If not, Algorithm 1 returns the empty matching. Excluding the cost of the preprocessing phase, the time complexity of Algorithm 1 is $O(n)$. So the overall running time is $O(n^2)$, as it is dominated by the preprocessing phase. We now discuss how to reduce the case of general a^* to the special case $a^* = 1$. The key is Lemma 3.16 below. Informally, Lemma 3.16 tells us that we can ignore all of the agents and objects in $[a^* - 1]$. Doing this, a^* is once again leftmost, and we can proceed as in the special case $a^* = 1$. Alternatively, we can run Algorithm 1 with $\mu_0 = \{(i, i) \mid i \in [a^* - 1]\}$ and $\mu_1 = \mu_0 + \{(a^*, b^*)\}$. We now proceed to prove Lemma 3.16.

Lemma 3.16. *Let $F = (n, \succ)$ be an OAF, let μ be a matching in* $\text{reach}(\chi_F)$*, let a belong to* $\text{agents}(\mu)$*, let b denote $\mu(a)$, and assume that $a < b$. Then there is a matching μ' in* $\text{reach}(\chi_F)$ *such that $\mu'(a) = b$ and $\mu'(a') = a'$ for all agents a' in $[a - 1]$.*

For any (α, β) in $\Psi(n, n)$, we say that a cancel operation is applicable to (α, β) if $w(\alpha) > 0$. The result of applying this operation is the pair of binary strings (α', β') that is the same as (α, β) except that the first appearing 1 in α (resp., β) is changed to a 0 in α' (resp., β'). It is easy to see that (α', β') belongs to $\Psi(n, n)$.

Observation 3.17. *Let (α, β) belong to $\Psi(n, n)$, let a and a' be agents in $[n]$ such that $\alpha_a = \alpha_{a'} = 1$ and $a > a'$, let (α', β') be the result of applying a cancel operation to (α, β), let μ denote $[\alpha, \beta]$, and let μ' denote $[\alpha', \beta']$, Then $\mu'(a) = \mu(a)$ and $\text{span}(\mu', a'')$ is contained in $\text{span}(\mu, a'')$ for all agents a'' in $[n]$.*

Proof. Lemma 3.10 implies that $\text{IR}(\succ, \mu)$ holds and there exists (α, β) in $\Psi(n, n)$ such that $\mu = [\alpha, \beta]$. Since $a < b$, Observation 3.1 implies that $\alpha_a = \beta_b = 1$. Let the number of 1 bits to the left of position a in α be equal to k. Thus the number of 1 bits to the left of position b in β is also k. Let (α', β') be the pair of binary strings in $\Psi(n, n)$ that results from applying k cancel operations to (α, β). Using Observation 3.17, it is straightforward to check that the matching $\mu' = [\alpha', \beta']$ satisfies the requirements of the lemma. □

Theorem 3.18 below summarizes the main result of this section.

Theorem 3.18. *Reachable object on paths can be solved in $O(n^2)$ time.*

3.4 Pareto-Efficient Reachability

Let $F = (n, \succ)$ be an OAF, and let $\chi = (F, \mu_0)$ be a configuration for which we wish to compute a Pareto-efficient matching. Below we describe a simple way to use Algorithm 1 to solve Pareto-efficient matching on paths in $O(n^3)$ time. We then explain how to improve the time bound to $O(n^2 \log n)$, and then to $O(n^2)$. In all cases we employ the same high-level strategy based on serial dictatorship. We begin by performing the $O(n^2)$-time preprocessing phase discussed in Section 3.3; we only need to perform this computation once. After the preprocessing phase, the output matching is computed in n stages numbered from 1 to n. In stage k, we determine the best possible match that we can provide to agent k while continuing to maintain the previously-determined matches for agents 1 through $k-1$. We now describe how to use Algorithm 1 to implement any given stage k in $O(n^2)$ time. In stage k, we call Algorithm 1 $O(n)$ times. In each of these calls, the input matching μ_0 contains the $k-1$ previously-determined agent-object pairs involving the agents in $[k-1]$. The calls to Algorithm 1 differ only in terms of the value assigned to the input object b_0 which, together with μ_0, determines μ_1. We vary b_0 over all values meeting the precondition $\max(\mu_0) < b_0 \leq \text{right}(\succ, |\mu_1|)$ associated with Algorithm 1; the number of such values is $O(n)$.

This allows us to determine, in $O(n^2)$ time, the rightmost feasible match, if any, for agent k. By Lemma 3.13, the leftmost potential match for agent k is $\text{hole}(\mu_0)$, and this option is only feasible if $\text{left}(\succ, k) \leq \text{hole}(\mu_0)$. Since $\text{reach}(\chi_F, \mu_0) \neq \emptyset$, we are guaranteed to find at least one candidate match for agent k in this process. If there is exactly one candidate, then we select it as the match of agent k. Otherwise, there are two candidates (leftmost and rightmost), and we select the candidate that agent k prefers.

The simple algorithm described above has a running time of $O(n^2)$ per stage, and hence $O(n^3)$ overall. To understand how to implement a stage more efficiently, it is useful to assign a color to the program state each time the condition of the while loop is evaluated. We color such a state red if $\mu = \emptyset$. If the state is red, then the execution is guaranteed to fail (i.e., return $\emptyset$) immediately. We color such a state green if $|\mu| = \max(\mu) > 0$. If the state is green, it is straightforward to prove that the program will proceed to assign every agent a in $\{|\mu|+1, \ldots, n\}$ to object a, and then will succeed (i.e., return a nonempty matching). This observation also implies that if the state is green after a given number of iterations, it remains green after each subsequent iteration. If a state is neither red nor green, we color it yellow.

We are now ready to see how to improve the running time of stage k to $O(n \log n)$. As a thought experiment, consider running the $O(n)$ executions of Algorithm 1 associated with the simple algorithm, but now in parallel. Within each of these executions, we color each successive agent in the set $\{k, \ldots, n\}$ white or black as follows: agent k is colored black; agent $|\mu|+1$ is colored white if $\text{left}(\succ, |\mu|+1) \leq \text{hole}(\mu)$, and black otherwise. The key observation is that as long as none of the parallel executions have terminated, they all agree on the coloring of the processed agents. It follows that if we compare two of the parallel executions, say executions A and B where execution A has a lower value for b_0 than execution B, then execution A can only transition to a green state at a strictly earlier iteration than execution B, and execution A cannot transition to a red state earlier than execution B. This implies that there is a threshold b_1 such that all executions with $b_0 \leq b_1$ succeed, and all of the remaining executions fail. This in turn means that we do not need to run all $O(n)$ of the parallel executions of Algorithm 1. Instead, we can use binary search to determine the threshold b_1 in $O(\log n)$ executions. This observation reduces the running time of a stage from $O(n^2)$ to $O(n \log n)$.

We now sketch how to further improve the running time of stage k to $O(n)$. To do so, we will use a single execution of a modified version of Algorithm 1 to compute the threshold b_1 discussed above. The high-level idea is to treat b_0 as a variable instead of a fixed value. Initially, we set b_0 to $\max(\mu_0)+1$, the minimum value satisfying the associated precondition of Algorithm 1. We also maintain a lower bound b_1' on the threshold b_1. We initialize b_1' to a low dummy value, such as 0. Each time the condition of the while loop is evaluated, we check whether the color of the current state is red, green, or yellow. If the color is yellow, we continue the execution without altering b_0 or b_1'. If the color is red, then we halt and output the threshold $b_1 = b_1'$. If the color is green, then we assign b_1' to the current value of b_0, and we increment b_0.

Unfortunately, we cannot simply increment b_0 and continue the execution. While incrementing b_0 has no impact on the white-black categorization of the agents processed so far, and on the matches of the white agents, it causes the match of each black agent to be incremented. This leads to two difficulties that we now discuss.

The first difficulty is that there can be a lot of black agents, making it expensive to maintain an explicit match for each black agent. Accordingly, when we color an agent black, we do not explicitly match that agent to a particular object. Instead, we maintain an ordered list of the black agents. At any given point in the execution, the black agents are implicitly matched (in the order specified by the list) to the contiguous block of objects that starts with the current value of b_0. Thus, when b_0 is incremented, the matches of the black agents are implicitly updated in constant time.

The second difficulty associated with incrementing b_0 is that if a black agent a is matched to object right$(\succ, a)$ just before the increment, then it is infeasible to shift the match of agent a to the right. If this happens, we need to recognize that executing Algorithm 1 from the beginning with the new higher value of b_0 results in a red state, and so we should terminate. To recognize such events, we introduce an integer variable called *slack*. We maintain the invariant that *slack* is equal to the maximum number of positions we can shift the list of black agents to the right without violating a constraint. We initialize *slack* to right$(\succ, |\mu_1|)$ minus the initial value $\max(\mu_0) + 1$ of b_0. When we color an agent a black, we update *slack* to the minimum of its current value and right$(\succ, a) - b_0 - \ell$, where ℓ denotes the number of previously-identified black agents. When we increment b_0, we decrement *slack* in order to maintain the invariant. If *slack* becomes negative, we recognize that the program should be in a red state, and we terminate.

Upon termination, it is straightforward to argue that the output threshold b_1 is correct. If b_1 is equal to the initial dummy value, then the sole candidate match for agent k is object hole(μ_0). If not, object b_1 is a candidate, and if hole(μ_0) is not equal to b_1 and left$(\succ, k) \leq$ hole(μ_0) then hole(μ_0) is a second candidate. If there are two candidates, we use the preferences of agent k to select between them.

Theorem 3.19 below summarizes the main result of this section.

Theorem 3.19. *Pareto-efficient matching on paths can be solved in $O(n^2)$ time.*

4 Pareto-Efficient Reachability on Generalized Stars

Throughout this section, let F denote an OAF associated with a generalized star G, let o denote the center object of G, let m denote the number of branches of G, and assume that the branches are indexed from 1 to m. For any i in $[m]$, let $\ell_i > 0$ denote the number of vertices on branch i. We refer to the objects on branch i as $\langle i, 1\rangle, \ldots, \langle i, \ell_i\rangle$, where object $\langle i, j\rangle$ is at distance j from the center. Let $\chi_0 = (F, \mu_0)$ denote the initial configuration, and let $n = 1 + \sum_{1 \leq i \leq m} \ell_i$ denote the total number of vertices in G.

Our algorithm uses serial dictatorship to compute a Pareto-efficient matching for configuration χ_0. For any sequence of agents $\sigma = a_1, \ldots, a_s$ we define serial(σ) as the cardinality-s matching of F in which agent a_1 (the first dictator) is matched to its best match b_1 in reach(χ_0, τ_0) where $\tau_0 = \emptyset$, agent a_2 (the second dictator) is matched to its best match b_2 in reach(χ_0, τ_1) where $\tau_1 = \tau_0 + (a_1, b_1)$, $\ldots$, and agent a_s (the sth dictator) is matched to its best match b_s in reach(χ_0, τ_{s-1}) where $\tau_{s-1} = \tau_{s-2} + (a_{s-1}, b_{s-1})$. Observe that for any permutation σ of the entire set of agents in F, serial(σ) is a Pareto-efficient matching of χ_0.

We iteratively grow a dictator sequence σ. We find it convenient to partition the iterations into two phases. The first phase ends when the current dictator is matched to the center object. The second phase reduces to solving a collection of disjoint path problems, one for each branch.

We begin by discussing the design and analysis of the first phase. We find it useful to introduce the concept of a "nice pair" for OAF F. For any configuration χ of the form (F, μ) and any matching τ of F, we say that the pair (χ, τ) is nice for F if the following conditions hold:

- The set reach(χ, τ) is nonempty.

- For any branch i in $[m]$, there is a (possibly empty) sequence of integers $1 \le j_1 < \cdots < j_s \le \ell_i$ such that $\tau(\chi(\langle i,t\rangle)) = \langle i, j_t\rangle$ for $1 \le t \le s$ and $\chi(\langle i,t\rangle)$ is unmatched in τ for $s < t \le \ell_i$. We refer to this (unique) sequence as $\mathrm{indices}(\chi, \tau, i)$.

The first phase iteratively updates a dictator sequence σ, a configuration χ, and a matching τ. We initialize the configuration χ to χ_0, the initial configuration of F. We initialize σ to the singleton sequence containing agent $\chi(o)$, the first dictator. We use Subroutine 1 of Section 4.1 to determine the best match of agent $\chi(o)$ in $\mathrm{reach}(\chi)$, call it b, and we initialize τ to $\{(\chi(o), b)\}$.

We then execute the while loop described below, which we claim satisfies the following loop invariant I: (χ, τ) is a nice pair for F, agent $\chi(o)$ is matched in τ, $\mathrm{reach}(\chi, \tau) = \mathrm{reach}(\chi_0, \tau)$, and $\tau = \mathrm{serial}(\sigma)$. It is straightforward to verify that invariant I holds after initialization of σ, χ, and τ. In Section 4.2, we prove that if I holds at the start of an iteration of the while loop, then I holds at the end of the iteration.

While $\tau(\chi(o)) \neq o$, we use the following steps to update σ, χ, and τ.

1. Since $\tau(\chi(o)) \neq o$, object $\tau(\chi(o))$ is of the form $\langle i, j_0\rangle$ for some i in $[m]$ and j_0 in $[\ell_i]$. Let $j_1 < \cdots < j_s$ denote $\mathrm{indices}(\chi, \tau, i)$, and let a denote the agent $\chi(\langle i, s+1\rangle)$.

2. Append agent a to the dictator sequence σ.

3. Use Subroutine 2 of Section 4.1 to set k to the maximum j such that object $\langle i, j\rangle$ is a possible match of agent a in $\mathrm{reach}(\chi, \tau)$, or to 0 if no such j exists.

4. If $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$, then perform the following steps.
   (a) Let μ denote the matching of F such that $\chi = (F, \mu)$, let μ^* denote the matching obtained from μ by applying $s+1$ swaps to move agent a from object $\langle i, s+1\rangle$ to the center object o, and let χ^* denote the configuration (F, μ^*).
   (b) Use Subroutine 1 of Section 4.1 to set b to the best match of agent a in $\mathrm{reach}(\chi^*, \tau)$.
   (c) If $k = 0$ or $b >_a \langle i, k\rangle$, then set χ to χ^*, τ to $\tau + (a, b)$, and k to -1.

5. If $k > 0$, then set τ to $\tau + (a, \langle i, k\rangle)$.

Upon termination of the first phase, invariant I holds and $\tau(\chi(o)) = o$. Thus, letting χ_1 denote the value of program variable χ at the end of the first phase, we know that (χ_1, τ) is a nice pair for F, $\tau(\chi_1(o)) = o$, $\mathrm{reach}(\chi_1, \tau) = \mathrm{reach}(\chi_0, \tau)$, and $\tau = \mathrm{serial}(\sigma)$.

In the second phase, we perform the following computation for each branch i (in arbitrary order). First, we let $j_1 < \cdots < j_s$ denote $\mathrm{indices}(\chi_1, \tau, i)$. Second, we perform the following steps for j ranging from $s+1$ to ℓ_i.

1. Let a denote agent $\chi_1(\langle i, j\rangle)$, and append a to σ.

2. Use Subroutine 3 of Section 4.1 to set b to the best match of agent a in $\mathrm{reach}(\chi_1, \tau)$.

3. Set τ to $\tau + (a, b)$.

At the end of the second phase, we output the matching τ.

Let I' denote the invariant "reach(χ_1, τ) = reach(χ_0, τ) and τ = serial(σ)". Thus invariant I' holds at the end of the first phase. Moreover, it is easy to see that invariant I' continues to hold immediately after each execution of step 3 in the second phase.

Since invariant I holds in the first phase and invariant I' holds in the second phase, the overall algorithm faithfully implements the serial dictatorship framework discussed at the beginning of this section. Thus the algorithm correctly computes a Pareto-efficient matching for configuration χ_0. In Section 4.1, we explain how to implement Subroutines 1, 2, and 3 so that the overall running time of the algorithm is $O(n^2 \log n)$.

4.1 Polynomial-Time Implementation

In this section we describe an efficient implementation of the two-phase algorithm presented in Section 4. Our description of the first phase make use of Subroutines 1 and 2, while our description of the second phase makes use of Subroutine 3. Below we discuss how to implement Subroutines 1, 2, and 3 efficiently. Our analysis of the time complexity of these subroutines assumes that a certain preprocessing phase has been performed. Specifically, for each agent a in F, we precompute the set of all objects b such that the sequence of objects $\chi_0(a) = b_1, \ldots, b_k = b$ on the unique simple path from $\chi_0(a)$ (the initial object of agent a) to b in G satisfies $b_1 <_a \cdots <_a b_k$. It is straightforward to compute each such set in $O(n)$ time, and hence the overall time complexity of the preprocessing phase is $O(n^2)$.

We now describe Subroutine 1. The input to Subroutine 1 is a nice pair (χ, τ) for F such that agent $a = \chi(o)$ is unmatched in τ. The output of Subroutine 1 is the best match of a in reach(χ, τ). Subroutine 1 works by considering each branch i in turn to compute the best branch-i match of a in reach(χ, τ). For a fixed i in $[m]$, the latter problem can be solved as follows. Consider the path P of objects consisting of the center object o plus branch i. By restricting the generalized star configuration χ to path P, we obtain a path configuration χ_P. Similarly, by restricting the matching τ to the agents associated with path P, we obtain a matching τ_P defined on path P. For any given j in $[\ell_i]$, it is easy to argue that object $\langle i, j \rangle$ is a possible match of a in reach(χ, τ) if and only if object $\langle i, j \rangle$ is a possible match of a in reach(χ_P, τ_P). Moreover, we can determine whether $\langle i, j \rangle$ is a possible match of a in reach(χ_P, τ_P) by performing at most one call to Algorithm 1 on path P. Given the results of the preprocessing phase, the additional time complexity required to determine whether $\langle i, j \rangle$ is a possible match of a in reach(χ_P, τ_P) is $O(\ell_i)$. Using binary search, we can determine the maximum j (if any) such that $\langle i, j \rangle$ is a possible match of a in reach(χ_P, τ_P) in $O(\ell_i \log \ell_i)$ time. (Remark: Letting $j_1 < \cdots < j_s$ denote indices(χ, τ, i), we can restrict the binary search to the interval $\{1, \ldots, j_1 - 1\}$ if $s > 0$.) Thus we can determine the best branch-i match of a (if any) in reach(χ, τ) in $O(\ell_i \log \ell_i)$ time, and hence we can determine the best match of a in reach(χ, τ) in $O(n \log n)$ time.

We now describe Subroutine 2. The input to Subroutine 2 is a nice pair (χ, τ) for F and an integer i in $[m]$ such that $s < \ell_i$ where $j_1 < \cdots < j_s$ denotes indices(χ, τ, i). The output k of Subroutine 2 is the maximum j such that $\langle i, j \rangle$ is a possible match of agent $a = \langle i, s+1 \rangle$ in reach(χ, τ), or 0 if no such j exists. As in Subroutine 1, we can reduce this task to a path problem. In the present case, we can restrict χ and τ to branch i, that is, we do not need to consider the extended path that includes the center object. Moreover, if $s > 0$ we can restrict the binary search for j to the interval $\{j_s+1,\dots,\ell_i\}$. Each iteration of the binary search involves a single call to Algorithm 1. Given the results of the preprocessing phase, the additional time complexity required for each such call is $O(\ell_i)$. Taking into account the binary search, this approach yields a time complexity of $O(\ell_i \log \ell_i)$.

Having discussed Subroutines 1 and 2, we can now establish an upper bound on the time complexity of the first phase. The first phase performs at most n iterations. The worst-case running time of each iteration is dominated by the cost of a possible call to Subroutine 1, and hence is $O(n \log n)$. Thus the overall running time of the first phase is $O(n^2 \log n)$.

We now discuss Subroutine 3 and the time complexity of the second phase. Since $\tau(\chi(o)) = o$ throughout the second phase, the sequence of calls to Subroutine 3 that we make for a given value of i can be resolved by restricting attention to the branch-i objects and their matched agents under configuration χ_1. Because (χ_1, τ) is a nice pair for F at the end of the first phase, and because we iterate over increasing values of j from $s+1$ to ℓ_i, the approach of Section 3.4 can be used to implement each successive call to Subroutine 3 in $O(\ell_i)$ time. Thus the time required to process branch i is $O(\ell_i^2)$, and the overall time complexity of the second phase is $\sum_{i\in[m]} O(\ell_i^2) = O(n^2)$.

Since the time complexity of the preprocessing phase is $O(n^2)$, the time complexity of the first phase is $O(n^2 \log n)$, and the time complexity of the second phase is $O(n^2)$, we conclude that the overall time complexity of the algorithm is $O(n^2 \log n)$.

4.2 The First Phase Invariant

The following sequence of lemmas pertain to an arbitrary iteration in the first phase. We assume that invariant I holds before the iteration, and we seek to prove that invariant I holds after the iteration. We use the symbols σ, χ, and τ (resp., σ', χ', τ') to refer to the values of the corresponding program variables at the start (resp., end) of the iteration. We use the symbols i, j_0, s, $j_1, \dots, j_s$, a, μ, μ^*, and χ^* to refer to the corresponding program variables; these variables do not change in value during the iteration. We use the symbol k to refer to the initial value of the corresponding program variable. The value of the program variable k can change at most once (see step 4(c)). We use the symbol k' to refer to the value of program variable k at the end of the iteration.

Lemma 4.1. *We have $j_t > t$ for all t in $[s]$.*

Proof. Since (χ, τ) is nice, we have $\text{reach}(\chi, \tau) \neq \emptyset$. Let μ' denote a matching in $\text{reach}(\chi, \tau)$. Since $\tau(\chi(\langle i, 1\rangle)) = \langle i, j_1\rangle$ and agent $\chi(o)$ cannot overtake agent $\chi(\langle i, 1\rangle)$ as we transform the matching associated with χ to μ', we have $1 \le j_0 < j_1$. Since $j_1 < \cdots < j_s$, we deduce that $j_2 > 2$, …, $j_s > s$. □

It follows from Lemma 4.1 that $s < \ell_i$ and hence that agent a is well-defined. Agent a serves as the dictator in this iteration.

Throughout the remainder of this section, let R denote the set of all matchings in $\text{reach}(\chi, \tau)$ that match a to an object in branch i, and let J denote the set of all integers j in $[\ell_i]$ such that object $\langle i, j\rangle$ is a possible match of agent a in R. Thus k is the maximum integer in J, or 0 if J is empty.

Lemma 4.2. *The following statements hold: (1) if $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ does not hold, then* $\text{reach}(\chi, \tau) = R$*; (2) if j belongs to J and $s = 0$, then $j > 1$; (3) if j belongs to J and $s > 0$ then $j > j_s$; (4) if $k > 0$ then $\langle i, k\rangle$ is the best match of agent a in R; (5) if $k = 0$ then $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ holds.*

Proof. We begin by proving part (1). Assume that $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ does not hold. It follows agent a cannot be matched to an object outside of branch i in $\text{reach}(\chi, \tau)$. Hence $\text{reach}(\chi, \tau) = R$, as required.

To establish parts (2) through (5), we consider two cases.

Case 1: $J = \emptyset$. In this case, $k = 0$ and hence parts (2), (3), and (4) hold vacuously. It remains to prove part (5). Assume for the sake of contradiction that $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ does not hold. It follows from part (1) that $\text{reach}(\chi, \tau) = R = \emptyset$, contradicting invariant I.

Case 2: $J \neq \emptyset$. Thus $k > 0$ and hence part (5) holds vacuously. In the proofs of parts (2), (3), and (4) below, let j belong to J, and let μ' be a matching in R such that $\mu'(a) = \langle i, j\rangle$.

We first argue that part (2) holds. Assume for the sake of contradiction that $s = 0$ and $j = 1$. Thus $\mu'(a) = \chi(a) = \langle i, 1\rangle$, and hence $j_0 > 1$. It follows that agent $\chi(o)$ overtakes the stationary agent a as we transform the matching associated with χ to μ', a contradiction.

Now we argue that part (3) holds. Assume that $s > 0$. We begin by proving that $j \geq s+1$. Assume for the sake of contradiction that $j < s+1$. Since (χ, τ) is nice, a is the closest branch-i agent to the center that is "inward-moving" in the sense that $\mu'(a)$ is closer to the center than $\chi(a)$. Since no inward-moving agent can overtake another inward-moving agent, and since agent $\chi(o)$ moves into branch i, we deduce that agent a moves out of branch i, a contradiction since j belongs to J. Thus $j \geq s+1$. Now we argue that $j > j_s$. Lemma 4.1 implies that agent $\chi(\langle i, s\rangle)$ moves outward to object $\langle i, j_s\rangle$. Since $j \geq s+1$, agent a is either stationary or outward-moving, and hence cannot be overtaken by the outward-moving agent $\chi(\langle i, s\rangle)$. It follows that $j > j_s$, as required.

Now we argue that part (4) holds. Since $j > j_s$ and Lemma 4.1 implies $j_s > s$, we have $j > s+1$. Thus agent a is outward-moving on branch i. It follows that $\langle i, k\rangle$ is the best match of agent a in R, as required. $\square$

Lemma 4.3. *Assume that $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ holds. Then $\mu \rightsquigarrow_F \mu^*$ and*

$$\text{reach}(\chi, \tau) \setminus R = \text{reach}(\chi^*, \tau) \neq \emptyset.$$

Proof. Since (χ, τ) is nice and $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ holds, Lemma 4.1 implies that the $s+1$ exchanges used to transform μ to μ^* are all Pareto-improving. Hence $\mu \rightsquigarrow_F \mu^*$,

The set $\text{reach}(\chi^*, \tau)$ is nonempty since it includes μ^*. It remains to prove that $\text{reach}(\chi, \tau) \setminus R = \text{reach}(\chi^*, \tau)$.

We first argue that $\text{reach}(\chi^*, \tau) \subseteq \text{reach}(\chi, \tau) \setminus R$. Let μ^{**} belong to $\text{reach}(\chi^*, \tau)$. Thus $\mu^* \rightsquigarrow_F \mu^{**}$. Since $\mu \rightsquigarrow_F \mu^*$, we conclude that $\mu \rightsquigarrow_F \mu^{**}$ and hence μ^{**} belongs to $\text{reach}(\chi, \tau)$. Since $\chi^*(o) = a$ and $\langle i, 1\rangle <_a o$, we deduce that $\mu^{**}(a)$ does not belong to branch i. Since μ^{**} belongs to $\text{reach}(\chi, \tau)$ and $\mu^{**}(a)$ does not belong to branch i, we conclude that μ^{**} belongs to $\text{reach}(\chi, \tau) \setminus R$.

Now we argue that $\text{reach}(\chi, \tau) \setminus R \subseteq \text{reach}(\chi^*, \tau)$. Let μ^{**} belong to $\text{reach}(\chi^*, \tau) \setminus R$. From our discussion of the reachable matching problem on trees in Section 7.1, we can obtain a sequence of swaps that transforms μ to μ^{**} by repeatedly performing any swap that moves the two participating agents closer to their matched objects under μ^{**}. Since $\mu^{**}(a)$ does not belong to branch i, this means that we can begin by performing the $s+1$ swaps that transform μ to μ^*. It follows that μ^{**} belongs to $\text{reach}(\chi^*, \tau)$, as required. □

Lemma 4.3 implies that if line 4(b) is executed, then $\text{reach}(\chi^*, \tau) \neq \emptyset$, and hence object b is well-defined.

Lemma 4.4. *Assume that $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ holds. If $k = 0$ or $b >_a \langle i, k\rangle$, then b is the best match of agent a in $\text{reach}(\chi, \tau)$ and*

$$\text{reach}(\chi, \tau') = \text{reach}(\chi', \tau').$$

Otherwise, $\langle i, k\rangle$ is the best match of agent a in $\text{reach}(\chi, \tau)$.

Proof. We consider two cases.

Case 1: $k = 0$. In this case, $R = \emptyset$ and $\chi' = \chi^*$. Hence Lemma 4.3 implies $\text{reach}(\chi^*, \tau) = \text{reach}(\chi, \tau)$. Thus object b is the best match of agent a in $\text{reach}(\chi, \tau)$ and $\text{reach}(\chi, \tau') = \text{reach}(\chi', \tau')$.

Case 2: $k > 0$. Since $k > 0$, object $\langle i, k\rangle$ is the best match of agent a in R. Lemma 4.3 implies that b is the best match of agent a in $\text{reach}(\chi^*, \tau) = \text{reach}(\chi, \tau) \setminus R$. We consider two subcases.

Case 2.1: $b >_a \langle i, k\rangle$. Thus b is the best match of agent a in $\text{reach}(\chi, \tau)$, $\chi' = \chi^*$, $\tau' = \tau + (a, b)$, and $\text{reach}(\chi, \tau') = \text{reach}(\chi', \tau')$.

Case 2.2: $b <_a \langle i, k\rangle$. Thus object $\langle i, k\rangle$ is the best match of agent a in $\text{reach}(\chi, \tau)$. □

The next lemma establishes that invariant I holds at the end of the iteration.

Lemma 4.5. *At the end of the iteration, (χ', τ') is a nice pair for F, agent $\chi'(o)$ is matched in τ', $\text{reach}(\chi', \tau') = \text{reach}(\chi_0, \tau')$, and $\tau' = \text{serial}(\sigma')$.*

Proof. Observe that either $k' = -1$ or $k' = k$. Below we consider these two cases separately.

Case 1: $k' = -1$. In this case, step 4(c) is executed and the associated if condition evaluates to true. Furthermore, the if condition associated with step 5 evaluates to false. Thus $\chi' = \chi^*$ and $\tau' = \tau + (a, b)$. It is straightforward to verify that the pair (χ', τ') is nice for F with $\text{indices}(\chi', \tau', i)$ equal to $j_0 < \cdots < j_s$, and that $\chi'(o) = a$ is matched in τ'. By Lemma 4.4, object b is the best match of agent a in $\text{reach}(\chi, \tau)$. Since invariant I implies $\tau = \text{serial}(\sigma)$ and $\text{reach}(\chi, \tau) = \text{reach}(\chi_0, \tau)$, we deduce that $\tau' = \text{serial}(\sigma')$. By Lemma 4.4, $\text{reach}(\chi, \tau') = \text{reach}(\chi', \tau')$. Invariant I implies $\text{reach}(\chi, \tau) = \text{reach}(\chi_0, \tau)$ and hence $\text{reach}(\chi, \tau') = \text{reach}(\chi_0, \tau')$. We conclude that $\text{reach}(\chi', \tau') = \text{reach}(\chi_0, \tau')$, as required.

Case 2: $k' = k$. We begin by proving that $k > 0$. Assume for the sake of contradiction that $k = 0$. Part (5) of Lemma 4.2 implies that $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ holds. Hence step 4(c) is executed, and since $k = 0$, the associated if condition evaluates to true. Hence $k' = -1$, contradicting the Case 2 assumption.

Since $k' = k > 0$, we deduce that $\chi' = \chi$ and $\tau' = \tau + (a, \langle i, k\rangle)$. It is straightforward to verify that (χ', τ') is nice for F with $\text{indices}(\chi', \tau', i)$ equal to $j_1 < \cdots < j_s < k$. Since $\chi' = \chi$ and invariant I holds, we deduce that agent $\chi'(o)$ is matched in τ'.

We claim that $\langle i, k\rangle$ is the best match of agent a in reach(χ, τ). If $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ does not hold, the claim follows from parts (1) and (4) of Lemma 4.2, so assume that $\langle i, s+1\rangle <_a \cdots <_a \langle i, 1\rangle <_a o$ holds. Thus the if condition of step 4 evaluates to true, and hence the if condition of step 4(c) is evaluated. Since $k' = k > 0$, the latter if condition evaluates to false, and hence Lemma 4.4 implies that the claim holds.

Invariant I implies reach$(\chi, \tau) =$ reach(χ_0, τ) and $\tau =$ serial(σ). Thus the claim of the preceding paragraph implies $\tau' =$ serial(σ').

Since $\chi' = \chi$, invariant I implies reach$(\chi', \tau) =$ reach(χ_0, τ) and hence reach$(\chi', \tau') =$ reach(χ_0, τ'). □

5 NP-Completeness of Reachable Object on Cliques

It is easy to see that the reachable object on cliques problem belongs to NP. In this section, we prove that the problem is NP-hard by presenting a polynomial-time reduction from the known NP-complete problem 2P1N-SAT to reachable object on cliques.

An instance of 2P1N-SAT is a propositional formula f over n variables $x_1, \ldots, x_n$ with the following properties: f is the conjunction of a number of clauses, where each clause is the disjunction of a number of literals, and each literal is either a variable or the negation of a variable; each variable occurs exactly three times in f, once in each of three distinct clauses; each variable x_i occurs twice as a positive literal (i.e., x_i) and once as a negative literal (i.e., $\neg x_i$).

Throughout the remainder of Section 5, let f denote a given instance of 2P1N-SAT with n variables $x_1, \ldots, x_n$ and m clauses $C_1, \ldots, C_m$.

In Section 5.1, we describe a polynomial-time procedure for transforming f into an instance I of reachable object on cliques. In Section 5.2, we prove that f is a positive instance of 2P1N-SAT if and only if I is a positive instance of reachable object on cliques.

5.1 Description of the Reduction

We now describe how we transform a 2P1N-SAT instance f into a corresponding instance I of reachable object on cliques. For each variable x_i in f, there are two agents $\hat{x}_i^1$ and $\hat{x}_i^2$ in I. For each clause C_j in f, there are three agents $\hat{u}_j, \hat{v}_j$ and $\hat{w}_j$ in I. Note that the name we use to refer to each agent in I includes a hat symbol. We adopt the convention that if the hat symbol is removed from the name of such an agent, we obtain the name of the initial endowment of that agent. For example, agents $\hat{u}_j$ and $\hat{x}_i^1$ are initially endowed with objects u_j and x_i^1, respectively. For convenience, we define $\hat{U}$ as the set of agents $\{\hat{u}_j \mid j \in [m]\}$, and we define U as the set of objects $\{u_j \mid j \in [m]\}$. We define $\hat{V}$, V, $\hat{W}$, W, $\hat{X}$, and X similarly.

Let A (resp., B) denote the set of all agents (resp., objects) in I. Let N denote $|B|$. Thus $N = 3m + 2n$. Let K_N denote the complete graph with vertex set B, and let E denote the edge set of K_N. Let μ_0 denote the initial matching of agents with objects.

Below we describe the preferences $\succ$ of the agents in A over the objects in B. Let $\chi = (F, \mu_0)$ denote the initial configuration of I, where $F = (A, B, \succ, E)$. Instance I asks the following reachability question: Is there a matching μ in reach(χ) such that $\mu(\hat{w}_m) = u_1$?

Let variable x_i appear in clauses $C_{p_i^1}$ and $C_{p_i^2}$ as a positive literal, where $p_i^1 < p_i^2$, and in clause C_{n_i} as a negative literal. The definition of 2P1N-SAT implies that p_i^1, p_i^2, and n_i are distinct.

Below we list the preferences of each agent in A. In doing so, we specify only the objects that an agent prefers to its initial endowment; the order of the remaining objects is immaterial. The initial endowment is shown in a box. For any i in $[n]$, the preferences of agent $\hat{x}_i^1$ are

$$\hat{x}_i^1 : x_i^2 \succ w_{p_i^1} \succ v_{p_i^1} \succ \boxed{x_i^1}$$

and the preferences of agent $\hat{x}_i^2$ are

$$\hat{x}_i^2 : w_{n_i} \succ v_{n_i} \succ w_{p_i^2} \succ v_{p_i^2} \succ x_i^1 \succ \boxed{x_i^2}$$

if $n_i < p_i^2$, and are

$$\hat{x}_i^2 : w_{p_i^2} \succ v_{p_i^2} \succ w_{n_i} \succ v_{n_i} \succ x_i^1 \succ \boxed{x_i^2}$$

otherwise.

For any j in $[m]$, the preferences of agent $\hat{u}_j$ are

$$\hat{u}_j : v_j \succ \boxed{u_j}.$$

For any j in $[m-1]$, the preferences of agent $\hat{w}_j$ are

$$\hat{w}_j : u_{j+1} \succ \boxed{w_j}.$$

The preferences of agent $\hat{w}_m$ are

$$\hat{w}_m : u_1 \succ v_1 \succ w_1 \succ u_2 \succ v_2 \succ w_2 \succ \ldots \succ u_m \succ v_m \succ \boxed{w_m}.$$

For any j in $[m]$, the preferences of agent $\hat{v}_j$ are

$$\hat{v}_j : \{x_i^1 \mid j \in \{p_i^1, n_i\}\} \cup \{x_i^2 \mid j = p_i^2\} \succ \boxed{v_j},$$

where the set of objects preceding v_j may be ordered arbitrarily.

This completes the description of the reachable object on cliques instance I.

5.2 Correctness of the Reduction

In this section, we prove that f is a positive instance of 2P1N-SAT if and only if I is a positive instance of reachable object on cliques. Lemma 5.4 establishes the only if direction. Lemmas 5.11 through 5.17 lay the groundwork for Lemma 5.18, which establishes the if direction.

For the purposes of our analysis, it is convenient to assign a nonnegative integer rank to each object in B, as follows. For any j in $[m]$, we define $\text{rank}(u_j)$ as $3j-2$, $\text{rank}(v_j)$ as $3j-1$, and $\text{rank}(w_j)$ as $3j$. The rank of any object in X is defined to be 0.

Observation 5.1 below can be justified by enumerating all those agents who like object x_i^1 at least as well as their initial endowment. Observations 5.2 and 5.3 can be justified in a similar manner.

Observation 5.1. *For any i in $[n]$ and any matching μ in* $\text{reach}(\chi)$*, agent $\mu^{-1}(x_i^1)$ belongs to $\{\hat{x}_i^1, \hat{x}_i^2, \hat{v}_{p_i^1}, \hat{v}_{n_i}\}$.*

Observation 5.2. *For any i in $[n]$ and any matching μ in* $\text{reach}(\chi)$*, agent $\mu^{-1}(x_i^2)$ belongs to $\{\hat{x}_i^1, \hat{x}_i^2, \hat{v}_{p_i^2}\}$.*

Observation 5.3. *For any j in $[m]$ and any matching μ in* $\text{reach}(\chi)$*, agents $\mu^{-1}(v_j)$ and $\mu^{-1}(w_j)$ belong to $\{\hat{u}_j, \hat{v}_j, \hat{w}_m\} \cup A_j$ and $\{\hat{w}_j, \hat{w}_m\} \cup A_j$, respectively, where A_j denotes*

$$\{\hat{x}_i^1 \mid j = p_i^1\} \cup \{\hat{x}_i^2 \mid j \in \{p_i^2, n_i\}\}.$$

Lemma 5.4. *Assume that 2P1N-SAT instance f is satisfiable. Then there is a matching μ in* $\text{reach}(\chi)$ *such that $\mu(\hat{w}_m) = u_1$ in the reachable object on cliques instance I.*

Proof. Let $\sigma : \{x_1, \ldots, x_n\} \to \{0, 1\}$ denote a satisfying assignment for f. We specify a sequence of matchings $\mu_0, \ldots, \mu_{3n+3m-1}$, which depends on σ, such that (1) $\mu_{3n+3m-1}(\hat{w}_m) = u_1$ and (2) $\mu_{k-1} = \mu_k$ or $\mu_{k-1} \to_F \mu_k$ for all k in $[3n + 3m - 1]$. We obtain this sequence in two phases.

In the first phase, we perform the following three steps for each i from 1 to n.

1. If $\sigma(x_i) = 1$ and $\mu_{3i-3}(\hat{v}_{p_i^1}) = v_{p_i^1}$, we set μ_{3i-2} to the matching obtained by swapping $\hat{v}_{p_i^1}$ with $\hat{x}_i^1$ in μ_{3i-3}. Otherwise, we set μ_{3i-2} to μ_{3i-3}.
2. If $\sigma(x_i) = 1$ and $\mu_{3i-2}(\hat{v}_{p_i^2}) = v_{p_i^2}$, we set μ_{3i-1} to the matching obtained by swapping $\hat{v}_{p_i^2}$ with $\hat{x}_i^2$ in μ_{3i-2}. Otherwise, we set μ_{3i-1} to μ_{3i-2}.
3. If $\sigma(x_i) = 0$ and $\mu_{3i-1}(\hat{v}_{n_i}) = v_{n_i}$, we set μ_{3i} to the matching obtained by first swapping $\hat{x}_i^1$ with $\hat{x}_i^2$, and then swapping $\hat{v}_{n_i}$ with $\hat{x}_i^2$, in μ_{3i-1}. Otherwise, we set μ_{3i} to μ_{3i-1}.

It is easy to check that all of the swaps in the first phase are valid.

Let A_j be as defined in the statement of Observation 5.3. We claim that at the end of the first phase, agent $\mu_{3n}^{-1}(v_j)$ belongs to A_j for all j in $[m]$. Assume for the sake of contradiction that for some j in $[m]$, agent $\mu_{3n}^{-1}(v_j)$ does not belong to A_j. Since agents $\hat{u}_j$ and $\hat{w}_m$ do not participate in any swap in the first phase, $\mu_{3n}(\hat{u}_j) = u_j$ and $\mu_{3n}(\hat{w}_m) = w_m$. Since $\mu_{3n}^{-1}(v_j)$ does not belong to $\{\hat{u}_j, \hat{w}_m\} \cup A_j$, Observation 5.3 implies that $\mu_{3n}(\hat{v}_j) = v_j$. Let variable x_i satisfy clause C_j. The swaps in the ith iteration of first phase imply that $\mu_{3i}^{-1}(v_j) \neq \hat{v}_j$. Since no agent in $\hat{V}$ participates in more than one swap in the first phase, we have $\mu_{3n}^{-1}(v_j) \neq \hat{v}_j$, a contradiction since $\mu_{3n}(\hat{v}_j) = v_j$. This completes the proof of the claim.

Note that there is a unique object of rank k for each k in $[3m]$. Using the claim stated above, together with the fact that no agent in $\hat{U} \cup \hat{W}$ participates in a swap in the first phase, it is easy to verify that for any rank k in $[3m - 1]$, there is an agent a such that $\text{rank}(\mu_{3n}(a)) = k$ and a prefers the object of rank $k + 1$ to the object of rank k. The preferences of agent $\hat{w}_m$ imply that for any rank k in $[3m - 1]$, agent $\hat{w}_m$ prefers the object of rank k to the object of rank $k + 1$. Moreover, $\text{rank}(\mu_{3n}(\hat{w}_m)) = 3m$. In the second phase we perform $3m - 1$ swaps, each involving agent $\hat{w}_m$. For k running from $3m - 1$ down to 1, we set $\mu_{3n+3m-k}$ to the matching obtained by swapping $\hat{w}_m$ with the agent matched to the object of rank k in $\mu_{3n+3m-k-1}$. The foregoing arguments show that all of these $3m - 1$ swaps are valid and $\text{rank}(\mu_{3n+3m-1}(\hat{w}_m)) = 1$. Thus $\mu_{3n+3m-1}(\hat{w}_m) = u_1$. □

Observation 5.5 below can be justified by using the preferences of agent $\hat{w}_m$ and the fact that if any agent a swaps from object b to b' then $b' \succ_a b$. Observations 5.6 through 5.10 can be justified similarly.

Observation 5.5. *For any matchings μ_1 and μ_2 in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$, we have* $\text{rank}(\mu_2(\hat{w}_m)) \leq \text{rank}(\mu_1(\hat{w}_m))$.

Observation 5.6. *For any j in $[m-1]$ and any matchings μ_1 and μ_2 in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$, we have* $\text{rank}(\mu_2(\hat{w}_j)) \leq \text{rank}(\mu_1(\hat{w}_j)) + 1$.

Observation 5.7. *For any j in $[m]$ and any matchings μ_1 and μ_2 in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$, we have* $\text{rank}(\mu_2(\hat{u}_j)) \leq \text{rank}(\mu_1(\hat{u}_j)) + 1$.

Observation 5.8. *For any j in $[m]$ and any matchings μ_1 and μ_2 in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$, we have* $\text{rank}(\mu_2(\hat{v}_j)) \leq \text{rank}(\mu_1(\hat{v}_j))$.

Observation 5.9. *For any i in $[n]$ and any matchings μ_1 and μ_2 in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$ and $\mu_1(\hat{x}_i^1)$ belongs to $B \setminus X$, we have* $\text{rank}(\mu_2(\hat{x}_i^1)) \leq \text{rank}(\mu_1(\hat{x}_i^1)) + 1$.

Observation 5.10. *For any i in $[n]$ and any matchings μ_1 and μ_2 in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$ and $\mu_1(\hat{x}_i^2)$ belongs to $B \setminus X$, we have* $\text{rank}(\mu_2(\hat{x}_i^2)) \leq \text{rank}(\mu_1(\hat{x}_i^2)) + 1$.

Lemma 5.11 below follows immediately from Observations 5.5 through 5.10.

Lemma 5.11. *For any agent a in A and any matchings μ_1 and μ_2 in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$ and $\mu_1(a)$ belongs to $B \setminus X$, we have* $\text{rank}(\mu_2(a)) \leq \text{rank}(\mu_1(a)) + 1$.

Lemma 5.12. *Let μ_1 and μ_2 be matchings in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$. Then*

$$\text{rank}(\mu_2(\hat{w}_m)) \geq \text{rank}(\mu_1(\hat{w}_m)) - 1.$$

Proof. Assume for the sake of contradiction that $\text{rank}(\mu_2(\hat{w}_m)) < \text{rank}(\mu_1(\hat{w}_m)) - 1$. The preferences of $\hat{w}_m$ imply that $\mu_1(\hat{w}_m)$ and $\mu_2(\hat{w}_m)$ belong to $B \setminus X$. Thus there is an agent a such that $\mu_1(a)$ belongs to $B \setminus X$ and $\text{rank}(\mu_2(a)) > \text{rank}(\mu_1(a)) + 1$, contradicting Lemma 5.11. □

Lemma 5.13. *Let i belong to $[n]$ and j belong to $[m]$. If $\hat{v}_j$ prefers x_i^1 to v_j, then $\hat{v}_j$ prefers v_j to x_i^2. Similarly, if $\hat{v}_j$ prefers x_i^2 to v_j, then $\hat{v}_j$ prefers v_j to x_i^1.*

Proof. Assume that $\hat{v}_j$ prefers x_i^1 to v_j. The preferences of $\hat{v}_j$ imply that $j \in \{p_i^1, n_i\}$. Since p_i^1, p_i^2, and n_i are distinct, $j \neq p_i^2$. Hence the preferences of $\hat{v}_j$ imply that $\hat{v}_j$ prefers v_j to x_i^2. We can use a similar argument to prove that if $\hat{v}_j$ prefers x_i^2 to v_j, then $\hat{v}_j$ prefers v_j to x_i^1. □

Lemma 5.14. *Let μ_1 and μ_2 be matchings in* $\text{reach}(\chi)$ *such that $\mu_1 \to_F \mu_2$ and $\mu_1(\hat{x}_i^2) = x_i^2$. Then $\mu_2(\hat{x}_i^2)$ belongs to $\{x_i^1, x_i^2, v_{p_i^2}\}$.*

Proof. By examining the preferences of $\hat{x}_i^2$ we deduce that object $\mu_2(\hat{x}_i^2)$ belongs to

$$\{w_{n_i}, w_{p_i^2}, v_{n_i}, v_{p_i^2}, x_i^1, x_i^2\}.$$

Assume for the sake of contradiction that $\mu_2(\hat{x}_i^2)$ belongs to $\{w_{n_i}, w_{p_i^2}, v_{n_i}\}$. We consider three cases.

Case 1: $\mu_2(\hat{x}_i^2) = w_{n_i}$. Thus $\mu_1^{-1}(w_{n_i}) = \mu_2^{-1}(x_i^2)$. Since $\mu_2^{-1}(x_i^2) \neq \hat{x}_i^2$, Observation 5.2 implies that $\mu_2^{-1}(x_i^2)$ belongs to $\{\hat{x}_i^1, \hat{v}_{p_i^2}\}$. We consider two cases.

Case 1.1: $\mu_2^{-1}(x_i^2) = \hat{x}_i^1$. Thus $\mu_1^{-1}(w_{n_i}) = \hat{x}_i^1$. By examining the preferences of $\hat{x}_i^1$, we deduce that $\mu_1(\hat{x}_i^1) \neq w_{n_i}$, a contradiction.

Case 1.2: $\mu_2^{-1}(x_i^2) = \hat{v}_{p_i^2}$. A contradiction follows using a similar argument as in Case 1.1.

Case 2: $\mu_2(\hat{x}_i^2) = w_{p_i^2}$. A contradiction follows using a similar argument as in Case 1.

Case 3: $\mu_2(\hat{x}_i^2) = v_{n_i}$. A contradiction follows using a similar argument as in Case 1.

Thus $\mu_2(\hat{x}_i^2)$ belongs to $\{x_i^1, x_i^2, v_{p_i^2}\}$. □

Throughout the remainder of Section 5.2, we say that an agent a is satisfied in a matching μ if $\mu(a)$ is the most preferred object of a. In Lemmas 5.15 through 5.18 below, let $\mu_0, \ldots, \mu_t$ be a sequence of matching such that $\mu_{s-1} \rightarrow_F \mu_s$ for all s in $[t]$, and for each i in $[n]$ let $P(i)$, (resp., $Q(i)$ and $R(i)$) denote the predicate that holds if there is an integer s in $[t]$ such that $\mu_s(\hat{x}_i^1) = v_{p_i^1}$ (resp., $\mu_s(\hat{x}_i^2) = v_{p_i^2}$, $\mu_s(\hat{x}_i^2) = v_{n_i}$). Lemmas 5.15 and 5.16 below present useful properties of these predicates.

Lemma 5.15. *Let i be an element of $[n]$ such that $R(i)$ holds. Then $Q(i)$ does not hold.*

Proof. Let s be an element of $[t]$ such that $\mu_s(\hat{x}_i^2) = v_{n_i}$; such an s exists since $R(i)$ holds. Assume that $Q(i)$ holds. Let s' be an element of $[t]$ such that $\mu_{s'}(\hat{x}_i^2) = v_{p_i^2}$; such an s' exists as $Q(i)$ holds. We consider two cases.

Case 1: $s' > s$. Let s'' be an integer such that $s \leq s'' \leq s'$. The preferences of agent $\hat{x}_i^2$ imply that $p_i^2 < n_i$ and $\mu_{s''}(\hat{x}_i^2)$ belongs to $\{w_{p_i^2}, v_{p_i^2}, w_{n_i}, v_{n_i}\}$. Hence $\text{rank}(\mu_{s''}(\hat{x}_i^2))$ belongs to $\{3p_i^2, 3p_i^2 - 1, 3n_i, 3n_i - 1\}$. Note that $\text{rank}(\mu_s(\hat{x}_i^2)) = 3n_i - 1$, and $\text{rank}(\mu_{s'}(\hat{x}_i^2)) = 3p_i^2 - 1$. Hence there is an s''' such that $s \leq s''' < s'$ and $\text{rank}(\mu_{s'''+1}(\hat{x}_i^2)) \leq \text{rank}(\mu_{s'''}(\hat{x}_i^2)) - 2$. It follows that there is an agent a such that $\mu_{s'''}(a)$ belongs to $B \setminus X$ and $\text{rank}(\mu_{s'''+1}(a)) \geq \text{rank}(\mu_{s'''}(a)) + 2$, contradicting Lemma 5.11.

Case 2: $s' < s$. We can derive a contradiction using a similar argument as in Case 1. □

Lemma 5.16. *Let i be an element of $[n]$ such that $R(i)$ holds. Then $P(i)$ does not hold.*

Proof. Let s be an element of $[t]$ such that $\mu_s(\hat{x}_i^2) = v_{n_i}$; such an s exists since $R(i)$ holds. We begin by proving the following claim: There is an integer s'' in $[s-1]$ such that $\mu_{s''}(\hat{x}_i^2) = x_i^1$. Assume for the sake of contradiction that there is no s'' in $[s-1]$ such that $\mu_{s''}(\hat{x}_i^2) = x_i^1$. Let s'' be the least index in $[s]$ such that $\mu_{s''}(\hat{x}_i^2) \neq x_i^2$. Since $\mu_{s''}(\hat{x}_i^2)$ does not belong to $\{x_i^1, x_i^2\}$, Lemma 5.14 implies that $\mu_{s''}(\hat{x}_i^2) = v_{p_i^2}$. Thus $Q(i)$ holds, contradicting Lemma 5.15. This completes the proof of the claim.

Having established the claim, we let s'' denote the least integer in $[s-1]$ such that $\mu_{s''}(\hat{x}_i^2) = x_i^1$. The preferences of agent $\hat{x}_i^2$ imply that $\mu_{s''-1}(\hat{x}_i^2) = x_i^2$. Let a be the agent $\mu_{s''-1}^{-1}(x_i^1)$. Since $a \neq \hat{x}_i^2$, Observation 5.1 implies that a belongs to $\{\hat{x}_i^1, \hat{v}_{p_i^1}, \hat{v}_{n_i}\}$. We consider two cases.

Case 1: $a \in \{\hat{v}_{p_i^1}, \hat{v}_{n_i}\}$. Lemma 5.13 implies that a does not prefer x_i^2 to their initially endowment. Hence $\mu_{s''}^{-1}(x_i^2) \neq a$, a contradiction.

Case 2: $a = \hat{x}_i^1$. Since $\mu_0(\hat{x}_i^1) = \mu_{s''-1}(\hat{x}_i^1) = x_i^1$, we deduce that $\mu_s(\hat{x}_i^1) = x_i^1 \neq v_{p_i^1}$ for all s' such that $0 \leq s' < s''$. Moreover, $\hat{x}_i^1$ is satisfied in $\mu_{s''}$ and hence $\mu_{s'}(\hat{x}_i^1) = x_i^2 \neq v_{p_i^1}$ for all s' such that $s'' \leq s' \leq t$. Hence $\mu_{s'}(\hat{x}_i^1) \neq v_{p_i^1}$ for all s' such that $0 \leq s' \leq t$. Thus $P(i)$ does not hold. ☐

Lemma 5.17. *Let j belong to $[m]$ and assume that $\mu_t(\hat{w}_m) = u_1$. Then there is an s in $[t-1]$ such that $\mu_s(\hat{w}_m) = w_j$ and $\mu_{s+1}(\hat{w}_m) = v_j$.*

Proof. The only object with rank $3j$ (resp., $3j-1$) is w_j (resp., v_j). Since $\text{rank}(\mu_0(\hat{w}_m)) = 3m$ and $\text{rank}(\mu_t(\hat{w}_m)) = 1$, Lemma 5.12 implies that for every rank k in $[3m-1]$, there is an integer s in $[t-1]$ such that $\text{rank}(\mu_s(\hat{w}_m)) = k+1$ and $\text{rank}(\mu_{s+1}(\hat{w}_m)) = k$. The lemma follows by choosing k to be $3j-1$. ☐

Lemma 5.18. *Assume that $\mu_t(\hat{w}_m) = u_1$. Then the 2P1N-SAT instance f is satisfiable.*

Proof. We construct an assignment $\sigma : \{x_1, \ldots, x_n\} \to \{0, 1\}$ for f as follows: for any i in $[n]$, we set $\sigma(x_i)$ to 1 if $P(i) \vee Q(i)$ holds, and to 0 otherwise.

We now show that σ satisfies f. Let j belong to $[m]$. Let s be an element of $[t-1]$ such that $\mu_s(\hat{w}_m) = w_j$ and $\mu_{s+1}(\hat{w}_m) = v_j$; such an s exists by Lemma 5.17. Thus there is an agent a such that $\mu_s(a) = v_j$ and $\mu_{s+1}(a) = w_j$. Since $a \neq \hat{w}_m$, Observation 5.3 implies that exactly one of the following three statements holds: (1) $j = p_i^1$ and $\mu_s(\hat{x}_i^1) = v_j$; (2) $j = p_i^2$ and $\mu_s(\hat{x}_i^2) = v_j$; (3) $j = n_i$ and $\mu_s(\hat{x}_i^2) = v_j$. We consider two cases.

Case 1: (1) or (2) holds. Then $P(i)$ or $Q(i)$ holds, respectively. Hence $\sigma(x_i) = 1$. By construction, the variable x_i appears as the positive literal x_i in clause C_j. Thus, C_j is satisfied.

Case 2: (3) holds. Then $R(i)$ holds. Lemmas 5.15 and 5.16 imply that $P(i)$ and $Q(i)$ do not hold. Hence $\sigma(x_i) = 0$. By construction, the variable x_i appears as the negative literal $\neg x_i$ in clause C_j. Thus, C_j is satisfied.

Since σ satisfies each clause C_j in f, σ satisfies f. ☐

Theorem 5.19. *Reachable object on cliques is NP-complete.*

Proof. In Section 5.1, we described a polynomial-time reduction from 2P1N-SAT instance f to reachable object on cliques instance I. Thus the theorem follows from Lemmas 5.4 and 5.18. ☐

6 Other NP-Completeness and NP-Hardness Results

We prove the remaining NP-completeness and NP-hardness results stated in Table 1. Specifically, we show that the reachable object problem on generalized stars is NP-complete, reachable matching problem on cliques is NP-complete, and Pareto-efficient matching problem on cliques is NP-hard. These results are proved by adapting the corresponding proofs of Gourvès et al. [10] and Müller and Bentert [14] for the object-moving model. The most significant changes are associated with the proof of the first result.

6.1 NP-Completeness for Reachable Object on Generalized Stars

Theorem 6.1. *Reachable object on generalized stars is NP-complete.*

Proof. It is easy to see that reachable object on generalized stars belongs to NP. We use a reduction from the problem 2P1N-SAT to establish that the reachable object on generalized stars is NP-complete. In an instance of 2P1N-SAT, we are given a propositional formula f that is the conjunction of m clauses $C_1, \dots, C_m$. Each clause C_i is the disjunction of a number of literals, where each literal is either a variable or the negation of a variable. The set of variables is $x_1, \dots, x_n$. For each variable x_j, the positive literal x_j appears in exactly two clauses, and the negative literal $\neg x_j$ appears in exactly one clause. We are asked to determine whether the formula f is satisfiable.

Given such a formula f, we construct a corresponding reachable object on generalized stars instance $I = (F, \mu)$ where $F = (A, B, \succ, E)$ as follows. We begin by describing the set of objects B. For each clause index i, there are two objects c'_i and c''_i in B. There is also a special object c''_0 in B. For each variable index j, there are five objects in B: "dummy" objects d_j and d'_j; an object n_j corresponding to the lone occurrence of the negative literal $\neg x_j$ in f; an object p_j corresponding to the first occurrence of the positive literal x_j in f (i.e., the occurrence associated with the lower-indexed clause); an object p'_j corresponding to the other occurrence of the positive literal x_j. Thus there are a total of $2m + 5n + 1$ objects in B.

Observe that our identifier for any given object in B includes a single lowercase letter. By changing this letter to upper case, we obtain our identifier for the agent initially matched to that object. So, for example, agent C''_0 is initially matched to object c''_0.

We now describe the edge set E. Object c''_m is the center object. We begin by describing $m + n + 1$ vertex-disjoint subgraphs of (B, E): a "clause gadget" for each clause index i, a "variable gadget" for each variable index j, and an additional gadget that we call the "staircase". Clause gadget i consists of the lone object c'_i. Variable gadget j is a path of length 3 containing the objects n_j, d_j, p'_j and p_j, in that order, and the lone object d'_j. The staircase is a path of length m containing the sequence of objects $c''_0, c''_1, \dots, c''_m$. We say that object c''_0 is at the bottom of the staircase, and that object c''_m is at the top of the staircase. The following additional $m + 2n$ edges are used to connect these $m + n + 1$ subgraphs into a generalized star: there is an edge from object c'_i to object c''_m for each clause index i; there is two edges from object p_j, d'_j to object c''_m, respectively, for each variable index j.

With regard to the agent preferences, all that matters is the set objects(I, a) associated with each agent a in A. It is straightforward to verify that we can choose agent preferences so that the following conditions are satisfied. First, for each clause index i, we have objects$(I, C'_i) = \{c'_i\} \cup \{c''_k \mid i - 1 \le k \le m\}$. Second, for each integer i such that $0 \le i \le m$, we have

$$\text{objects}(I, C''_i) = \{c''_k \mid i \le k \le m\} \cup \{p_j \mid j \in [n]\} \cup \{p'_j \mid j \in [n]\}.$$

Finally, for each variable index j, the following properties hold: objects(I, D_j) is equal to $\{d_j, p'_j, p_j, d'_j, c''_m\}$; objects$(I, D'_j)$ is equal to $\{d'_j, p_j, p'_j, c''_m\}$; objects$(I, N_j)$ is equal to $\{n_j, d_j, p'_j, p_j, c''_m, c'_i\}$ where i is the index of the clause that contains the negative literal x_j; objects(I, P_j) is equal to $\{p_j, p'_j, d_j, c''_m, c'_i\}$ where i is the index of the clause that contains the first occurrence of the positive literal x_j; objects(I, P'_j) is equal to $\{p'_j, d_j, n_j, p_j, c''_m, c'_i\}$

where i is the index of the clause that contains the second occurrence of the positive literal x_j.

We claim that agent C''_0 (which starts out at the bottom of the staircase) can reach object c''_m (at the top of the staircase) if and only if f is satisfiable.

We begin by addressing the "if" direction of the claim. Assume that f is satisfiable. Fix a satisfying assignment for f, and for each clause index i, let ℓ_i denote a literal in C_i that is set to true by this satisfying assignment. Let L_i denote the agent that corresponds to literal ℓ_i, as follows: if ℓ_i is the negative literal $\neg x_j$, then L_i is equal to N_j; if ℓ_i is the first occurrence of the positive literal x_j, then ℓ_i is equal to P_j; if ℓ_i is the second occurrence of the positive literal x_j, then ℓ_i is equal to P'_j. Note that the L_i's are all distinct.

For any variable x_j, let $\mathcal{L}_j = \{L_i \mid 1 \leq i \leq m\} \cap \{N_j, P_j, P'_j\}$. Note that if $|\mathcal{L}_j| \geq 2$ then $\mathcal{L}_j = \{P_j, P'_j\}$.

For each clause index i, $1 \leq i \leq m$, let $f(i)$ denote the unique variable index j such that $L_i \in \{N_j, P_j, P'_j\}$.

We now describe how to perform a sequence of swaps that result in agent C''_0 being matched to object c''_m. We perform these swaps in m phases. We will define each phase so that a certain invariant holds. Specifically, we will ensure that the following conditions hold after k phases have been completed, $0 \leq k \leq m$: (1) the sequence of agents associated with the $m+1$ staircase objects, $c''_0, \ldots, c''_m$ is $C'_1, \ldots, C'_k, C''_0, \ldots, C''_{m-k}$; (2) for any clause index i such that $k < i \leq m$, the agent matched to object c'_i is C'_i; (3) for any variable index j such that $\mathcal{L}_j \not\subseteq \{L_i \mid 1 \leq i \leq k\}$, the agents matched to the objects n_j, d_j, p'_j, and d'_j are N_j, D_j, P'_j, and D'_j, respectively, and the agent matched to object p_j is P_j if $\{L_i \mid 1 \leq i \leq k\} \cap \mathcal{L}_j = \emptyset$ and belongs to $\{C''_{m-i+1} \mid 1 \leq i \leq k\}$ otherwise.

Notice that if we can prove the invariant holds after m phases, then condition (1) of the invariant implies that agent C''_0 is matched to object c''_m, as desired. It is easy to see that the invariant holds at the outset (i.e., after 0 phases). Let k be an integer such that $1 \leq k \leq m$, and assume that the invariant holds after $k-1$ phases. It remains to describe how to implement phase k so that the claimed invariant holds after phase k.

Let j denote $f(k)$. We implement phase k in three stages. In the first stage, we perform swaps within variable gadget j to move agent L_k to the center object c''_m. To see how to do this, consider the following cases. Notice that Condition (2) of the invariant implies the agent matched with the center object c''_m is C''_{m-k+1}.

Case 1: $L_k = N_j$. Thus $\mathcal{L}_j = \{N_j\}$. Condition (3) of the invariant implies that the agents matched to the objects n_j, d_j, p'_j, p_j, and d'_j are N_j, D_j, P'_j, P_j, and D'_j, respectively. Using 8 swaps within agents of variable gadget j and agent C''_{m-k+1}, we can rearrange these six agents so that the agents matched to the objects n_j, d_j, p'_j, p_j, c''_m, and d'_j are P'_j, P_j, C''_{m-k+1}, D'_j, N_j, and D_j, respectively. Thus agent L_k is matched to the center object c''_m, as required.

Case 2: $L_k = P_j$.Thus $\{P_j\} \subseteq \mathcal{L}_j \subseteq \{P_j, P'_j\}$ Condition (3) of the invariant implies that L_k is matched to object p_j, so swap L_k with C''_{m-k+1} as required.

Case 3: $L_k = P'_j$ and $\mathcal{L}_j = \{P'_j\}$.

Condition (3) of the invariant implies that the agents matched to the objects n_j, d_j, p'_j, p_j, and d'_j are N_j, D_j, P'_j, P_j, and D'_j, respectively. By swapping P'_j with P_j and then with C''_{m-k+1}, we can ensure that L_k is matched to object c''_m, as required.

Case 4: $L_k = P'_j$ and $\mathcal{L}_j = \{P_j, P'_j\}$. Thus $\{L_i \mid 1 \leq i < k\} \cap \mathcal{L}_j = \{P_j\}$. Condition (3) of the invariant implies that the agents matched to the objects n_j, d_j, p'_j, d'_j, and p_j are N_j, D_j, P'_j, D'_j, and an agent in the set $\{C''_{m-i+1} \mid 1 \leq i < k\}$, respectively. By swapping agent P'_j with the agent matched to p_j and then with C''_{m-k+1}, we can ensure that L_k is matched to object c''_m, as required.

At the start of the second stage, the agent matched with the center object c''_m is L_k (due to the first stage), and the agent matched with object c'_k is C'_k (due to condition (2) of the invariant). In the second stage, we use one swap to rearrange these two agents so that the agents matched to the objects c''_m and c'_k are C'_k and L_k.

At the start of the third stage, the sequence of agents associated with the $m+1$ staircase objects $c''_0, \ldots, c''_m$ is

$$C'_1, \ldots, C'_{k-1}, C''_0, \ldots, C''_{m-k}, C'_k,$$

(due to condition (1) of the invariant and the second stage). In the third stage, we perform $m-k+1$ swaps to move agent C'_k down from the top of the staircase to object c''_{k-1}.

It remains to verify that the invariant holds after phase k. The third stage ensures that condition (1) of the invariant holds after phase k. Condition (2) of the invariant holds after phase k because it held before phase k and the only clause gadget involved in a swap in phase k is clause gadget k. After phase k, condition (3) of the invariant only makes a nontrivial claim about the allocation of a variable gadget j for which $L_k = P_j$ and $\mathcal{L}_j = \{P_j, P'_j\}$. In this case, after phase k, the agents matched to the objects n_j, d_j, p'_j, and d'_j are N_j, D_j, P'_j, C''_{m-k+1}, and D'_j, respectively, so the associated claim is satisfied. Moreover, the claims made in condition (3) after phase k that concern other variable gadgets follow from condition (3) before phase k since the only variable gadget involved in any swaps in phase k is variable gadget j.

We now address the "only if" direction. Assume that a sequence S of valid swaps results in agent C''_0 being matched to object c''_m. Thus there are integers $0 = t_0 < t_1 < \cdots < t_m$ such that for $0 \leq k \leq m$, agent C''_0 first becomes matched to object c''_k at "time" t_k, i.e., immediately after the first t_k swaps of S have been performed. For any integer k such that $0 \leq k \leq m$, let $Q(k)$ denote the predicate "at time t_k, agent C'_i is matched to object c''_{i-1} for all i such that $1 \leq i \leq k$". We now use induction on k to prove that $Q(k)$ holds for $0 \leq k \leq m$. It is easy to see that $Q(0)$ holds. Let k be an integer such that $1 \leq k \leq m$ and assume that $Q(k-1)$ holds. We need to prove that $Q(k)$ holds. Since $Q(k-1)$ holds, each agent in $\{C_i \mid 1 \leq i < k\}$ is matched to its favorite object at time t_{k-1}, and hence does not move thereafter. Thus, to establish that $Q(k)$ holds, it is sufficient to prove that the agent, call it a, moving from object c''_k to object c''_{k-1} in swap t_k of S (which moves agent C''_0 from object c''_{k-1} to object c''_k) is C'_k. Since c''_{k-1} belongs to objects(I, a), we deduce that a belongs to

$$\{C'_i \mid 1 \leq i \leq k\} \cup \{C''_i \mid 0 \leq i < k\}.$$

As discussed above, for $1 \leq i < k$, agent C'_i is permanently matched to object c''_{i-1} as of time t_{k-1}. It follows that a does not belong to $\{C'_i \mid 1 \leq i < k\}$. It also follows that each agent in $\{C''_i \mid 1 \leq i < k\}$ moved up the staircase from object c''_{k-1} to object c''_k prior to time t_{k-1}, and hence can never return to object c''_{k-1}; thus agent a does not belong to $\{C''_i \mid 1 \leq i < k\}$. Since agent a is not equal to C''_0, we conclude that agent a is equal to C'_k, as required. This completes our proof by induction that $Q(k)$ holds for $0 \leq k \leq m$.

Since $Q(m)$ holds, we know that for each clause index i, the sequence of swaps S causes agent C_i' to move away from its initial object c_i'. For any clause index i, let A_i denote $\{a \in A \mid c_i' \in \text{objects}(I, a)\} - C_i'$. We can only swap agent C_i' away from its initial object c_i' in favor of some agent in A_i. Our reduction ensures that A_i is equal to the set of agents corresponding to literals satisfying clause i, in the following sense: for each negative literal $\neg x_j$ appearing in C_i, the agent N_j belongs to A_i; for each positive literal x_j such that the first (resp., second) occurrence of x_j appears in C_i, the agent P_j (resp., P_j') belongs to A_i. For each clause index i, let L_i denote the agent in A_i that swaps with agent C_i' when C_i' moves away from its initial object c_i', and let ℓ_i denote the literal corresponding to L_i. We claim that it is possible to find a truth assignment for f that simultaneously sets all of the literals ℓ_i to true, and thus satisfies f. To prove this, it suffices to show that for any variable index j, if N_j belongs to $\{L_i \mid 1 \leq i \leq m\}$ then $\{L_i \mid 1 \leq i \leq m\} \cap \{P_j, P_j'\} = \emptyset$. Below we prove that the following stronger claim holds: For any variable index j, if a sequence of swaps causes agent N_j to leave variable gadget j (i.e., to move from object p_j to object c_m'') then agents P_j and P_j' remain in variable gadget j under this sequence of swaps.

To prove the latter claim, let us fix a sequence of swaps S that causes agent N_j to leave variable gadget j. The only agent $a \neq N_j$ such that object n_j belongs to objects(I, a) is P_j'. Since N_j moves away from its initial object n_j under S, we deduce that S includes two swaps moving agent P_j' first to object d_j and then to object n_j. Since object n_j is a leaf, agent P_j' remains matched to object n_j thereafter. Since agent N_j does not remain matched to object d_j, it is eventually swapped to object p_j'. Since agent D_j has previously moved from object d_j to object p_j', it cannot move back to object d_j. The only agent $a \notin \{N_j, D_j, P_j'\}$ such that d_j belongs to objects(I, a) is agent P_j. Thus, the swap that moves agent N_j from object to d_j to object p_j' moves agent P_j from object p_j' to object d_j. Since agent P_j' is permanently matched to object n_j, we conclude that agent P_j is permanently matched to object d_j. Thus if agent N_j leaves variable gadget j (indeed, if N_j merely reaches object p_j'), then neither agent P_j nor agent P_j' leaves variable gadget j. $\square$

6.2 NP-Completeness for Reachable Matching on Cliques

We begin by proving in Lemma 6.2 that reachable matching on general graphs is NP-complete. We use Lemma 6.2 to establish that reachable matching on cliques is also NP-complete.

Lemma 6.2. *Reachable matching on general graphs is NP-complete.*

Proof. It is easy to see that reachable matching on general graphs belongs to NP. We use a reduction from reachable object on generalized stars to reachable matching on general graphs to establish that reachable matching on general graphs is NP-complete.

Fix an arbitrary reachable object on generalized stars instance I. Without loss of generality, we can assume that the associated configuration $\chi = (F, \mu)$ is such that $F = (A, B, \succ, E)$, $A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $\chi(a_i) = b_i$ for $1 \leq i \leq n$. We can also assume without loss of generality that our goal is to determine whether there is a matching μ_1 in reach(χ) such that $\mu_1(a_1) = b_n$.

Below we describe how to transform reachable object on generalized stars instance I into a reachable matching on general graphs instance I' such that I is a positive instance of reachable object on generalized stars if and only if I' is a positive instance of reachable matching on general graphs . The reachable matching on general graphs instance I' has two associated configurations $\chi' = (F', \mu')$ and $\chi'' = (F', \mu'')$, where $F' = (A', B', \succ', E')$. The set of agents A' is equal to $A \cup A^*$ where $A^* = \{a_1^*, \ldots, a_n^*\}$. The set of objects B' is equal to $B \cup B^*$ where $B^* = \{b_1^*, \ldots, b_n^*\}$. The perfect matching μ' from A' to B' satisfies $\mu'(a_i) = b_i$ and $\mu'(a_i^*) = b_i^*$ for $1 \leq i \leq n$. The subgraph of (B', E') induced by the set of objects B is equal to (B, E). The subgraph of (B', E') induced by the set of objects B^* is a clique. There are n edges connecting these two subgraphs: there is an edge from object b_i to object b_i^* for $1 \leq i \leq n$. The agent preferences $\succ'$ are defined as follows.

- For any integer i such that $1 \leq i \leq n$, the most preferred object of agent a_i^* is b_i, followed by object b_i^*, followed by the remaining objects in B' in arbitrary order.
- The most preferred object of agent a_1 is b_n^*, followed by the objects in B in the order specified by the preferences of agent a_1 under $\succ$, followed by the objects in $B^* - b_n^*$ in arbitrary order.
- The most preferred objects of agent a_n are $b_1^*, \ldots, b_{n-1}^*$, followed by the objects in B in the order specified by the preferences of agent a_n under $\succ$, followed by object b_n^*.
- For any integer i such that $1 < i < n$, the most preferred objects of a_i are $b_i^*, \ldots, b_{n-1}^*$, followed by $b_{i-1}^*, \ldots, b_1^*$, followed by the objects in B in the order specified by the preferences of agent a_i under $\succ$, followed by object b_n^*.

The perfect matching μ'' associated with configuration χ'' maps each agent in A' to its most preferred object in B'. (Note that μ'' is a perfect matching from A' to B', since no two agents in A' share the same most preferred object.)

It is easy to see that we can construct instance I' in polynomial time in the size of instance I. It remains to argue that instance I is a positive instance of reachable object on generalized stars if and only if I' is a positive instance of reachable matching on general graphs.

We begin by addressing the "only if" direction. Assume that instance I is a positive instance of reachable object on generalized stars. Thus there is a configuration χ_1 in reach(χ) such that $\chi_1(a_1) = b_n$. Our construction of the agent preferences therefore ensures the existence of a configuration χ_1' in reach(χ') such that $\chi_1'(a_i) = \chi_1(a_i)$ and $\chi_1'(a_i^*) = \chi(a_i^*) = b_i^*$ for $1 \leq i \leq n$.

It is easy to check that a swap across edge (b_i, b_i^*) can be applied to configuration χ_1' for $1 \leq i \leq n$. Let χ_2' denote the configuration obtained by applying these n swaps to χ_1'. Thus χ_2' belongs to reach(χ'). Furthermore, it is easy to check that each agent in $A^* + a_1$ is matched in χ_2' to its most preferred object under $\succ'$.

Next, we iteratively construct a sequence of $n - 1$ configurations $\chi_3', \ldots, \chi_{n+1}'$ such that configuration χ_k' satisfies the following properties for $3 \leq k \leq n+1$: χ_k' belongs to reach(χ'); every agent in $A \cup \{a_1, \ldots, a_{k-2}\} + a_n$ is matched in χ_k' to its most preferred object in configuration χ_k'. We begin by applying zero or one swaps to configuration χ_2' to obtain configuration χ_3'. If $\chi_2'(a_n) = b_1^*$, then we define χ_3' as χ_2'. If not, then $\chi_2'(a_i) = b_1^*$ for some i in $\{2, \ldots, n - 1\}$. The preferences of agents a_i and a_n ensure that a swap between these two agents can be applied to configuration χ'_2. We define χ'_3 as the configuration that results from applying this swap. It easy to see that configuration χ'_3 belongs to reach(χ') and that every agent in $A \cup \{a_1, a_n\}$ is matched in χ'_3 to its most preferred object under $\succ'$.

Now fix an integer k such that $4 \leq k \leq n+1$, and inductively assume that we have constructed a configuration χ'_{k-1} in reach(χ') such that every agent in $A \cup \{a_1, \ldots, a_{k-3}\} + a_n$ is matched to its most preferred object under $\succ'$. We apply zero or one swaps to configuration χ'_{k-1} to obtain configuration χ'_k. If $\chi'_{k-1}(a_{k-2}) = b^*_{k-2}$, then we define χ'_k as χ'_{k-1}. If not, then $\chi'_{k-1}(a_i) = b^*_{k-2}$ for some i in $\{k-1, \ldots, n-1\}$. The preferences of agents a_{k-2} and a_i ensure that a swap between these two agents can be applied to configuration χ'_{k-1}. We define χ'_k as the configuration that results from applying this swap. It easy to see that configuration χ'_k belongs to reach(χ') and that every agent in $A \cup \{a_1, \ldots, a_{k-2}\} + a_n$ is matched in χ'_k to its most preferred object under $\succ'$.

Since χ'_{n+1} belongs to reach(χ') and every agent in A' is matched in χ'_{n+1} to its most preferred object under $\succ'$, we conclude that $\chi'_{n+1} = \chi''$ and hence that I' is a positive instance of reachable matching on general graphs.

Now we address the "if" direction. Assume that instance I' is a positive instance of reachable matching on general graphs. Thus χ'' belongs to reach(χ'), and hence there is a sequence of swaps S that transforms configuration χ' into configuration χ''.

By examining the preferences of the agents in A^*, we deduce that each agent in A^* participates in exactly one swap in S, and that the other agent participating in each of these swaps belongs to A. By examining the preferences of the agents in A, we deduce that agent a_1 is the agent that swaps with agent a^*_n, and that once an agent in A becomes matched to an object in B^*, it remains matched to an object in B^* thereafter. It follows that there is a permutation π of $\{1, \ldots, n\}$ such that $\pi(n) = 1$ and a^*_i swaps with $a_{\pi(i)}$ for $1 \leq i \leq n$.

For any integer k such that $0 \leq k \leq |S|$, let χ'_k denote the configuration reached by applying the first k swaps of sequence S to configuration χ'. Thus $\chi' = \chi'_0$, $\chi'' = \chi'_{|S|}$, and χ'_k is of the form (F', μ'_k) where μ'_k is a perfect matching from A' to B'.

For any integer k such that $0 \leq k \leq |S|$, we use the perfect matching μ'_k to construct a perfect matching μ_k from A to B, as follows: for each agent a^*_i in A^* such that $\mu'_k(a^*_i)$ belongs to B^*, we define $\mu_k(a_{\pi(i)})$ as $\mu'_k(a_{\pi(i)})$; for each agent a^*_i in A^* such that $\mu'_k(a^*_i)$ belongs to B, we define $\mu_k(a_{\pi(i)})$ as $\mu'_k(a^*_i)$.

For any integer k such that $0 \leq k \leq |S|$, we define χ_k as the configuration (F, μ_k). It is straightforward to prove by induction on k that χ_k belongs to reach(χ) for $0 \leq k \leq |S|$.

Let ℓ denote the least integer such that $\chi_\ell^{-1}(b_n) = a^*_n$. We know that ℓ exists since $\chi_{|S|}^{-1}(b_n) = a^*_n$, and that ℓ is positive since $\chi_0^{-1}(b_n) = a_n$. As discussed earlier, agent a_1 is the only agent that participates in a swap with agent a^*_n. Hence $\chi_{\ell-1}^{-1}(b_n) = a_1$. Since $\chi_{\ell-1}$ belongs to reach(χ), we conclude that I is a positive instance of reachable object on generalized stars, as required. □

It is easy to see that the reachable matching on cliques problem belongs to NP. We use a reduction from reachable object on cliques to reachable matching on cliques to establish that reachable matching on cliques is NP-complete. This reduction is similar to the one used in the proof of Lemma 6.2.

Fix an arbitrary reachable object on cliques instance I. Without loss of generality, we can assume that the associated configuration $\chi = (F, \mu)$ is such that $F = (A, B, \succ, E)$,

$A = \{a_1, \ldots, a_n\}$, $B = \{b_1, \ldots, b_n\}$, and $\chi(a_i) = b_i$ for $1 \le i \le n$. We can also assume without loss of generality that our goal is to determine whether there is a matching μ_1 in $\text{reach}(\chi)$ such that $\mu_1(a_1) = b_n$.

We now describe how to transform reachable object on cliques instance I into a reachable matching on cliques instance I'. Instance I' has two associated configurations $\chi' = (F', \mu')$ and $\chi'' = (F', \mu'')$, where $F' = (A', B', \succ', E')$. The set of agents A' is equal to $A \cup A^*$ where $A^* = \{a_1^*, \ldots, a_n^*\}$. The set of objects B' is equal to $B \cup B^*$ where $B^* = \{b_1^*, \ldots, b_n^*\}$. Let K_{2n} denote the complete graph with vertex set B', and let E' denote the edge set of K_{2n}. The agent preferences $\succ'$ and the matchings μ' and μ'' are as described in the proof of Lemma 6.2.

Let $\hat{E}$ denote the union of three sets of edges: $\{(b_i, b_j) \mid i, j \in [n] \wedge i \neq j\}$; $\{(b_i, b_i^*) \mid i \in [n]\}$; $\{(b_i^*, b_j^*) \mid i, j \in [n] \wedge i \neq j\}$. Lemma 6.3 below establishes that if a swap occurs on an edge e in I', then e belongs to $\hat{E}$.

Lemma 6.3. *Let i and j be elements of $[n]$ such that $i \neq j$. Let μ_1 and μ_2 be matchings in* $\text{reach}(\chi')$ *such that $\mu_1 \to_F \mu_2$. Then $\mu_2^{-1}(b_i) \neq \mu_1^{-1}(b_j^*)$.*

Proof. We consider two cases.

Case 1: $\mu_1^{-1}(b_j^*)$ belongs to A^*. By examining the preferences of agents in A^*, we deduce that $\mu_1^{-1}(b_j^*) = a_j^*$. The only object that agent a_j^* prefers to b_j^* is b_j. Hence $\mu_2^{-1}(b_i) \neq a_j^* = \mu_1^{-1}(b_j^*)$.

Case 2: $\mu_1^{-1}(b_j^*)$ belongs to A. By examining the preferences of agents in A, we deduce that $\mu_2(\mu_1^{-1}(b_j^*))$ belongs to B^*. Hence $\mu_2^{-1}(b_i) \neq \mu_1^{-1}(b_j^*)$. □

Using Lemma 6.3, along with the same reasoning as in the proof of Lemma 6.2, we deduce that I' is a positive instance of reachable matching on cliques if and only if I is a positive instance of reachable object on cliques. Thus Theorem 6.4 below holds.

Theorem 6.4. *Reachable matching on cliques is NP-complete.*

6.3 NP-Hardness for Pareto-Efficiency on Cliques

Theorem 6.5. *Pareto-efficient matching on cliques is NP-hard.*

Proof. We use the same reduction as we used in Section 6.2 to establish the NP-completeness of reachable matching on cliques. In analyzing that reduction, we proved that a given instance of reachable object on cliques is positive if and only if every agent gets its most preferred object in the corresponding instance of reachable matching on cliques. Therefore an efficient algorithm for computing a Pareto-efficient matching on cliques yields an efficient algorithm for reachable object on cliques. Since reachable object on cliques is NP-complete, we deduce that Pareto-efficient matching on cliques is NP-hard. □

7 Other Polynomial-Time Bounds

In this section, we briefly discuss simple algorithms that serve to justify the remaining polynomial-time entries in Table 1.

For reachable matching on trees, the corresponding algorithm of Gourvès et al. [10] for the object-moving model can also be used for the agent-moving model. In particular, for the agent-moving model, Section 7.1 shows that reachable matching problem on trees can be solved in $O(n^2)$ time.

For the other two polynomial-time table entries for stars, observe that once an agent swaps away from the center object, it cannot participate in another swap. This observation severely restricts the swap dynamics, making it easy to establish that the reachable object problem on stars can be solved in $O(n^2)$ time and Pareto-efficient matching problem on stars can be solved in $O(n)$ time.

7.1 Reachable Matching on Trees

Theorem 7.1. *Reachable matching on trees can be solved in $O(n^2)$ time.*

Proof Sketch. Let μ and μ' be the initial and target perfect matchings in an instance of the reachable matching problem on a tree. To solve the problem, we use the approach presented by Gourvès et al. [10] for the object-moving model. Observe that every agent a moves along a unique dipath in the tree from $\mu(a)$ to $\mu'(a)$. Therefore, it suffices to check whether there is a sequence of swaps such that each agent moves according to its dipath. Let us say that an edge is good if the two agents currently matched to the endpoints of the edge both need to cross the edge as the next step along their respective dipaths. Using essentially the same argument as in the proof of Proposition 3 in Gourvès et al. [10], we can show if the target matching is reachable and is not equal to the current matching, then at least one good edge exists. It follows easily that if the target matching is reachable, then we can reach it by repeatedly performing a swap across an arbitrary good edge.

We now discuss how to find good edges efficiently so that the overall running time is $O(n^2)$. We begin by traversing the tree to construct a set containing all of the edges that are good in the initial state. The order of the edges within this set is immaterial, so it can be implemented using a simple data structure such as a list or stack. Then, while the set of good edges is nonempty, we perform the following steps. First, we remove an arbitrary good edge from the set, and perform the associated swap. It is easy to see that the other edges in the good set remain good after the swap. Furthermore, up to two edges can become good as a result of the swap, and a simple local search can be used to identify these edges in constant time. Any newly-identified good edges are added to the set of good edges, and we proceed to the next iteration.

Initialization of the set of good edges takes $O(n)$ time and each iteration of the loop performs one swap and takes constant time. Since the number of swaps is $O(n^2)$, the claimed time bound follows. □

7.2 Algorithms for Stars

In this section, we solve the reachable object and Pareto-efficient matching problems on stars. We refer to the object located at the center of the star as the center object, and to the remaining objects as leaf objects. We refer to the agent that is currently matched to the center object as the center agent O, and to the remaining agents as leaf agents.

7.2.1 Reachable Object on Stars

Theorem 7.2. *Reachable object on stars can be solved in $O(n^2)$ time.*

Proof Sketch. Let a be the given agent and let b the given target object in an instance of the reachable object problem. Notice that there is a unique dipath for a to follow to reach b, and the dipath has at most two edges.

There are three cases for the dipath, from center to leaf, from leaf to center, and from leaf to leaf. The center to leaf case is straightforward as the only way for agent a to reach object b is to perform a single swap involving both of them.

For the leaf to center case, we use the approach presented by Gourvès et al. [10] for the object-moving model. The problem reduces to the search of a path in a digraph $G = (A, E)$ where $(a, a') \in E$ with $a \in A, a' \in A \setminus \{O\}$ if and only if a and a' can rationally trade when a is at the center position and a' is in its initial position. There is a path from O to a in G if and only if a can move to the center position. It takes $O(n^2)$ time to construct G and solve this path problem.

We reduce the leaf to leaf case to the leaf to center case, as follows. First, we solve a leaf to center problem to determine whether agent a can be moved to the center without moving the initial owner of object b. If this is possible, then we check whether agent a and the initial owner of object b can trade their objects. This procedure works correctly because the ownership of any leaf object can change at most once in any valid sequence of swaps. To see this, observe that once an agent moves from the center to a leaf, it can never return to the center. Accordingly, for agent a to move to object b, the initial owner of b needs to remain stationary until a reaches the center. □

7.2.2 Pareto Efficiency on Stars

Theorem 7.3. *Pareto-efficient matching on stars can be solved in $O(n)$ time.*

Proof Sketch. We use an algorithm based on serial dictatorship [1]. First, we prune any leaf agent with its associated object if the agent does not prefer the center object to its own object, since this leaf agent will never be involved in any swap. Therefore, the center agent can swap with any remaining leaf agent only if the leaf agent holds one of its preferred objects. If the top preference of the center agent out of the remaining objects is the center object, then the current matching is Pareto-efficient as no swaps can occur. Otherwise, the top preference of the center agent a out of the remaining objects is a leaf object b, and we can apply the swap operation that moves agent a to object b. The previous owner of object b becomes the new center agent. Then, we prune agent a and object b and recurs on the new center agent. It is easy to see that this whole process takes $O(n)$ time, and when each agent is pruned, it holds its favorite object among the remaining objects. It follows that the matching returned by this process is not Pareto-dominated by any other matching, and hence is Pareto-efficient. □

8 Concluding Remarks

In this paper, we have introduced the agent-moving model, and we have revisited the collection of problems listed in Table 1, which were previously considered in the context of the object-moving model. In all cases where a polynomial-time algorithm or hardness result has been established for the object-moving model, we have established a corresponding result for the agent-moving model.

In addition, we have presented a polynomial-time algorithm for Pareto Efficiency on generalized stars in the agent-moving model, a problem that remains open in the object-moving model. It is natural to ask whether our techniques can be extended to address this open problem. Our algorithm relies on the polynomial-time solvability of the Reachable Object problem for the center agent, which allows us to compute an object that is matched to the center agent in some Pareto-efficient matching. In the object-moving model, no polynomial-time algorithm is known to compute an agent that is matched to the center object in some Pareto-efficient matching. (We do know how to compute the agents that can be reached by the center object in polynomial time, but it isn't clear how to use this information to compute a Pareto-efficient matching in polynomial time.) An interesting direction for future research in the agent-moving model is to determine whether our techniques for solving Pareto Efficiency on generalized stars can be extended to trees. It would also be interesting to study strategic aspects of this model.

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