The Obnoxious Facility Location Game with Dichotomous Preferences

Fu Li C. Gregory Plaxton Vaibhav B. Sinha

September 2021

Abstract

We consider a facility location game in which $n$ agents reside at known locations on a path, and $k$ heterogeneous facilities are to be constructed on the path. Each agent is adversely affected by some subset of the facilities, and is unaffected by the others. We design two classes of mechanisms for choosing the facility locations given the reported agent preferences: utilitarian mechanisms that strive to maximize social welfare (i.e., to be efficient), and egalitarian mechanisms that strive to maximize the minimum welfare. For the utilitarian objective, we present a weakly group-strategyproof efficient mechanism for up to three facilities, we give a strongly group-strategyproof mechanism that guarantees at least half of the optimal social welfare for arbitrary $k$, and we prove that no strongly group-strategyproof mechanism achieves an approximation ratio of $5/4$ for one facility. For the egalitarian objective, we present a strategyproof egalitarian mechanism for arbitrary $k$, and we prove that no weakly group-strategyproof mechanism achieves a $o(\sqrt{n})$ approximation ratio for two facilities. We extend our egalitarian results to the case where the agents are located on a cycle, and we extend our first egalitarian result to the case where the agents are located in the unit square.

1 Introduction

The facility location game (FLG) was introduced by Procaccia and Tannenholtz [21]. In this setting, a central planner wants to build a facility that serves agents located on a path. The agents report their locations, which are fed to a mechanism that decides where the facility should be built. Procaccia and Tannenholtz studied two different objectives that the planner seeks to minimize: the sum of the distances from the facility to all agents and the maximum distance of any agent to the facility.
Every agent aims to maximize their welfare, which increases as their distance to the facility decreases. An agent or a coalition of agents can misreport their location(s) to try to increase their welfare. It is natural to seek strategyproof (SP) or group-strategyproof (GSP) mechanisms, which incentivize truthful reporting. Often such mechanisms cannot simultaneously optimize the planner’s objective. In these cases, it is desirable to approximately optimize the planner’s objective.

In real scenarios, an agent might dislike a certain facility, such as a power plant, and want to stay away from it. This variant, called the obnoxious facility location game (OFLG), was introduced by Cheng et al., who studied the problem of building an obnoxious facility on a path [6]. In the present paper, we consider the problem of building multiple obnoxious facilities on a path. With multiple facilities, there are different ways to define the welfare function. For example, in the case of two facilities, the welfare of the agent can be the sum, minimum, or maximum of the distances to the two facilities. In our work, as all the facilities are obnoxious, a natural choice for welfare is the minimum distance to any obnoxious facility: the closest facility to an agent causes them the most annoyance, and if it is far away, then the agent is satisfied.

A facility might not be universally obnoxious. Consider, for example, a school or sports stadium. An agent with no children might consider a school to be obnoxious due to the associated noise and traffic, while an agent with children might not consider it to be obnoxious. Another agent who is not interested in sports might similarly consider a stadium to be obnoxious. We assume that each agent has dichotomous preferences; they dislike some subset of the facilities and are indifferent to the others. Each agent reports a subset of facilities to the planner. As the dislikes are private information, the reported subset might not be the subset of facilities that the agent truly dislikes. On the other hand, we assume that the agent locations are public and cannot be misreported.

In this paper, we study a variant of FLG, which we call DOFLG (Dichotomous Obnoxious Facility Location Game), that combines the three aspects mentioned above: multiple (heterogeneous) obnoxious facilities, minimum distance as welfare, and dichotomous preferences. We seek to design mechanisms that perform well with respect to either a utilitarian or egalitarian objective. The utilitarian objective is to maximize the social welfare, that is, the total welfare of all the agents. A mechanism that maximizes social welfare is said to be efficient. The egalitarian objective is to maximize the minimum welfare of any agent. For both objectives, we seek mechanisms that are SP, or better yet, weakly or strongly group-strategyproof (WGSP / SGSP).

1.1 Our contributions

We study DOFLG with $n$ agents. In Section 4, we consider the utilitarian objective. We present 2-approximate SGSP mechanisms for any number of facilities when the agents are located on a path, cycle, or square. We obtain the following two additional results for the path setting. In the first main result of the paper, we obtain a mechanism that is WGSP for any number of facilities and efficient for up to three facilities. To show that this mechanism is WGSP, we relate it to
Table 1: Summary of our results for DOFLG when the agents are located on a path. The heading LB (resp., UB) stands for lower (resp., upper) bound. The results in the egalitarian column also hold when the agents are located on a cycle. Boldface results hold when the agents are located on a path, cycle, or square.

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<td>LB</td>
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<td>SP</td>
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<td>WGSP</td>
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a weighted approval voting mechanism. To prove its efficiency, we identify two crucial properties that the welfare function satisfies, and we use an exchange argument. For the path setting, we also show that no SGSP mechanism can achieve an approximation ratio better than $5/4$, even for one facility.

In Section 5, we consider the egalitarian objective. We provide SP mechanisms for any number of facilities when the agents are located on a path, cycle, or square. In the second main result of the paper, we prove that the approximation ratio achieved by any WGSP mechanism is $\Omega(\sqrt{n})$, even for two facilities. Also, we present a straightforward $O(n)$-approximate WGSP mechanism. Both of the results for WGSP mechanisms hold for DOFLG when the agents are located on a path or cycle. Table 1 summarizes our results.

1.2 Related work

FLG was introduced by Procaccia and Tannenholtz [21]. Many generalizations and extensions of FLG have been studied [1, 8, 11, 12, 13, 14, 19, 27]; here we highlight some of the most relevant work. Cheng et al. introduced OFLG and presented a WGSP mechanism to build a single facility on a path [6]. Later they extended the model to cycles and trees [7]. A complete characterization of single-facility SP/WGSP mechanisms for paths has been developed [16, 17]. Duan et al. studied the problem of locating two obnoxious facilities at least distance $d$ apart [9]. Other variants of OFLG have been considered [5, 15, 20, 25].

Agent preferences over the facilities were introduced to FLG in [10] and [28]. Serafino and Ventre studied FLG for building two facilities where each agent likes a subset of the facilities [22]. Anastasiadis and Deligkas extended this model to allow the agents to like, dislike, or be indifferent to the facilities [2]. The aforementioned works address linear (sum) welfare function. Yuan et al. studied non-linear welfare functions (max and min) for building two non-obnoxious facilities [26]; their results have subsequently been strengthened [4, 18]. In the present paper, we initiate the study of a non-linear welfare function (min) for building multiple obnoxious facilities.
2 Preliminaries

The problems considered in this paper involve a set of agents located on a path, cycle, or square. In the path (resp., cycle, square) setting, we assume without loss of generality that the path (resp., cycle, square) is the unit interval (resp., unit-circumference circle, unit square). We map the points on the unit-circumference circle to [0, 1), in the natural manner. Thus, in the path (resp., cycle, square) setting, each agent \( i \) is located in [0, 1] (resp., [0, 1), [0, 1])\(^2\). The distance between any two points \( x \) and \( y \) is denoted \( \Delta(x, y) \). In the path and square settings, \( \Delta(x, y) \) is defined as the Euclidean distance between \( x \) and \( y \). In the cycle setting, \( \Delta(x, y) \) is defined as the length of the shorter arc between \( x \) and \( y \). In all settings, we index the agents from 1. Each agent has a specific location in the path, cycle, or square. A location profile \( \mathbf{x} \) is a vector \((x_1, \ldots, x_n)\) of points, where \( n \) denotes the number of agents and \( x_i \) is the location of agent \( i \). Sections 4.1 and 5.1 (resp., Sections 4.2 and 5.2, Sections 4.3 and 5.3) present our results for the path (resp., cycle, square) setting.

Consider a set of agents 1 through \( n \) and a set of facilities \( \mathcal{F} \), where we assume that each agent dislikes (equally) certain facilities in \( \mathcal{F} \) and is indifferent to the rest. In this context, we define an aversion profile \( \mathbf{a} \) as a vector \((a_1, \ldots, a_n)\) where each component \( a_i \) is a subset of \( \mathcal{F} \). We say that such an aversion profile is true if each component \( a_i \) is equal to the subset of \( \mathcal{F} \) disliked by agent \( i \). In this paper, we also consider reported aversion profiles where each component \( a_i \) is equal to the set of facilities that agent \( i \) claims to dislike. Since agents can lie, a reported aversion profile need not be true. For any aversion profile \( \mathbf{a} \) and any subset \( C \) of agents \([n], \mathbf{a}_C \) (resp., \( \mathbf{a}_{-C} \)) denotes the aversion profile for the agents in (resp., not in) \( C \). For a singleton set of agents \( \{i\} \), we abbreviate \( \mathbf{a}_{-\{i\}} \) as \( \mathbf{a}_{-i} \).

An instance of the dichotomous obnoxious facility location (DOFL) problem is given by a tuple \((n, k, \mathbf{x}, \mathbf{a})\) where \( n \) denotes the number of agents, there is a set of \( k \) facilities \( \mathcal{F} = \{F_1, \ldots, F_k\} \) to be built, \( \mathbf{x} = (x_1, \ldots, x_n) \) is a location profile for the agents, and \( \mathbf{a} = (a_1, \ldots, a_n) \) is an aversion profile (true or reported) for the agents with respect to \( \mathcal{F} \). A solution to such a DOFL instance is a vector \( \mathbf{y} = (y_1, \ldots, y_k) \) where component \( y_j \) specifies the point at which to build \( F_j \). We say that a DOFL instance is true (resp., reported) if the associated aversion profile is true (resp., reported). For any DOFL instance \( I = (n, k, \mathbf{x}, \mathbf{a}) \) and any \( j \) in \([k]\), we define haters\((I, j)\) as \( \{i \in [n] \mid F_j \in a_i\} \), and indiff\((I)\) as \( \{i \in [n] \mid a_i = \emptyset\} \).

For any DOFL instance \( I = (n, k, \mathbf{x}, \mathbf{a}) \) and any associated solution \( \mathbf{y} \), we define the welfare of agent \( i \), denoted \( w(I, i, \mathbf{y}) \), as \( \min_{j: F_j \in a_i} \Delta(x_i, y_j) \), i.e., the minimum distance from \( x_i \) to any facility in \( a_i \). Remark: If \( a_i \) is empty, we define \( w(I, i, \mathbf{y}) \) as \( 1/2 \) in the cycle setting, \( \max(\Delta(x_i, 0), \Delta(x_i, 1)) \) in the path setting, and the maximum distance from \( x_i \) to a corner in the square setting.

The foregoing definition of agent welfare is suitable for true DOFL instances, and is only meaningful for reported DOFL instances where the associated aversion profile is close to true. In this paper, reported aversion profiles arise in the context of mechanisms that incentivize truthful reporting, so it is reason-
able to expect such aversion profiles to be close to true. We define the social welfare (resp., minimum welfare) as the sum (resp., minimum) of the individual agent welfares. When the facilities are built at $y$, the social welfare and minimum welfare are denoted by $SW(I, y)$ and $MW(I, y)$, respectively. Thus $SW(I, y) = \sum_{i \in [n]} w(I, i, y)$ and $MW(I, y) = \min_{i \in [n]} w(I, i, y)$.

**Definition 1.** For $\alpha \geq 1$, a DOFL algorithm $A$ is $\alpha$-efficient if for any DOFL instance $I$,

$$\max_y SW(I, y) \leq \alpha SW(I, A(I)).$$

Similarly, $A$ is $\alpha$-egalitarian if for any DOFL instance $I$,

$$\max_y MW(I, y) \leq \alpha MW(I, A(I)).$$

A 1-efficient (resp., 1-egalitarian) DOFL algorithm, is said to be efficient (resp., egalitarian).

We are now ready to define a DOFL-related game, which we call DOFLG. It is convenient to describe a DOFLG instance in terms of a pair $(I, I')$ of DOFL instances where $I = (n, k, x, a)$ is true and $I' = (n, k, x, a')$ is reported. There are $n$ agents indexed from 1 to $n$, and a planner. There is a set of $k$ facilities $F = \{F_1, \ldots, F_k\}$ to be built. The numbers $n$ and $k$ are publicly known, as is the location profile $x$ of the agents. Each component $a_i$ of the true aversion profile $a$ is known only to agent $i$. Each agent $i$ submits component $a'_i$ of the reported aversion profile $a'$ to the planner. The planner, who does not have access to $a$, runs a DOFL algorithm, call it $A$, to map $I'$ to a solution. The input-output behavior of $A$ defines a DOFLG mechanism, call it $M$; in the special case where $k = 1$, we say that $M$ is a single-facility DOFLG mechanism.

We would like to choose $A$ so that $M$ enjoys strong game-theoretic properties. We say that $M$ is $\alpha$-efficient (resp., $\alpha$-egalitarian, efficient, egalitarian) if $A$ is $\alpha$-efficient (resp., $\alpha$-egalitarian, efficient, egalitarian). As indicated earlier, such properties (which depend on the notion of agent welfare) are only meaningful if the reported aversion profile is close to true. To encourage truthful reporting, we require our mechanisms to be SP, as defined below; we also consider the stronger properties WGSP and SGSP.

The SP property says that no agent can increase their welfare by lying about their dislikes.

**Definition 2.** A DOFLG mechanism $M$ is SP if for any DOFLG instance $(I, I')$ with $I = (n, k, x, a)$, and $I' = (n, k, x, a')$, and any agent $i$ in $[n]$ such that $a'_i = (a_{-i}, a'_i)$, we have

$$w(I, i, M(I)) \geq w(I, i, M(I')).$$

The WGSP property says that if a non-empty coalition $C \subseteq [n]$ of agents lies, then at least one agent in $C$ does not increase their welfare.
Definition 3. A DOFLG mechanism $M$ is WGSP if for any DOFLG instance $(I, I')$ with $I = (n, k, x, a)$, and $I' = (n, k, x, a')$, and any non-empty coalition $C \subseteq [n]$ such that $a' = (a - C, a'_C)$, there exists an agent $i$ in $C$ such that

$$w(I, i, M(I)) \geq w(I, i, M(I')).$$

The SGSP property says that if a coalition $C \subseteq [n]$ of agents lies and some agent in $C$ increases their welfare then some agent in $C$ decreases their welfare.

Definition 4. A DOFLG mechanism $M$ is SGSP if for any DOFLG instance $(I, I')$ with $I = (n, k, x, a)$, and $I' = (n, k, x, a')$, and any coalition $C \subseteq [n]$ such that $a' = (a - C, a'_C)$, if there exists an agent $i$ in $C$ such that

$$w(I, i, M(I)) < w(I, i, M(I')),$$

then there exists an agent $i'$ in $C$ such that

$$w(I, i', M(I)) > w(I, i', M(I')).$$

Every SGSP mechanism is WGSP and every WGSP mechanism is SP.

3 Weighted Approval Voting

Before studying efficient mechanisms for our problem, we review a variant of the approval voting mechanism [3]. An instance of Dichotomous Voting (DV) is a tuple $(m, n, C, w^+, w^-)$ where $m$ voters $1, \ldots, m$ have to elect a candidate among the set of candidates $C = \{c_1, \ldots, c_n\}$. Each voter $i$ has dichotomous preferences, that is, voter $i$ partitions all of the candidates into two equivalence classes: a top (most preferred) tier $C_i$ and a bottom tier $C_i = C \setminus C_i$. Each voter $i$ has associated (and publicly known) weights $w_i^+ \geq w_i^- \geq 0$. The symbols $C$, $w^+$, and $w^-$ denote length-$m$ vectors with $i$th element $C_i$, $w_i^+$, and $w_i^-$, respectively. We now present our weighted approval voting mechanism.\(^1\)

Mechanism 1. Given a DV instance $(m, n, C, w^+, w^-)$, every voter $i$ votes by partitioning $C$ into $C_i'$ and $C_i''$. Let the weight function $w$ be such that for voter $i$ and candidate $c_j$, $w(i, j) = w_i^+$ if $c_j$ is in $C_i'$ and $w(i, j) = w_i^-$ otherwise. For any $j$ in $[n]$, we define $A(j) = \sum_{i \in [m]} w(i, j)$ as the approval of candidate $c_j$. The candidate $c_j$ with highest approval $A(j)$ is declared the winner. Ties are broken according to a fixed ordering of the candidates (e.g., in favor of lower indices).

We note that the approval voting mechanism can be obtained from the weighted approval voting mechanism by setting weights $w_i^+$ to 1 and $w_i^-$ to 0 for all voters $i$. In Section 2, we defined SP, WGSP, and SGSP in the DOFLG setting. These definitions are easily generalized to the voting setting. Brams

\(^1\)Our mechanism differs from the homonymous mechanism of Massó et al., which has weights for the candidates instead of the voters [24].
and Fishburn proved that the approval voting mechanism is SP [3]. Below we prove that our weighted approval voting mechanism is WGSP (and hence also SP).

**Theorem 1.** Mechanism 1 is WGSP.

**Proof.** Assume for the sake of contradiction that there is an instance in which a coalition of voters $U$ with true preferences $\{(C_i, \overline{C}_i)\}_{i \in U}$ all benefit by misreporting their preferences as $\{(C'_i, \overline{C}'_i)\}_{i \in U}$. For any candidate $c_j$, let $A(j)$ denote the approval of $c_j$ when coalition $U$ reports truthfully, and let $A'(j)$ denote the approval of $c_j$ when coalition $U$ misreports.

Let $c_k$ be the winning candidate when coalition $U$ reports truthfully, and let $c_\ell$ be the winning candidate when coalition $U$ misreports. Since every voter in $U$ benefits when the coalition misreports, we know that $c_k$ belongs to $\bigcap_{i \in U} \overline{C}_i$ and $c_\ell$ belongs to $\bigcap_{i \in U} C_i$.

Since $c_k$ belongs to $\bigcap_{i \in U} \overline{C}_i$, we deduce that $A'(k) = A(k) + \sum_{i \in U: c_k \in C_i} w_i^+ - w_i^-$ and hence $A'(k) \geq A(k)$. Similarly, since $c_\ell$ belongs to $\bigcap_{i \in U} C_i$, we deduce that $A'(\ell) = A(\ell) + \sum_{i \in U: c_\ell \in C_i} w_i^- - w_i^+$ and hence $A(\ell) \geq A'(\ell)$.

Since $c_k$ wins when coalition $U$ truthfully, one of the following two cases is applicable.

**Case 1:** $A(k) > A(\ell)$. Since $A'(k) \geq A(k)$ and $A(\ell) \geq A'(\ell)$, the case condition implies that $A'(k) > A'(\ell)$. Hence $c_\ell$ does not win when coalition $U$ misreports, a contradiction.

**Case 2:** $A(k) = A(\ell)$ and $c_k$ has higher priority than $c_\ell$. Since $A'(k) \geq A(k)$ and $A(\ell) \geq A'(\ell)$, the case condition implies that $A'(k) \geq A'(\ell)$ and $c_k$ has higher priority than $c_\ell$. Hence $c_\ell$ does not win when coalition $U$ misreports, a contradiction. \qed

**Theorem 2.** Mechanism 1 is not SGSP.

The above theorem can be established by adapting the instance shown in Section 4 to prove that Mechanism 2 is not SGSP.

4 Efficient Mechanisms

4.1 Efficient mechanisms for the unit interval

We now present our efficient mechanism for DOFLG.

**Mechanism 2.** For a given reported DOFL instance $I = (n, k, x, a)$, output the lexicographically least solution $y$ in $\{0, 1\}^k$ that maximizes the social welfare $SW(I, y)$.

**Theorem 3.** Mechanism 2 is WGSP.

**Proof.** To establish this theorem, we show that Mechanism 2 can be equivalently expressed in terms of the approval voting mechanism. Hence Theorem 1 implies the theorem.
Let \((I, I')\) denote a DOFLG instance where \(I = (n, k, x, a)\) and \(I' = (n, k, x, a')\). We view each agent \(i \in [n]\) as a voter, and each \(y\) in \([0, 1]^k\) as a candidate. We obtain the top-tier candidates \(C_i\) of voter \(i\), and their reported top-tier candidates \(C'_i\), from \(a_i\) and \(a'_i\), respectively. Assume without loss of generality that \(x_i \leq 1/2\) (the other case can be handled similarly). Set \(C_i = \{y = (y_1, \ldots, y_k) \in [0, 1]^k \mid y_j = 1\ \text{for all}\ F_j \in a_i\}\) and similarly \(C'_i = \{y = (y_1, \ldots, y_k) \in [0, 1]^k \mid y_j = 1\ \text{for all}\ F_j \in a'_i\}\). Also set \(w^+_i = 1 - x_i\) and \(w^-_i = x_i\). With this notation, it is easy to see that \(A(y) = SW(I', y)\), and that choosing the \(y\) with the highest social welfare in Mechanism 2 is the same as electing the candidate with the highest approval in Mechanism 1. □

We show that Mechanism 2 is efficient for \(k = 3\). First, we note a well-known result about the 1-Maxian problem. In this problem, there are \(n\) points located at \(z_1, \ldots, z_n\) in the interval \([a, b]\), and the task is to choose a point in \([a, b]\) such that the sum of the distances from that point to all \(z_i\)s is maximized.

**Lemma 1** (Optimality of the 1-Maxian Problem). Let \([a, b]\) be a real interval, let \(z_1, \ldots, z_n\) belong to \([a, b]\), and let \(f(z)\) denote \(\sum_{i \in [n]} |z - z_i|\). Then \(\max_{z \in [a, b]} f(z)\) belongs to \(\{f(a), f(b)\}\).

Before proving the main theorem, we establish Lemma 2, which follows from Lemma 1.

**Lemma 2.** Let \(I = (n, k, x, a)\) denote the reported DOFL instance, let \(Y\) denote the set of all \(y\) in \([0, 1]\) such that it is efficient to build all \(k\) facilities at \(y\), and assume that \(Y\) is non-empty. Then \(Y \cap \{0, 1\}\) is non-empty.

**Proof.** Let \(U\) denote \(\text{indiff}(I)\). When all of the facilities are built at \(y\),

\[
\text{SW}(I, (y, \ldots, y)) = \sum_{i \in [n] \setminus U} |x_i - y| + \sum_{i \in U} w(I, i, y).
\]

Since \(Y\) is non-empty, \(\max_y \text{SW}(I, (y, \ldots, y)) = \max_y \text{SW}(I, y)\). Moreover, since \(\sum_{i \in U} w(I, i, y)\) does not depend on \(y\), Lemma 1 implies that

\[
\max_y (\text{SW}(I, (0, \ldots, 0)), \text{SW}(I, (1, \ldots, 1))) = \max_y \text{SW}(I, (y, \ldots, y))
\]

Thus, if \(\text{SW}(I, (0, \ldots, 0)) \geq \text{SW}(I, (1, \ldots, 1))\), it is efficient to build all \(k\) facilities at 0. Otherwise, it is efficient to build all \(k\) facilities at 1. □

**Theorem 4.** Mechanism 2 is efficient for \(k = 3\).

**Proof.** Let \(I = (n, k, x, a)\) denote the reported DOFL instance and let \(y^* = (y_1, y_2, y_3)\) be an efficient solution for \(I\) such that \(y_1 \leq y_2 \leq y_3\).

Consider fixing variables \(y_1\) and \(y_3\) in the social welfare function \(\text{SW}(I, y)\). That is, we have

\[
\text{SW}(I, y)|_{y_1 = y^*_1, y_3 = y^*_3} = \sum_{i \in [n]} w(I, i, y)|_{y_1 = y^*_1, y_3 = y^*_3}.
\]
For convenience, let $SW(y_2)$ denote $SW(I,y)|_{y_1=y_1',y_3=y_3'}$ and let $w_i(y_2)$ denote $w(I,i,y)|_{y_1=y_1',y_3=y_3'}$ for each agent $i$.

Claim 1: For each agent $i$, the welfare function $w_i(y_2)$ with $y_2 \in [y_1',y_3']$ satisfies at least one of the following two properties:

1. $w_i(y_2) = |y_2 - x_i|$;
2. $w_i(y_1') = w_i(y_3') = \max_{y \in [y_1',y_3']} w_i(y)$.

Proof: Consider an agent $i$. We consider five cases.

Case 1: $F_2 \notin a_i$. Since the welfare of agent $i$ is independent of the location of $F_2$, $w_i$ is a constant function. Hence property 2 is satisfied.

Case 2: $a_i = \{F_2\}$. By definition, we have $w_i(y_2) = |y_2 - x_i|$. Hence property 1 is satisfied.

Case 3: $a_i = \{F_1,F_2\}$. By definition, we have $w_i(y_2) = \min(|y_1' - x_i|,|y_2 - x_i|)$. Notice that $w_i(y_1') = \min(|y_1' - x_i|,|y_1' - x_i|) = |y_1' - x_i| = \max_{y \in [y_1',y_3']} w_i(y)$.

Moreover, $w_i(y_3') = \min(|y_3' - x_i|,|y_3' - x_i|)$. We consider two cases.

Case 3.1: $|y_1' - x_i| > |y_3' - x_i|$. Then $w_i(y_3') = |y_3' - x_i|$ and hence $w_i(y_2) = |y_2 - x_i|$ for all $y_2$ in $[y_1',y_3']$, that is, $w_i(y_2)$ satisfies property 1.

Case 3.2: $|y_1' - x_i| \leq |y_3' - x_i|$. Then $w_i(y_2) = |y_1' - x_i| = \max_{y \in [y_1',y_3']} w_i(y) = w_i(y_1')$ and hence $w_i(y_2)$ satisfies property 2.

Case 4: $a_i = \{F_2,F_3\}$. This case is symmetric to Case 3 and can be handled similarly.

Case 5: $a_i = \{F_1,F_2,F_3\}$. By definition, we have $w_i(y_2) = \min(|y_1' - x_i|,|y_2 - x_i|,|y_3' - x_i|)$. Notice that $w_i(y_1') = w_i(y_3') = \min(|y_1' - x_i|,|y_3' - x_i|)$.

Also notice that for any $y_2$ in $[y_1',y_3']$, $w_i(y_2) = \min(|y_1' - x_i|,|y_2 - x_i|,|y_3' - x_i|) \leq \min(|y_1' - x_i|,|y_3' - x_i|) = w_i(y_1')$. Hence property 1 holds.

This concludes our proof of Claim 1.

Claim 2: There is a solution that optimizes $\max_y SW(I,y)$ and builds facilities in at most two locations.

Proof: We establish the claim by proving that either $SW(I,(y_1',y_1',y_3')) \geq SW(I,y^*)$ or $SW(I,(y_1',y_2',y_3')) \geq SW(I,y^*)$.

Claim 1 implies that the set of agents $[n]$ can be partitioned into two sets $(S,\overline{S})$ such that $w_i(y_2)$ satisfies property 1 for all $i \in S$, and $w_i(y_2)$ satisfies property 2 for all $i \in \overline{S}$. Thus, we have $SW(y_2) = \sum_{i \in [n]} w_i(y_2) = \sum_{i \in S} w_i(y_2) + \sum_{i \in \overline{S}} w_i(y_2)$. By Lemma 1, there is a $b$ in $\{y_1',y_3'\}$ such that $\sum_{i \in S} w_i(b) \geq \sum_{i \in S} w_i(y_2)$ for all $y_2$ in $[y_1',y_3']$. For any $i \in \overline{S}$, we deduce from property 2 that $w_i(b) \geq w_i(y_2)$ for all $y_2$ in $[y_1',y_3']$. Therefore, $SW(b) \geq SW(y_2)$ for all $y_2$ in $[y_1',y_3']$. This completes our proof of Claim 2.

Having established Claim 2, we can assume without loss of generality that $y_2^* = y_3^*$. A similar argument as above can be used to prove that either $(0,y_2^*,y_2^*)$ or $(y_2^*,y_2^*,y_2^*)$ is an efficient solution. Now if $(0,y_2^*,y_2^*)$ is efficient, then one can use a similar argument to prove that either $(0,0,0)$ or $(0,1,1)$ is efficient. And if $(y_2^*,y_2^*,y_2^*)$ is efficient, then by applying Lemma 2 with $k = 3$, we deduce that either $(0,0,0)$ or $(1,1,1)$ is efficient. Thus, there is a 0-1 efficient solution. The efficiency of Mechanism 2 follows. □
When $k = 2$ (resp., 1), we can add one (resp., two) dummy facilities and use Theorem 4 to establish that Mechanism 2 is efficient for $k = 2$ (resp., 1). Theorem 5 below provides a lower bound on the approximation ratio of any SGSP efficient mechanism; this result implies that Mechanism 2 is not SGSP.

**Theorem 5.** There is no SGSP $\alpha$-efficient DOFLG mechanism with $\alpha < 5/4$.

**Proof.** Let $n$ be a large even integer. We construct two $(3n + 1)$-agent single-facility DOFLG instances $(I, I)$ and $(I, I')$. In both $(I, I)$ and $(I, I')$, agent 1 is located at 0 and dislikes $\{F_1\}$, $n/2$ agents are located at 1 and dislike $\{F_1\}$, and the remaining $n$ agents, which we denote by the set $U$, are located at 0 and dislike $\emptyset$. In $I$, all agents report truthfully, while in $I'$, all agents in $U$ report $\{F_1\}$ and the remaining agents report truthfully.

Let the maximum social welfare for instances $I$ and $I'$ be OPT and OPT', respectively. It is easy to see that OPT $= 3n/2$ and OPT' $= n + 1$ (obtained by building $F_1$ at 0 and 1, respectively). Let the social welfare achieved by some SGSP DOFLG mechanism $M$ on these instances be ALG and ALG', respectively.

Let $M$ build $F_1$ at $y$ on $I$. It follows that $\text{ALG} = y + \frac{3n}{2} - \frac{ny}{2}$. If the agents in $U$ and agent 1 form a coalition in $I$ and the agents in $U$ report $\{F_1\}$, then the instance becomes $I'$. Thus, as $M$ is SGSP, $M$ cannot build $F_1$ to the right of $y$ in $I'$. Using this fact, it is easy to see that $\text{ALG}' \leq (n+1)y + \frac{n}{2}(1-y) = \frac{ny}{2} + \frac{n}{2} + y$.

Using $\text{OPT} = \frac{3n}{2}$ and $\text{ALG} = y + \frac{3n}{2} - \frac{ny}{2}$, we obtain

$$\alpha \geq \frac{3n}{y + \frac{3n}{2} - \frac{ny}{2}}. \quad (1)$$

Similarly, using $\text{OPT}' = n + 1$ and $\text{ALG}' \leq \frac{ny}{2} + \frac{n}{2} + y$, we obtain

$$\alpha \geq \frac{n + 1}{\frac{ny}{2} + \frac{n}{2} + y}. \quad (2)$$

Let $f(y)$ denote

$$\max \left( \frac{\frac{3n}{2}}{y + \frac{3n}{2} - \frac{ny}{2}}, \frac{n + 1}{\frac{ny}{2} + \frac{n}{2} + y} \right).$$

From (1) and (2) we deduce that $\alpha \geq f(y)$. Let $y^*$ denote a value of $y$ in $[0, 1]$ minimizing $f(y)$. It is easy to verify that $y^*$ satisfies $f(y^*) = \frac{5n^2 + 4n - 4}{4(n(n + 1))}$. Thus, $\alpha \geq f(y^*)$. As $n$ approaches infinity, $f(y^*)$ approaches $5/4$. Thus, for any SGSP $\alpha$-efficient mechanism, we have $\alpha \geq 5/4$. \qed

In view of Theorem 5, it is natural to try to determine the minimum value of $\alpha$ for which an SGSP $\alpha$-efficient DOFLG mechanism exists. Below we present a 2-efficient SGSP mechanism. It remains an interesting open problem to improve the approximation ratio of 2, or to establish a tighter lower bound for the approximation ratio.

**Mechanism 3.** Let $(n, k, x, a)$ denote the reported DOFL instance. Build all facilities at 0 if $\sum_{i \in [n]} x_i \geq \sum_{i \in [n]} (1 - x_i)$; otherwise, build all facilities at 1.
Theorem 6. Mechanism 3 is SGSP.

Proof. Reported dislikes do not affect the locations at which the facilities are built. Hence the theorem follows. □

Theorem 7. Mechanism 3 is 2-efficient.

Proof. Let \( I = (n, k, x, a) \) denote the reported DOFL instance. Let \( ALG \) denote the social welfare obtained by Mechanism 3 on this instance, and let \( OPT \) denote the maximum possible social welfare on this instance. We need to prove that \( 2 \cdot ALG \geq OPT \).

Assume without loss of generality that Mechanism 3 builds all facilities at 0. (A symmetric argument handles the case where all facilities are built at 1). Then the welfare of an agent \( i \) not in indiff\( (I) \) is \( x_i \) and the welfare of an agent \( i' \) in indiff\( (I) \) is \( \max(x_{i'}, 1-x_{i'}) \geq x_{i'} \). Thus, \( ALG \geq \sum_{i \in [n]} x_i \). As Mechanism 3 builds the facilities at 0 and not 1, we have \( \sum_{i \in [n]} x_i \geq \sum_{i \in [n]} (1-x_i) \), which implies that \( \sum_{i \in [n]} x_i \geq n/2 \). Combining the above two inequalities, we have \( ALG \geq n/2 \). Since no agent has welfare greater than 1, we have \( n \geq OPT \). Thus, \( 2 \cdot ALG \geq n \geq OPT \), as required. □

We now establish that the analysis of Theorem 7 is tight by exhibiting a two-facility DOFL instance on which Mechanism 3 achieves half of the optimal social welfare. For the reported DOFL instance \( I = (2, 2, (0, 1), (\{F_1\}, \{F_2\})) \), it is easy to verify that the optimal social welfare is \( SW(I, (1, 0)) = 2 \), while the social welfare obtained by Mechanism 3 is \( SW(I, (0, 0)) = 1 \).

4.2 Efficient mechanism for the cycle

Now we present a simple adaptation of Mechanism 3 to the case where the agents are located on a cycle.

Mechanism 4. Let \( (n, k, x, a) \) denote the reported DOFL instance. Build all facilities at 0 if
\[
\sum_{i \in [n]} \Delta(x_i, 0) \geq \sum_{i \in [n]} \Delta(x_i, 1/2);
\]
otherwise, build all facilities at 1/2.

As with Mechanism 3, reported dislikes do not affect the locations at which Mechanism 4 builds the facilities. Hence Mechanism 4 is SGSP.

Theorem 8. Mechanism 4 is SGSP.

Theorem 9. Mechanism 4 is 2-efficient.

Proof. We sketch a proof that is similar to our proof of Theorem 7. Let \( I = (n, k, x, a) \) denote the reported DOFL instance. Let \( ALG \) denote the social welfare obtained by Mechanism 4 on this instance, and let \( OPT \) denote the
maximum possible social welfare on this instance. We need to prove that $2 \cdot ALG \geq OPT$.

Assume without loss of generality that Mechanism 4 builds all facilities at 0. (A symmetric argument handles the case where all facilities are built at 1/2). Using similar arguments, we obtain $ALG \geq \sum_{i \in [n]} \Delta(x_i, 0)$. Also, we have $\sum_{i \in [n]} \Delta(x_i, 0) \geq \sum_{i \in [n]} \Delta(x_i, 1/2)$, and $\Delta(x_i, 0) + \Delta(x_i, 1/2) \geq 1/2$ for all agents $i$, implying that $\sum_{i \in [n]} \Delta(x_i, 0) \geq n/4$. Thus $ALG \geq n/4$. Since no agent has welfare greater than $1/2$, we have $n/2 \geq OPT$. Thus, $2 \cdot ALG \geq n/2 \geq OPT$, as required.

### 4.3 Efficient mechanism for the unit square

We now show how to adapt Mechanism 3 to the case where the agents are located in the unit square.

**Mechanism 5.** Let $(n, k, x, a)$ denote the reported DOFL instance. For each point $p$ in $\{0, 1\}^2$, let $d_p$ denote $\sum_{i \in [n]} \Delta(x_i, p)$. Let $q$ be the point in $\{0, 1\}^2$ that maximizes $d_q$, breaking ties lexicographically. Build all facilities at $q$.

As in the case of Mechanism 3, reported dislikes do not affect the locations at which Mechanism 5 builds the facilities. Hence Mechanism 5 is SGSP.

**Theorem 10.** Mechanism 5 is SGSP.

**Theorem 11.** Mechanism 5 is 2-efficient.

**Proof.** We sketch a proof that is similar to our proof of Theorem 7. Let $I = (n, k, x, a)$ denote the reported DOFL instance. Let $ALG$ denote the social welfare obtained by Mechanism 5 on this instance, and let $OPT$ denote the maximum possible social welfare on this instance. We need to prove that $2 \cdot ALG \geq OPT$.

Assume without loss of generality that Mechanism 5 builds all facilities at $(0, 0)$. (A symmetric argument handles other cases). Using similar arguments, we obtain $ALG \geq \sum_{i \in [n]} \Delta(x_i, (0, 0))$. Also, we have

$$\sum_{i \in [n]} \Delta(x_i, (0, 0)) \geq \max_{p \in \{(0,1),(1,0),(1,1)\}} \sum_{i \in [n]} \Delta(x_i, p),$$

and

$$\Delta(x_i, (0, 0)) + \Delta(x_i, (0, 1)) + \Delta(x_i, (1, 0)) + \Delta(x_i, (1, 1)) \geq 2\sqrt{2}$$

for all agents $i$, implying that $\sum_{i \in [n]} \Delta(x_i, (0, 0)) \geq n/\sqrt{2}$. Thus $ALG \geq n/\sqrt{2}$. Since no agent has welfare greater than $\sqrt{2}$, we have $\sqrt{2}n \geq OPT$. Thus, $2 \cdot ALG \geq \sqrt{2}n \geq OPT$, as required. \qed
5 Egalitarian Mechanisms

We now design egalitarian mechanisms for DOFLG when the agents are located on an interval, cycle, or square.

In Definition 5 below, we introduce a simple way to convert a single-facility DOFLG mechanism into a DOFLG mechanism. Observe that for single-facility DOFLG mechanisms, specifying the input DOFL instance $I = (n, 1, x, a)$ is equivalent to specifying $(n, 1, x, \text{haters}(I, 1))$.

**Definition 5.** For any single-facility DOFLG mechanism $M$, $\text{Parallel}(M)$ denotes the DOFLG mechanism that takes as input a DOFL instance $I = (n, k, x, a)$ and outputs $y = (y_1, \ldots, y_k)$, where $y_j$ is the location at which $M$ builds the facility on input $(n, 1, x, \text{haters}(I, j))$.

Lemmas 3 and 4 below reduce the task of designing a SP egalitarian DOFLG mechanism to the single-facility case.

**Lemma 3.** Let $M$ be a SP single-facility DOFLG mechanism. Then $\text{Parallel}(M)$ is a SP DOFLG mechanism.

*Proof.* Let $(I, I')$ denote a DOFLG instance with $I = (n, k, x, a)$, $I' = (n, k, x, a')$, and let $i$ be an agent such that $a'_i = (a_i - i, a'_i)$. Let $y = (y_1, \ldots, y_k)$ (resp., $y' = (y'_1, \ldots, y'_k)$) denote $\text{Parallel}(M)(I)$ (resp., $\text{Parallel}(M)(I')$). Since $M$ is SP, we have $\Delta(x_i, y_j) \geq \Delta(x_i, y'_j)$ for each facility $F_j$ in $a_i$. Thus $w(I, i, y) \geq w(I, i, y')$, implying that agent $i$ does not benefit by reporting $a'_i$ instead of $a_i$. \hfill $\square$

**Lemma 4.** Let $M$ be an egalitarian single-facility DOFLG mechanism. Then $\text{Parallel}(M)$ is an egalitarian DOFLG mechanism.

*Proof.* Let $I = (n, k, x, a)$ denote the reported DOFL instance. Let an optimal solution be $y^* = (y'_1, \ldots, y'_k)$ and let the optimal (maximum) value of the minimum welfare be $\text{OPT} = \text{MW}(I, y^*)$. Let $\text{Parallel}(M)$ build the facilities at $y' = (y'_1, \ldots, y'_k)$, resulting in a minimum welfare $\text{ALG} = \text{MW}(I, y')$. For each facility $F_j$, we define $\text{OPT}_j$ (resp., $\text{ALG}_j$), as the distance from $y'_j$ (resp., $y_j$) to the nearest agent in $\text{haters}(I, j)$ (or $\infty$ if $\text{haters}(I, j)$ is empty).

We have,

$$\text{OPT} = \min \left( \min_{j} \min_{i \in \text{indiff}(I)} w(I, i, y^*) \right)$$

and

$$\text{ALG} = \min \left( \min_{j} \min_{i \in \text{indiff}(I)} w(I, i, y') \right).$$

Since $M$ is egalitarian, we have $\text{OPT}_j = \text{ALG}_j$ for all $j$. The welfare of agents in indiff$(I)$ does not depend on the locations of the facilities. Thus, $\text{ALG} = \text{OPT}$ implying that $\text{Parallel}(M)$ is egalitarian. \hfill $\square$
5.1 Egalitarian mechanisms for the unit interval

We begin by describing a SP egalitarian mechanism for single-facility DOFLG when the agents are located in the unit interval.

**Mechanism 6.** Let \( I = (n, 1, x, a) \) denote the reported DOFL instance and let \( H \) denote haters(\( I, 1 \)). If \( H \) is empty, build \( F_1 \) at 0. Otherwise, let \( H \) contain \( \ell \) agents \( z_1, \ldots, z_\ell \) such that \( x_{z_1} \leq x_{z_2} \leq \cdots \leq x_{z_\ell} \). Let \( d_1 = x_{z_1} \) and \( d_3 = 1 - x_{z_\ell} \). If \( \ell = 1 \), then build \( F_1 \) at 0 if \( d_1 \geq d_3 \), and at 1 otherwise. If \( \ell > 1 \), let \( m \) be the midpoint of the leftmost largest interval between consecutive agents in \( H \). Formally, \( m = (x_{z_o} + x_{z_{o+1}})/2 \), where \( o \) is the smallest number in \( [\ell - 1] \) such that \( x_{z_{o+1}} - x_{z_o} = \max_{j \in [\ell - 1]}(x_{z_{j+1}} - x_{z_j}) \). Let \( d_2 = m - x_{z_o} \). Then build facility \( F_1 \) at 0 if \( d_1 \geq d_2 \) and \( d_1 \geq d_3 \), at \( m \) if \( d_2 \geq d_3 \), and at 1 otherwise.

The following lemma shows that Mechanism 6 is SP. It is established by examining how the location of the facility changes when an agent misreports.

**Lemma 5.** Mechanism 6 is SP for single-facility DOFLG.

*Proof.* Let \((I, I')\) denote a single-facility DOFL instance with \( I = (n, 1, x, a) \), \( I' = (n, 1, x, a') \), and let \( i \) be an agent such that \( a' = (a_{-i}, a'_i) \). If \( F_1 \) does not belong to \( a_i \), the welfare of agent \( i \) is independent of the location of \( F_1 \) and agent \( i \) does not benefit by reporting \( a'_i \). Moreover, if \( F_1 \) is in \( a_i \cap a'_i \), then the location of \( F_1 \) does not change by reporting \( a'_i \) instead of \( a_i \), and again, agent \( i \) does not benefit by reporting \( a'_i \). Thus for the remainder of the proof, we assume that \( F_1 \) is in \( a_i \setminus a'_i \).

Let Mechanism 6 build \( F_1 \) at \( y \) when agent \( i \) reports truthfully. We assume that \( y < x_i \) (the other case can be handled symmetrically). Let \( H = \text{haters}(I, 1) \) denote the set of agents who dislike \( \{F_1\} \). Note that Mechanism 6 does not build \( F_1 \) at the location of any agent in \( H \), that is, \( y \neq x_{i'} \) for any \( i' \) in \( H \). Let \( d_1 \), \( d_2 \), and \( d_3 \) be as defined in Mechanism 6 when all agents report truthfully. We consider two cases based on whether there is an agent from \( H \) between \( y \) and \( x_i \).

Case 1: No agent in \( H - i \) is located in \([y, x_i]\). We consider two cases.

Case 1.1: \( y = 0 \). Thus \( d_1 = x_i \). When agent \( i \) reports \( a'_i \), \( F_1 \) is built at 0, which does not benefit agent \( i \).

Case 1.2: \( y \neq 0 \). Thus \( d_2 = x_i - y \), there is an agent \( i' \) in \( H \) at \( y - d_2 \), and there are no agents in \( H \) in \([y - d_2, y + d_2]\). We consider two cases.

Case 1.2.1: No agent in \( H \) is located to the right of \( x_i \). Hence \( x_i \geq 1 - d_2 \). Thus when agent \( i \) reports \( a'_i \), \( F_1 \) is built at 1, which does not benefit agent \( i \).

Case 1.2.2: There is an agent in \( H \) located to the right of \( x_i \). Let \( i'' \) be the first agent to the right of \( x_i \), breaking ties arbitrarily. Then \( x_{i''} - x_i \leq 2d_2 \). Thus when agent \( i \) reports \( a'_i \), \( F_1 \) is built in \([y, x_i]\), which does not benefit agent \( i \).

Case 2: There is an agent in \( H - i \) in \([y, x_i]\). Let \( i' \) be the first agent to the right of \( y \) in \( H - i \). Let \( d \) denote \( d_1 = x_{i'} \) if \( y = 0 \), and \( d_2 = x_{i'} - y \) otherwise. We deduce that \( x_i - y \geq d \). We consider two cases.
Case 2.1: No agent in $H$ is located to the right of $x_i$. Hence $x_i \geq 1 - d$. Thus when agent $i$ reports $a'_i$, $F_1$ is either built at $y$ or at $1$, neither of which benefits agent $i$.

Case 2.2: There is an agent in $H$ located to the right of $x_i$. Let $b$ be the first agent to the right of $x_i$, breaking ties arbitrarily. Let agent $a$ be the agent located in $[0, x_i]$ that is closest to agent $i$, breaking ties arbitrarily. We deduce that $x_i - x_a \leq 2d$ and $x_b - x_i \leq 2d$. When agent $i$ reports $a'_i$, $F_1$ is built at $y$ or in $[x_i - d, x_i + d]$, neither of which benefits agent $i$.

Thus agent $i$ does not benefit by reporting $a'_i$.

Lemma 6. Mechanism 6 is egalitarian for single-facility DOFLG.

Proof. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance and let $H$ denote haters$(I, 1)$. The welfare of any agent in $[n] \setminus H$ is independent of the location of the facility. Thus, a mechanism is egalitarian if it maximizes the minimum welfare of any agent in $H$. Mechanism 6 ignores the agents that are not in $H$. Thus it is sufficient to show that Mechanism 6 maximizes the minimum welfare on instances where all agents belong to $H$. Hence for the remainder of the proof, we assume that all agents belong to $H$. Let $y^*$ denote an optimal location for the facility and let OPT denote MW$(I, y^*)$. Let $y'$ denote the location where Mechanism 6 builds the facility and let ALG denote MW$(I, y')$. Below we establish that $ALG \geq OPT$, which implies that Mechanism 6 is egalitarian.

If $H$ is empty then trivially Mechanism 6 is egalitarian. For the remainder of the proof, assume that $H$ is non-empty. We say that an agent in $H$ is tight if it is as close to $y^*$ as any other agent in $H$. Thus for any tight agent $i$, $OPT = |y^* - x_i|$. Similarly, ALG is the distance between $y'$ and a closest agent in $H$.

If $y^* = 0$, consider any tight agent $i$. Then no agent in $H$ is located in $[0, x_i)$. It follows that $d_1 = x_i = OPT$. As $ALG \geq d_1$, we have $ALG \geq OPT$. A symmetric argument can be made for the case $y^* = 1$.

It remains to consider the case where $y^*$ does not belong to $\{0, 1\}$. Assume that $x_i < y^*$ where $i$ is a tight agent (the other case can be handled symmetrically). We have $OPT = y^* - x_i$. Thus no agent in $H$ is located in $(x_i = y^* - OPT, y^* + OPT)$. We consider two cases.

Case 1: There is no agent $i'$ to the right of $y^*$. Thus $d_3 \geq OPT$. Since $ALG \geq d_3$, we have $ALG \geq OPT$.

Case 2: There is an agent in $H$ to the right of $y^*$. Consider the leftmost such agent $i'$. Since $x_{i'} \geq y^* + OPT$, we have $d_2 \geq OPT$. Since $ALG \geq d_2$, we have $ALG \geq OPT$.

We define Mechanism 7 as the DOFLG mechanism Parallel$(M)$, where $M$ denotes Mechanism 6. Using Lemmas 3 through 6, we immediately obtain Theorem 12 below.

Theorem 12. Mechanism 7 is SP and egalitarian.

Below we provide a lower bound on the approximation ratio of any WGSP egalitarian mechanism. Theorem 13 implies that Mechanism 7 is not WGSP.
\textbf{Theorem 13.} Let $M$ be a WGSP $\alpha$-egalitarian mechanism. Then $\alpha$ is $\Omega(\sqrt{n})$, where $n$ is the number of agents.

\textit{Proof.} Let $q$ be a large even integer, let $p$ denote $q^2 + 1$, and let $U$ (resp., $V$) denote the set of all integers $i$ such that $0 < i < q^2/2$ (resp., $q^2/2 < i < q^2$).

We construct two $(p + 3)$-agent two-facility DOFLG instances $(I, I)$ and $(I, I')$. In both $(I, I)$ and $(I, I')$, there is an agent located at $i/q^2$, called agent $i$, for each $i$ in $U \cup V$, and there are two agents each at 0, 1/2, and 1. In $I$, each agent $i$ such that $i$ is in $U$ dislikes $\{F_2\}$, each agent $i$ such that $i$ is in $V$ dislikes $\{F_1\}$, one agent at 0 (resp., 1/2, 1) dislikes $\{F_1\}$, and the other agent at 0 (resp., 1/2, 1) dislikes $\{F_2\}$. In $I'$, agents $i$ such that $i$ is in $U \setminus \{1, \ldots, q - 1\}$, have alternating reports: agent $q$ reports $\{F_1\}$, agent $q + 1$ reports $\{F_2\}$, agent $q + 2$ reports $\{F_1\}$, and so on. Symmetrically, the agents $i$ such that $i$ is in $V \setminus \{q^2 - q + 1, \ldots, q^2 - 1\}$, have alternating reports: agent $q^2 - q$ reports $\{F_2\}$, agent $q^2 - q - 1$ reports $\{F_1\}$, agent $q^2 - q - 2$ reports $\{F_2\}$, and so on. All other agents in $I'$ report truthfully.

Let the optimal minimum welfare for DOFL instance $I$ (resp., $I'$) be OPT (resp., OPT'). It is easy to see that OPT = 1/4 and OPT' = $\frac{1}{2q}$ (obtained by building the facilities at $(1/4, 3/4)$ and $(\frac{1}{2q}, 1 - \frac{1}{2q})$, respectively). Let ALG (resp., ALG') denote the minimum welfare achieved by $M$ on instance $I$ (resp., $I'$). Below we prove that either OPT/ALG $\geq \frac{q}{4}$ or OPT'/ALG' $\geq q/2$.

Let $M$ build facilities at $(y_1, y_2)$ (resp., $(y'_1, y'_2)$) on instance $I$ (resp., $I'$). We consider two cases.

Case 1: $0 \leq y_1' < 1/Q$ and $1 - 1/q < y_2' \leq 1$. Let $S$ denote the set of agents who lie in $I'$. If $y_1' < y_1$ and $y_2' > y_2$, then all agents in $S$ benefit by lying. Hence for $M$ to be WGSP, either $y_1' \geq y_1$ or $y_2' \leq y_2$. Let us assume that $y_1' \geq y_1$; the other case can be handled symmetrically. Since $y_1' < 1/q$, we have $y_1 < 1/q$. Note that there is an agent at 0 who reported $\{F_1\}$. Thus ALG $\leq y_1 < 1/q$. Hence OPT/ALG $\geq \frac{q}{4}$.

Case 2: $y_1' \geq 1/q$ or $y_2' \leq 1 - 1/q$. If $y_1' \geq 1/q$, then at least one agent within distance $1/q^2$ of $y_1'$ reported $\{F_1\}$ in $I'$. A similar observation holds for the case $y_2' \leq 1 - 1/q$. Thus ALG' $\leq 1/q^2$. Hence OPT'/ALG' $\geq q/2$.

The preceding case analysis shows that $\alpha \geq q/4$. Since $q = \sqrt{p - 1} = \sqrt{n - 4}$, the theorem holds. $\Box$

The following variant of Mechanism 7 is SGSP. In this variant, we first replace the reported dislikes of all agents with $\{F_1\}$ and use Mechanism 6 to determine where to build $F_1$. Then we build all of the remaining facilities at the same location as $F_1$. This mechanism is SGSP because it disregards the reported aversion profile. We claim that this mechanism is 2$(n + 1)$-egalitarian, where $n$ denotes the number of agents. To prove this claim, we first observe that when Mechanism 6 is run as a subroutine within this mechanism, we have $\max(d_1, 2d_2, d_3) \geq 1/(n + 1)$. Thus the minimum welfare achieved by the mechanism is at least $1/(2(n + 1))$. Since the optimal minimum welfare is at most 1, the claim holds.
5.2 Egalitarian mechanisms for the cycle

In this section, we present egalitarian mechanisms for the case where the agents are located on the unit-circumference circle $C$. We denote the point antipodal to $u$ on $C$ by $\hat{u}$. We begin by considering the natural adaptation of Mechanism 6 to a cycle.

**Mechanism 8.** Let $I = (n, 1, x, a)$ denote the reported DOFL instance and let $H$ denote haters($I, 1$). If $H$ is empty build facility $F_1$ at 0. If $H$ has only one agent $i$, then build $F_1$ at $\hat{x}_i$. Otherwise, build $F_1$ at the midpoint of the largest gap between any two consecutive agents from $H$. Formally, let $H$ have $\ell$ agents $z_0, \ldots, z_{\ell - 1}$ such that $x_{z_0} \leq x_{z_1} \leq \cdots \leq x_{z_{\ell - 1}}$. Let $\oplus$ denote addition modulo $\ell$. Build $F_1$ at the midpoint of $x_{z_o}$ and $x_{z_{\ell - 1}}$ where $o$ is the smallest number in $\{0, \ldots, \ell - 1\}$ such that $\Delta(x_{z_{o + 1}}, x_{z_o}) = \max_{j \in \{0, \ldots, \ell - 1\}} \Delta(x_{z_j}, x_{z_j}).$

**Lemma 7.** Mechanism 8 is SP for single-facility DOFLG.

**Proof.** Let $(I, I')$ denote a single-facility DOFLG instance with $I = (n, 1, x, a)$, $I' = (n, 1, x, a')$, and let $i$ be an agent such that $a' = (a_-, a'_+)$. Using the same arguments as in the proof of Lemma 5, we restrict our attention to the case where $F_1$ is in $a_i \setminus a'_i$.

Let Mechanism 8 build $F_1$ at $y$ when agent $i$ reports truthfully. Let $H =$ haters($I, 1$) denote the set of agents who dislike $\{F_1\}$. Note that Mechanism 8 does not build $F_1$ at the location of any agent in $H$, that is, $y \neq x_i$ for any $i' \in H$. Let the arc of $C$ that goes clockwise from $y$ to $x_i$ be $r_1$ and let the arc of $C$ that goes anti-clockwise from $y$ to $x_i$ be $r_2$. Both arcs $r_1$ and $r_2$ include the end-points $y$ and $x_i$. We consider three cases.

Case 1: No agent in $H - i$ is located on $r_1$ or $r_2$. Hence $H = \{i\}$. Thus $y = \hat{x}_i$, and $\Delta(x_i, y) = 1/2$. When agent $i$ reports $a'_i$, $F_1$ is built at 0. Since $\Delta(x_i, 0) \leq 1/2$, reporting $a'_i$ does not benefit agent $i$.

Case 2: There are agents in $H - i$ located on $r_1$ or $r_2$, but not both. Without loss of generality, we assume that there is an agent in $H - i$ located on $r_2$ (and that there are no agents in $H - i$ on $r_1$). Let $i'$ be the closest agent to $y$ in $H - i$ on $r_2$. Let $d' = \Delta(y, x_{i'})$. Thus, $y$ is the midpoint of $x_{i'}$ and $x_i$. Hence $d' = \Delta(x_i, y)$. Let $i''$ be the closest agent in $H - i$ in the clockwise direction from $x_i$. Thus, $\Delta(x_{i''}, x_i) \leq 2d'$. Since $\Delta(x_{i''}, x_i) \leq 2d'$, when agent $i$ reports $a'_i$, $F_1$ is built on $r_1$, which does not benefit agent $i$.

Case 3: There are agents in $H - i$ on $r_1$ and $r_2$. Let the closest agent from $y$ in $H - i$ on $r_2$ (resp., $r_1$) be $a$ (resp., $b$), respectively. We have $\Delta(x_a, y) = \Delta(y, x_b)$. Let $d'$ denote $\Delta(x_a, y)$. Note that $\Delta(x_i, y) \geq d'$. Let the first agent from $H - i$ in the anti-clockwise and clockwise direction from $x_i$ be $i'$ and $i''$. We have, $\Delta(x_i, x_{i'}) \leq 2d'$ and $\Delta(x_i, x_{i''}) \leq 2d'$. Thus, when agent $i$ reports $a'_i$, either $F_1$ is built at $y$ or $F_1$ is built within distance $d'$ of $x_i$, neither of which benefits agent $i$.

Thus, agent $i$ does not benefit by reporting $a'_i$. $\square$

**Lemma 8.** Mechanism 8 is egalitarian for single-facility DOFLG.
Proof. Let \( I = (n, 1, \mathbf{x}, \mathbf{a}) \) denote the reported DOFL instance and let \( H \) denote haters\((I, 1)\). Using the same arguments as in the proof of Lemma 6, we assume that all agents belong to \( H \). Let \( y^* \) denote an optimal location for the facility and let \( \text{OPT} \) denote \( \text{MW}(I, y^*) \). Let \( y' \) denote the location at which Mechanism 8 builds the facility and let \( \text{ALG} \) denote \( \text{MW}(I, y') \). Below we establish that \( \text{ALG} \geq \text{OPT} \), which implies that Mechanism 8 is egalitarian.

If \( H \) is empty, then trivially Mechanism 8 is egalitarian. For the remainder of the proof, assume that \( H \) is non-empty. We say that an agent in \( H \) is tight if it is as close to \( y^* \) as any other agent in \( H \). Thus for any tight agent \( i \), \( \text{OPT} = \Delta(y^*, x_i) \).

Let \( i \) be a tight agent. Assume that in the shorter arc between \( x_i \) and \( y^* \), \( x_i \) is anti-clockwise from \( y^* \) (the other case can be handled similarly). Thus \( \text{OPT} = \Delta(x_i, y^*) \). Let \( i' \) be the closest agent in \( H \) in the clockwise direction from \( y^* \). The definition of \( i' \) implies that \( \Delta(x_i, y^*) \geq \text{OPT} \). It follows that \( \Delta(x_i, x_{i'}) \geq 2 \cdot \text{OPT} \). Since \( \Delta(x_i, x_{i'}) \) is the gap between two consecutive agents in \( H \) and Mechanism 8 builds the facility at the midpoint of the largest gap, we deduce that \( \text{ALG} \geq \text{OPT} \).

We define Mechanism 9 as the DOFLG mechanism \( \text{Parallel}(M) \), where \( M \) denotes Mechanism 8. Using Lemmas 3, 4, 7, and 8, we immediately obtain Theorem 14 below.

**Theorem 14.** Mechanism 9 is SP and egalitarian.

Theorem 15 below extends Theorem 13 to the case of the cycle. Theorem 15 implies that Mechanism 9 is not WGSP.

**Theorem 15.** Let \( M \) be a WGSP \( \alpha \)-egalitarian mechanism. Then \( \alpha \) is \( \Omega(\sqrt{n}) \), where \( n \) is the number of agents.

**Proof.** It is easy to verify that the construction used in the proof of Theorem 13 also works for the cycle and establishes the same lower bound. (We identify the point 1 with the point 0.)

The following variant of Mechanism 9 is SGSP. As in the SGSP mechanism for the case when the agents are located in the unit interval, in this variant, we first replace the reported dislikes of all agents with \( \{F_1\} \) and use Mechanism 8 to determine where to build \( F_1 \). Then we build all of the remaining facilities at the same location as \( F_1 \). This mechanism is SGSP because it disregards the reported aversion profile. We claim that this mechanism is \( n \)-egalitarian, where \( n \) denotes the number of agents. To prove this claim, we first observe that the largest gap between two consecutive agents who report to dislike \( \{F_1\} \) is at least \( 1/n \). Thus the minimum welfare achieved by the mechanism is at least \( 1/(2n) \). Since the optimal minimum welfare is at most \( 1/2 \), the claim holds.

### 5.3 An egalitarian mechanism for the unit square

We extend Mechanism 6 to a SP egalitarian mechanism for single-facility DOFLG when the agents are located in the unit square. Let \( S \) denote \([0, 1]^2\), let \( B \)
denote the boundary of $S$, and let $x_i = (a_i, b_i)$ denote the location of agent $i$. For convenience, we assume that all agents are located at distinct points; the results below generalize easily to the case where this assumption does not hold.

**Mechanism 10.** Let $I = (n, 1, x, a)$ denote the reported DOFL instance and let $H$ denote haters($I, 1$). If $H$ is empty, build $F_1$ at $(0, 0)$. Otherwise, construct the Voronoi diagram $D$ associated with the locations of the agents in $H$. Let $V$ be the union of the following three sets of vertices: the vertices of $D$ in the interior of $S$; the points of intersection between $B$ and $D$; the four vertices of $S$. For each $v$ in $V$, let $d_v$ denote the minimum distance from $v$ to any agent in $H$. Build $F_1$ at a vertex $v$ maximizing $d_v$, breaking ties first by $x$-coordinate and then by $y$-coordinate.

Toussaint presents an efficient $O(n \log n)$ algorithm to find the optimal $v$ in Mechanism 10 [23]. The following lemma establishes that Mechanism 10 is egalitarian. The lemma is shown using Theorem 2 of [23] for the largest empty circle with location constraints problem.

**Lemma 9.** Mechanism 10 is egalitarian for single-facility DOFLG.

**Proof.** Let $I = (n, 1, x, a)$ denote the reported DOFL instance and let $H$ denote haters($I, 1$). Using the same arguments as in the proof of Lemma 6, we assume that all agents belong to $H$. Let $y^*$ denote an optimal location for the facility and let OPT denote MW($I, y^*$). Let $y'$ denote the location where Mechanism 10 builds the facility and let ALG denote MW($I, y'$). Below we establish that $\text{ALG} = \text{OPT}$, which implies that Mechanism 10 is egalitarian.

If $H$ is empty, then clearly $\text{ALG} = \text{OPT}$. Otherwise, finding the optimal location at which to build facility $F_1$ is equivalent to finding the maximum-radius circle centered in the interior or on the boundary of $S$ such that the interior of the circle has no points from $\{x_i \mid i \in H\}$. This problem is known as Largest Empty Circle with Location Constraint [23]. Toussaint [23], Theorem 2 shows that the optimal center for the circle is either a vertex of the Voronoi diagram of $S$, a point of intersection of $D$ with $B$, or a vertex of $S$. Hence $\text{ALG} = \text{OPT}$. \hfill $\square$

We use a case analysis to establish Lemma 10 below. The most interesting case deals with an agent $i$ who dislikes $F_1$ but does not report it. In this case, the key insight is that when agent $i$ misreports, facility $F_1$ is built either (1) at the same location as when agent $i$ reports truthfully, or (2) inside or on the boundary of the Voronoi region that contains $x_i$ when agent $i$ reports truthfully.

**Lemma 10.** Mechanism 10 is SP for single-facility DOFLG.

**Proof.** Let $(I, I')$ denote a single-facility DOFLG instance with $I = (n, 1, x, a)$, $I' = (n, 1, x, a')$, and let $i$ be an agent such that $a' = (a_{-i}, a_i')$. Using the same

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2Toussiant assumes that no three points are collinear and no four points are cocircular, but the result continues to hold without these assumptions.
arguments as in the proof of Lemma 5, we restrict our attention to the case
where $F_1$ is in $a_i \setminus a'_i$.

Let Mechanism 10 build $F_1$ at $y$ ($y'$, resp.) when agent $i$ reports $a_i$ ($a'_i$, resp.). Let $H = \text{haters}(I, 1)$ denote the set of agents who dislike $\{F_1\}$. Note that Mechanism 10 does not build $F_1$ at the location of any agent in $H$, that is, $y \neq x_i'$ for all $i'$ in $H$. When all agents report truthfully, the Voronoi diagram of $S$ consists of $|H|$ non-overlapping polygons with each polygon containing one agent. Let $P$ be the polygon containing agent $i$ when all agents report truthfully. When agent $i$ reports $a'_i$, $F_1$ is built either at $y$ or inside $P$ (boundary inclusive). If it is built at $y$, agent $i$ does not benefit. Thus, for the remainder of the proof, we assume that $y'$ belongs to the interior of $P$.

Let $\text{OPT}$ (resp., $\text{OPT}'$) be the closest distance of any agent from $H$ (resp., $H - i$) to the point $y$ (resp., $y'$). Let $d$ and $d'$ denote $\Delta(x_i, y)$ and $\Delta(x_i, y')$, respectively. Hence $d \geq \text{OPT}$. Since the distance from $y$ to any agent in $H - i$ is at least $\text{OPT}'$, we have $\text{OPT}' \geq \text{OPT}$.

Assume for the sake of contradiction that Mechanism 10 is not SP. Hence $d' > d$. We begin by showing that $\text{OPT} = \text{OPT}'$. Suppose $\text{OPT} \neq \text{OPT}'$. Since $\text{OPT}' \geq \text{OPT}$, we have $\text{OPT}' > \text{OPT}$. Let us examine the situation when $F_1$ is built at $y'$ when agent $i$ reports truthfully. We have $d' > d$. Moreover, the agent in $H - i$ closest to $y'$ is at distance $\text{OPT}'$ from $y'$. Since $d' > d$ and $\text{OPT}' > \text{OPT}$, we have $\text{MW}(I, y') > \text{MW}(I, y)$. This contradicts Lemma 9, since it implies that Mechanism 10 is egalitarian. Thus $\text{OPT} = \text{OPT}'$.

Recall that $y'$ belongs to $P$. Hence the closest agent in $H$ to $y'$ is agent $i$. Thus $d' \leq \text{OPT}'$. We have established that $\text{OPT} \leq d$. Since $\text{OPT} \leq d$, $d < d'$, and $d' \leq \text{OPT}'$, we obtain $\text{OPT} < \text{OPT}'$, which contradicts $\text{OPT} = \text{OPT}'$. Thus $d' \leq d$, and hence agent $i$ does not benefit by reporting $a'_i$. \qed

We define Mechanism 11 as the DOFLG mechanism Parallel($M$), where $M$ denotes Mechanism 10. Using Lemmas 3, 4, 9, and 10, we immediately obtain Theorem 16 below.

**Theorem 16.** Mechanism 11 is SP and egalitarian.

### 6 Concluding Remarks

In this paper, we studied the obnoxious facility location game with dichotomous preferences. This game generalizes obnoxious facility location game to more realistic scenarios. All of the mechanisms presented in this paper run in polynomial time, except that the running time of Mechanism 2 has exponential dependence on $k$ (and polynomial dependence on $n$). We can extend the results of Section 5.3 to obtain an analogue of Theorem 16 that holds for arbitrary convex polygon. We showed that Mechanism 2 is WGSP for all $k$ and is efficient for $k \leq 3$. Properties 1 and 2 in the proof of our associated theorem, Theorem 4, do not hold for $k > 3$. Nevertheless, we conjecture that Mechanism 2 is efficient
for all $k$. It remains an interesting open problem to reduce the gap between the $\Omega(\sqrt{n})$ and $O(n)$ bounds on the approximation ratio $\alpha$ of WGSP $\alpha$-egalitarian mechanisms.

References


