The Obnoxious Facility Location Game with Dichotomous Preferences*

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Abstract

We consider a facility location game in which n agents reside at known locations on a path, and k heterogeneous facilities are to be constructed on the path. Each agent is adversely affected by some subset of the facilities, and is unaffected by the others. We design two classes of mechanisms for choosing the facility locations given the reported agent preferences: utilitarian mechanisms that strive to maximize social welfare (i.e., to be efficient), and egalitarian mechanisms that strive to maximize the minimum welfare. For the utilitarian objective, we present a weakly group-strategyproof efficient mechanism for up to three facilities, we give strongly group-strategyproof mechanisms that achieve approximation ratios of 5/3 and 2 for $k = 1$ and $k > 1$, respectively, and we prove that no strongly group-strategyproof mechanism achieves an approximation ratio less than 5/3 for the case of a single facility. For the egalitarian objective, we present a strategyproof egalitarian mechanism for arbitrary k, and we prove that no weakly group-strategyproof mechanism achieves a $o(\sqrt{n})$ approximation ratio for two facilities. We extend our egalitarian results to the case where the agents are located on a cycle, and we extend our first egalitarian result to the case where the agents are located in the unit square.

1 Introduction

The facility location game (FLG) was introduced by Procaccia and Tennenholtz [22]. In this setting, a central planner wants to build a facility that serves agents located on a path. The agents report their locations, which are fed to a mechanism that decides where the facility should be built. Procaccia and Tennenholtz studied two different objectives that the planner seeks to minimize: the sum of the distances from the facility to all agents, and the maximum distance of any agent to the facility.

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Every agent aims to maximize their welfare, which increases as their distance to the facility decreases. An agent or a coalition of agents can misreport their location(s) to try to increase their welfare. The strategyproof (SP) property says that no agent can increase their welfare by lying about their dislikes. The weakly group-strategyproof (WGSP) property says that if a non-empty coalition of agents lies, then at least one agent in the coalition does not increase their welfare. The strongly group-strategyproof (SGSP) property says that if a coalition of agents lies and some agent in the coalition increases their welfare, then some agent in the coalition decreases their welfare. It is natural to seek SP, WGSP, or SGSP mechanisms, which incentivize truthful reporting. Often such mechanisms cannot simultaneously optimize the planner's objective. In these cases, it is desirable to approximately optimize the planner's objective.

In real scenarios, an agent might dislike a certain facility, such as a power plant, and want to stay away from it. This variant, called the obnoxious facility location game (OFLG), was introduced by Cheng et al., who studied the problem of building an obnoxious facility on a path [7]. In the present paper, we consider the problem of building multiple obnoxious facilities on a path. With multiple facilities, there are different ways to define the welfare function. For example, in the case of two facilities, the welfare of the agent might be the sum, minimum, or maximum of the distances to the two facilities. In our work, as all the facilities are obnoxious, a natural choice for welfare is the minimum distance to any obnoxious facility: the closest facility to an agent causes them the most annoyance, and if it is far away, then the agent is satisfied.

A facility might not be universally obnoxious. Consider, for example, a school or sports stadium. An agent with no children might consider a school to be obnoxious due to the associated noise and traffic, while an agent with children might not consider it to be obnoxious. Another agent who is not interested in sports might similarly consider a stadium to be obnoxious. We assume that each agent has dichotomous preferences; they dislike some subset of the facilities and are indifferent to the others. Each agent reports a subset of facilities to the planner. As the dislikes are private information, the reported subset might not be the subset of facilities that the agent truly dislikes. On the other hand, we assume that the agent locations are public and cannot be misreported.

In this paper, we study a variant of FLG, which we call DOFLG (Dichotomous Obnoxious Facility Location Game), that combines the three aspects mentioned above: multiple (heterogeneous) obnoxious facilities, minimum distance as welfare, and dichotomous preferences. We seek to design mechanisms that perform well with respect to either a utilitarian or egalitarian objective. The utilitarian objective is to maximize the social welfare, that is, the total welfare of all agents. A mechanism that maximizes social welfare is said to be efficient. The egalitarian objective is to maximize the minimum welfare of any agent. For both objectives, we seek mechanisms that are SP, or better yet, weakly or strongly group-strategyproof (WGSP / SGSP).

1.1 Our contributions

We study DOFLG with n agents. In Section 4, we consider the utilitarian objective. We present 2-approximate SGSP mechanisms for any number of facilities when the agents are located on a path, cycle, or square. We obtain the following two additional results for the path setting. We obtain a mechanism that is WGSP for any number of facilities and efficient for up to three facilities. To show that this mechanism is WGSP, we relate it to a weighted approval voting mechanism. To prove its efficiency, we identify two crucial properties that the welfare function satisfies, and we use an exchange argument. For the path setting, we show that no SGSP mechanism can achieve an approximation ratio better than 5/3, even in the single-facility case, and we present a single-facility SGSP mechanism that achieves an approximation ratio of 5/3, matching the lower bound. The argument underlying our 5/3 lower bound demonstrates that any single-facility SGSP mechanism needs to essentially disregard the agent preferences; in other words, the location of the facility has to be (essentially) determined by the agent locations.

The single-facility mechanism that we use to establish the matching 5/3 upper bound disregards the agent preferences entirely, and hence is SGSP. Our proof of the 5/3 upper bound is by far the most technical argument in the paper. Given the agent locations, we first use a sequence of lemmas to characterize the best possible approximation ratio that can be guaranteed (for all possible choices of the agent preferences) if the mechanism locates the facility at the left endpoint, right endpoint, or center of the path. (We also give a fast algorithm for computing these three approximation ratios, which allows for a fast implementation of our mechanism.) We exploit this characterization to show that it is sufficient to bound the approximation ratio achieved by the mechanism on instances where all of the agents to the left (resp., right) of the center are located at no more than two distinct locations. We then show that it is sufficient to further restrict our attention to "balanced" instances where the average agent location (i.e., the center of gravity of the agents) is at the center. Under these restrictions, we are able to show that if the mechanism cannot guarantee a 5/3 approximation ratio by building the facility at the left or right endpoint, then it can guarantee a 5/3 approximation ratio by building at the center.

In Section 5, we consider the egalitarian objective. We provide optimal SP mechanisms for any number of facilities when the agents are located on a path, cycle, or square. We prove that the approximation ratio achieved by any WGSP mechanism is $\Omega(\sqrt{n})$, even for two facilities. Also, we present a straightforward $O(n)$-approximate WGSP mechanism. Both of the results for WGSP mechanisms hold for DOFLG when the agents are located on a path or cycle. Table 1 summarizes our results.

1.2 Related work

Procaccia and Tannenholtz [22] introduced FLG. Many generalizations and extensions of FLG have been studied [1, 9, 12, 13, 14, 15, 20, 28]; here we

Table 1: Summary of our results for DOFLG when the agents are located on a path. The heading LB (resp., UB) stands for lower (resp., upper) bound. The results in the egalitarian column also hold when the agents are located on a cycle. Boldface results hold when the agents are located on a path, cycle, or square. The tight 5/3 upper bound for the SGSP utilitarian case holds when there is a single facility and the agents are located on a path, while the upper bound of 2 holds for an arbitrary number of facilities when the agents are located on a path, cycle, or square.

<table>
<tr><th rowspan="2"></th><th colspan="2">Utilitarian</th><th colspan="2">Egalitarian</th></tr>
<tr><th>LB</th><th>UB</th><th>LB</th><th>UB</th></tr>
<tr><td>SP</td><td rowspan="2">1</td><td rowspan="2">1 for $k \leq 3$</td><td>1</td><td>1</td></tr>
<tr><td>WGSP</td><td rowspan="2">$\Omega(\sqrt{n})$</td><td rowspan="2">$O(n)$</td></tr>
<tr><td>SGSP</td><td>5/3</td><td>5/3 for $k = 1$
2</td></tr>
</table>

highlight some of the most relevant work. Cheng et al. introduced OFLG and presented a WGSP mechanism to build a single facility on a path [7]. Later they extended the model to cycles and trees [8]. A complete characterization of single-facility SP/WGSP mechanisms for paths has been developed [17, 18]. Duan et al. studied the problem of locating two obnoxious facilities at least distance d apart [10]. Other variants of OFLG have been considered [6, 16, 21, 26].

Agent preferences over the facilities were introduced to FLG in [11] and [29]. Serafino and Ventre studied FLG for building two facilities where each agent likes a subset of the facilities [23]. Anastasiadis and Deligkas extended this model to allow the agents to like, dislike, or be indifferent to the facilities [2]. The aforementioned works address linear (sum) welfare functions. Yuan et al. studied non-linear welfare functions (maximum and minimum) for building two non-obnoxious facilities [27]; their results have subsequently been strengthened [5, 19]. In the present paper, we initiate the study of a non-linear welfare function (minimum) for building multiple obnoxious facilities.

2 Preliminaries

The problems considered in this paper involve a set of agents located on a path, cycle, or square. In the path (resp., cycle, square) setting, we assume without loss of generality that the path (resp., cycle, square) is the unit interval (resp., unit-circumference circle, unit square). We map the points on the unit-circumference circle to $[0, 1)$, in the natural manner. Thus, in the path (resp., cycle, square) setting, each agent i is located in $[0, 1]$ (resp., $[0, 1)$, $[0, 1]^2$). The distance between any two points x and y is denoted $\Delta(x, y)$. In the path and square settings, $\Delta(x, y)$ is defined as the Euclidean distance between x and y. In the cycle setting, $\Delta(x, y)$, is defined as the length of the shorter arc between

x and y. In all settings, we index the agents from 1. Each agent has a specific location in the path, cycle, or square. A *location profile* $\mathbf{x}$ is a vector $(x_1, \ldots, x_n)$ of points, where n denotes the number of agents and x_i is the location of agent i. Sections 4.1 and 5.1 (resp., Sections 4.2 and 5.2, Sections 4.3 and 5.3) present our results for the path (resp., cycle, square) setting.

Consider a set of agents 1 through n and a set of facilities $\mathcal{F}$, where we assume that each agent dislikes (equally) certain facilities in $\mathcal{F}$ and is indifferent to the rest. In this context, we define an *aversion profile* $\mathbf{a}$ as a vector $(a_1, \ldots, a_n)$ where each component a_i is a subset of $\mathcal{F}$. We say that such an aversion profile is *true* if each component a_i is equal to the subset of $\mathcal{F}$ disliked by agent i. In this paper, we also consider *reported* aversion profiles where each component a_i is equal to the set of facilities that agent i claims to dislike. Since agents can lie, a reported aversion profile need not be true. For any aversion profile $\mathbf{a}$ and any subset C of agents $[n]$, $\mathbf{a}_C$ (resp., $\mathbf{a}_{-C}$) denotes the aversion profile for the agents in (resp., not in) C. For a singleton set of agents $\{i\}$, we abbreviate $\mathbf{a}_{-\{i\}}$ as $\mathbf{a}_{-i}$.

An instance of the dichotomous obnoxious facility location (DOFL) problem is given by a tuple $(n, k, \mathbf{x}, \mathbf{a})$ where n denotes the number of agents, there is a set of k facilities $\mathcal{F} = \{F_1, \ldots, F_k\}$ to be built, $\mathbf{x} = (x_1, \ldots, x_n)$ is a location profile for the agents, and $\mathbf{a} = (a_1, \ldots, a_n)$ is an aversion profile (true or reported) for the agents with respect to $\mathcal{F}$. A solution to such a DOFL instance is a vector $\mathbf{y} = (y_1, \ldots, y_k)$ where component y_j specifies the point at which to build F_j. We say that a DOFL instance is true (resp., reported) if the associated aversion profile is true (resp., reported). For any DOFL instance $I = (n, k, \mathbf{x}, \mathbf{a})$ and any j in $[k]$, we define $\text{haters}(I, j)$ as $\{i \in [n] \mid F_j \in a_i\}$, and $\text{indiff}(I)$ as $\{i \in [n] \mid a_i = \emptyset\}$.

For any DOFL instance $I = (n, k, \mathbf{x}, \mathbf{a})$ and any associated solution $\mathbf{y}$, we define the *welfare* of agent i, denoted $w(I, i, \mathbf{y})$, as the minimum distance from x_i to any facility in a_i, that is, $\min_{j:F_j \in a_i} \Delta(x_i, y_j)$. Remark: If a_i is empty, we define $w(I, i, \mathbf{y})$ as $1/2$ in the cycle setting, $\max(\Delta(x_i, 0), \Delta(x_i, 1))$ in the path setting, and the maximum distance from x_i to a corner in the square setting.

The foregoing definition of agent welfare is suitable for true DOFL instances, and is only meaningful for reported DOFL instances where the associated aversion profile is close to true. In this paper, reported aversion profiles arise in the context of mechanisms that incentivize truthful reporting, so it is reasonable to expect such aversion profiles to be close to true. We define the *social welfare* (resp., *minimum welfare*) as the sum (resp., minimum) of the individual agent welfares. When the facilities are built at $\mathbf{y}$, the social welfare and minimum welfare are denoted by $\text{SW}(I, \mathbf{y})$ and $\text{MW}(I, \mathbf{y})$, respectively. Thus $\text{SW}(I, \mathbf{y}) = \sum_{i \in [n]} w(I, i, \mathbf{y})$ and $\text{MW}(I, \mathbf{y}) = \min_{i \in [n]} w(I, i, \mathbf{y})$.

Definition 1. For $\alpha \geq 1$, a DOFL algorithm A is α-efficient if for any DOFL instance I,

$$\max_{\mathbf{y}} \text{SW}(I, \mathbf{y}) \leq \alpha \, \text{SW}(I, A(I)).$$

Similarly, A is α-egalitarian if for any DOFL instance I,

$$\max_{\mathbf{y}} \mathrm{MW}(I, \mathbf{y}) \leq \alpha \, \mathrm{MW}(I, A(I)).$$

A 1-efficient (resp., 1-egalitarian) DOFL algorithm, is said to be efficient (resp., egalitarian).

We are now ready to define a DOFL-related game, which we call DOFLG. It is convenient to describe a DOFLG instance in terms of a pair (I, I') of DOFL instances where $I = (n, k, \mathbf{x}, \mathbf{a})$ is true and $I' = (n, k, \mathbf{x}, \mathbf{a}')$ is reported. There are n agents indexed from 1 to n, and a planner. There is a set of k facilities $\mathcal{F} = \{F_1, \ldots, F_k\}$ to be built. The numbers n and k are publicly known, as is the location profile $\mathbf{x}$ of the agents. Each component a_i of the true aversion profile $\mathbf{a}$ is known only to agent i. Each agent i submits component a'_i of the reported aversion profile $\mathbf{a}'$ to the planner. The planner, who does not have access to $\mathbf{a}$, runs a DOFL algorithm, call it A, to map I' to a solution. The input-output behavior of A defines a DOFLG mechanism, call it M; in the special case where $k = 1$, we say that M is a single-facility DOFLG mechanism. We would like to choose A so that M enjoys strong game-theoretic properties. We say that M is α-efficient (resp., α-egalitarian, efficient, egalitarian) if A is α-efficient (resp., α-egalitarian, efficient, egalitarian). As indicated earlier, such properties (which depend on the notion of agent welfare) are only meaningful if the reported aversion profile is close to true. To encourage truthful reporting, we require our mechanisms to be SP, as defined below; we also consider the stronger properties WGSP and SGSP.

The SP property says that no agent can increase their welfare by lying about their dislikes, regardless of the fixed aversion profile reported by the remaining agents.

Definition 2. A DOFLG mechanism M is SP if for any DOFLG instance (I, I') with $I = (n, k, \mathbf{x}, \mathbf{a})$, and $I' = (n, k, \mathbf{x}, \mathbf{a}')$, and any agent i in $[n]$ such that $\mathbf{a}' = (\mathbf{a}_{-i}, a'_i)$, we have

$$w(I, i, M(I)) \geq w(I, i, M(I')).$$

The WGSP property says that if a non-empty coalition $C \subseteq [n]$ of agents lies, then at least one agent in C does not increase their welfare.

Definition 3. A DOFLG mechanism M is WGSP if for any DOFLG instance (I, I') with $I = (n, k, \mathbf{x}, \mathbf{a})$, and $I' = (n, k, \mathbf{x}, \mathbf{a}')$, and any non-empty coalition $C \subseteq [n]$ such that $\mathbf{a}' = (\mathbf{a}_{-C}, \mathbf{a}'_C)$, there exists an agent i in C such that

$$w(I, i, M(I)) \geq w(I, i, M(I')).$$

The SGSP property says that if a coalition $C \subseteq [n]$ of agents lies and some agent in C increases their welfare then some agent in C decreases their welfare.

Definition 4. A DOFLG mechanism M is SGSP if for any DOFLG instance (I, I') with $I = (n, k, \mathbf{x}, \mathbf{a})$, and $I' = (n, k, \mathbf{x}, \mathbf{a}')$, and any coalition $C \subseteq [n]$ such that $\mathbf{a}' = (\mathbf{a}_{-C}, \mathbf{a}'_C)$, if there exists an agent i in C such that

$$w(I, i, M(I)) < w(I, i, M(I')),$$

then there exists an agent i' in C such that

$$w(I, i', M(I)) > w(I, i', M(I')).$$

Every SGSP mechanism is WGSP and every WGSP mechanism is SP.

3 Weighted Approval Voting

Before studying efficient mechanisms for our problem, we review a variant of the approval voting mechanism [4]. An instance of the Dichotomous Voting (DV) problem is a tuple $(m, n, \mathbf{C}, \mathbf{w}^+, \mathbf{w}^-)$ where m voters $1, \ldots, m$ have to elect a candidate among a set of candidates $C = \{c_1, \ldots, c_n\}$. Each voter i has dichotomous preferences, that is, voter i partitions all of the candidates into two equivalence classes: a top (most preferred) tier C_i and a bottom tier $\overline{C_i} = C \setminus C_i$. Each voter i has associated (and publicly known) weights $w_i^+ \geq w_i^- \geq 0$. The symbols $\mathbf{C}$, $\mathbf{w}^+$, and $\mathbf{w}^-$ denote length-m vectors with ith element C_i, w_i^+, and w_i^-, respectively. We now present our weighted approval voting mechanism.[1]

Mechanism 1. Given a DV instance $(m, n, \mathbf{C}, \mathbf{w}^+, \mathbf{w}^-)$, every voter i votes by partitioning C into C'_i and $\overline{C'_i}$. Let the weight function w be such that for voter i and candidate c_j, $w(i, j) = w_i^+$ if c_j is in C'_i and $w(i, j) = w_i^-$ otherwise. For any j in $[n]$, we define $A(j) = \sum_{i \in [m]} w(i, j)$ as the approval of candidate c_j. The candidate c_j with highest approval $A(j)$ is declared the winner. Ties are broken according to a fixed ordering of the candidates (e.g., in favor of lower indices).

We note that the approval voting mechanism can be obtained from the weighted approval voting mechanism by setting weights w_i^+ to 1 and w_i^- to 0 for all voters i. In Section 2, we defined SP, WGSP, and SGSP in the DOFLG setting. These definitions are easily generalized to the voting setting. Brams and Fishburn proved that the approval voting mechanism is SP [4]. Below we prove that our weighted approval voting mechanism is WGSP (and hence also SP).

Theorem 1. Mechanism 1 is WGSP.

Proof. Assume for the sake of contradiction that there is an instance in which a coalition of voters U with true preferences $\{(C_i, \overline{C_i})\}_{i \in U}$ all benefit by misreporting their preferences as $\{(C'_i, \overline{C'_i})\}_{i \in U}$. For any candidate c_j, let $A(j)$ denote

[1] Our mechanism differs from the homonymous mechanism of Massó et al., which has weights for the candidates instead of the voters [25].

the approval of c_j when coalition U reports truthfully, and let $A'(j)$ denote the approval of c_j when coalition U misreports.

Let c_k be the winning candidate when coalition U reports truthfully, and let c_ℓ be the winning candidate when coalition U misreports. Since every voter in U benefits when the coalition misreports, we know that c_k belongs to $\bigcap_{i\in U}\overline{C_i}$ and c_ℓ belongs to $\bigcap_{i\in U} C_i$.

Since c_k belongs to $\bigcap_{i\in U}\overline{C_i}$, we deduce that $A'(k) = A(k)+\sum_{i\in U:c_k\in C'_i} w_i^+ - w_i^-$ and hence $A'(k) \geq A(k)$. Similarly, since c_ℓ belongs to $\bigcap_{i\in U} C_i$, we deduce that $A'(\ell) = A(\ell) + \sum_{i\in U:c_\ell\in\overline{C'_i}} w_i^- - w_i^+$ and hence $A(\ell) \geq A(\ell')$.

Since c_k wins when coalition U reports truthfully, one of the following two cases applies.

Case 1: $A(k) > A(\ell)$. Since $A'(k) \geq A(k)$ and $A(\ell) \geq A'(\ell)$, the case condition implies that $A'(k) > A'(\ell)$. Hence c_ℓ does not win when coalition U misreports, a contradiction.

Case 2: $A(k) = A(\ell)$ and c_k has higher priority than c_ℓ. Since $A'(k) \geq A(k)$ and $A(\ell) \geq A(\ell')$, the case condition implies that $A'(k) \geq A(\ell')$ and c_k has higher priority than c_ℓ. Hence c_ℓ does not win when coalition U misreports, a contradiction. □

Theorem 2. Mechanism 1 is not SGSP.

Proof. Let I be a DV instance with 5 voters, candidates c_1 and c_2, and weights $w_i^+ = 1$ and $w_i^- = 0$ for all i in $\{1, \ldots, 5\}$. Each voter in I votes truthfully, and their votes are: $C_1 = \{c_1\}$, $C_2 = C_3 = \{c_1, c_2\}$, and $C_4 = C_5 = \{c_2\}$. Thus $A(1) = 3$ and $A(2) = 4$, and Mechanism 1 declares c_2 the winner. Let I' be the DV instance with the same voters, candidates, and weights as in I. Voters $1, 2$, and 3 form a coalition and vote $\{c_1\}$, while voters 4 and 5 vote $\{c_2\}$. Then $A(1) = 3$ and $A(2) = 2$, and Mechanism 1 declares c_1 the winner. This result benefits voter 1, without any loss to voters 2 and 3. Thus Mechanism 1 is not SGSP. □

4 Efficient Mechanisms

In this section, we present efficient mechanisms for DOFLG. In Section 4.1, we address the case where the agents are located in the unit interval. In Section 4.2 (resp., Section 4.3), we consider the case where the agents are located on a cycle (resp., square).

4.1 The unit interval

We now present our efficient mechanism for DOFLG.

Mechanism 2. For a given reported DOFL instance $I = (n, k, \mathbf{x}, \mathbf{a})$, output the lexicographically least solution $\mathbf{y}$ in $\{0, 1\}^k$ that maximizes the social welfare $\mathrm{SW}(I, \mathbf{y})$.

Mechanism 2 runs in $O(nk2^k)$ time, and hence runs in polynomial time when k is $O(\log n)$.

Theorem 3. Mechanism 2 is WGSP.

Proof. To establish this theorem, we show that Mechanism 2 can be equivalently expressed in terms of the approval voting mechanism. Hence Theorem 1 implies the theorem.

Let (I, I') denote a DOFLG instance where $I = (n, k, \mathbf{x}, \mathbf{a})$ and $I' = (n, k, \mathbf{x}, \mathbf{a}')$. We view each agent $i \in [n]$ as a voter, and each $\mathbf{y}$ in $\{0,1\}^k$ as a candidate. We obtain the top-tier candidates C_i of voter i, and their reported top-tier candidates C'_i, from a_i and a'_i, respectively. Assume without loss of generality that $x_i \leq 1/2$ (the other case can be handled similarly). Set $C_i = \{\mathbf{y} = (y_1, \ldots, y_k) \in \{0,1\}^k \mid y_j = 1 \text{ for all } F_j \in a_i\}$ and similarly $C'_i = \{\mathbf{y} = (y_1, \ldots, y_k) \in \{0,1\}^k \mid y_j = 1 \text{ for all } F_j \in a'_i\}$. Also set $w_i^+ = 1 - x_i$ and $w_i^- = x_i$. With this notation, it is easy to see that $A(\mathbf{y}) = \mathrm{SW}(I', \mathbf{y})$, and that choosing the $\mathbf{y}$ with the highest social welfare in Mechanism 2 is the same as electing the candidate with the highest approval in Mechanism 1. □

We show that Mechanism 2 is efficient for $k = 3$. First, we note a well-known result about the 1-Maxian problem. In this problem, there are n points located at $z_1, \ldots, z_n$ in the interval $[a, b]$, and the task is to choose a point in $[a, b]$ such that the sum of the distances from that point to all of the z_i's is maximized. This result follows from the fact that the sum of convex functions is convex, and that a convex function on a closed interval is maximized at the one of the endpoints of the interval [3].

Lemma 1. Let $[a, b]$ be a real interval, let $z_1, \ldots, z_n$ belong to $[a, b]$, and let $f(z)$ denote $\sum_{i\in[n]} |z - z_i|$. Then $\max_{z\in[a,b]} f(z)$ belongs to $\{f(a), f(b)\}$.

Before proving the main theorem, we establish Lemma 2, which follows from Lemma 1.

Lemma 2. Let $I = (n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance, let Y denote the set of all y in $[0, 1]$ such that it is efficient to build all k facilities at y, and assume that Y is non-empty. Then $Y \cap \{0, 1\}$ is non-empty.

Proof. Let U denote $\mathrm{indiff}(I)$. When all of the facilities are built at y,

$$\mathrm{SW}(I, (y, \ldots, y)) = \sum_{i\in[n]\setminus U} |x_i - y| + \sum_{i\in U} w(I, i, y).$$

Since Y is non-empty, $\max_y \mathrm{SW}(I, (y, \ldots, y)) = \max_{\mathbf{y}} \mathrm{SW}(I, \mathbf{y})$. Moreover, since $\sum_{i\in U} w(I, i, y)$ does not depend on y, Lemma 1 implies that

$$\max(\mathrm{SW}(I, (0, \ldots, 0)), \mathrm{SW}(I, (1, \ldots, 1))) = \max_y \mathrm{SW}(I, (y, \ldots, y)).$$

Thus, if $\mathrm{SW}(I, (0, \ldots, 0)) \geq \mathrm{SW}(I, (1, \ldots, 1))$, it is efficient to build all k facilities at 0. Otherwise, it is efficient to build all k facilities at 1. □

Theorem 4. Mechanism 2 is efficient for $k = 3$.

Proof. Let $I = (n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance and let $\mathbf{y}^* = (y_1^*, y_2^*, y_3^*)$ be an efficient solution for I such that $y_1^* \leq y_2^* \leq y_3^*$.

Consider fixing variables y_1 and y_3 in the social welfare function $\mathrm{SW}(I, \mathbf{y})$. That is, we have

$$\mathrm{SW}(I, \mathbf{y})|_{y_1=y_1^*, y_3=y_3^*} = \sum_{i\in[n]} w(I, i, \mathbf{y})|_{y_1=y_1^*, y_3=y_3^*}.$$

For convenience, let $\mathrm{SW}(y_2)$ denote $\mathrm{SW}(I, \mathbf{y})|_{y_1=y_1^*, y_3=y_3^*}$ and let $w_i(y_2)$ denote $w(I, i, \mathbf{y})|_{y_1=y_1^*, y_3=y_3^*}$ for each agent i.

Claim 1: For each agent i, the welfare function $w_i(y_2)$ with $y_2 \in [y_1^*, y_3^*]$ satisfies either (1) $w_i(y_2) = |y_2 - x_i|$ or (2) $w_i(y_1^*) = w_i(y_3^*) = \max_{y\in[y_1^*, y_3^*]} w_i(y)$.

Proof: Fix an agent i. We consider five cases.

Case 1: $F_2 \notin a_i$. Since the welfare of agent i is independent of the location of F_2, w_i is a constant function. Hence (2) is satisfied.

Case 2: $a_i = \{F_2\}$. By definition, we have $w_i(y_2) = |y_2 - x_i|$. Hence (1) is satisfied.

Case 3: $a_i = \{F_1, F_2\}$. By definition, we have $w_i(y_2) = \min(|y_1^* - x_i|, |y_2 - x_i|)$. Notice that $w_i(y_1^*) = |y_1^* - x_i|$. Since $\min(|y_1^* - x_i|, |y_2 - x_i|) \leq |y_1^* - x_i|$ for all y_2 in $[y_1^*, y_3^*]$, we have $w_i(y_1^*) = |y_1^* - x_i| = \max_{y\in[y_1^*, y_3^*]} w_i(y)$. Moreover, $w_i(y_3^*) = \min(|y_1^* - x_i|, |y_3^* - x_i|)$. We consider two cases.

Case 3.1: $|y_1^* - x_i| > |y_3^* - x_i|$. Then x_i belongs to $((y_1^* + y_3^*)/2, 1]$. Hence $|y_2 - x_i| \leq |y_1^* - x_i|$ for all y_2 in $[y_1^*, y_3^*]$. Thus $w_i(y_2) = |y_2 - x_i|$ for all y_2 in $[y_1^*, y_3^*]$, that is, $w_i(y_2)$ satisfies (1).

Case 3.2: $|y_1^* - x_i| \leq |y_3^* - x_i|$. Then $w_i(y_3^*) = |y_1^* - x_i| = \max_{y\in[y_1^*, y_3^*]} w_i(y) = w_i(y_1^*)$ and hence $w_i(y_2)$ satisfies (2).

Case 4: $a_i = \{F_2, F_3\}$. This case is symmetric to Case 3, and can be handled similarly.

Case 5: $a_i = \{F_1, F_2, F_3\}$. By definition, we have $w_i(y_2) = \min(|y_1^* - x_i|, |y_2 - x_i|, |y_3^* - x_i|)$. Notice that $w_i(y_1^*) = w_i(y_3^*) = \min(|y_1^* - x_i|, |y_3^* - x_i|)$. Also notice that for any y_2 in $[y_1^*, y_3^*]$, we have $w_i(y_2) = \min(|y_1^* - x_i|, |y_2 - x_i|, |y_3^* - x_i|) \leq \min(|y_1^* - x_i|, |y_3^* - x_i|) = w_i(y_1^*)$. Hence (2) holds.

This concludes our proof of Claim 1.

Claim 2: There is a solution that optimizes $\max_{\mathbf{y}} \mathrm{SW}(I, \mathbf{y})$ and builds facilities in at most two locations.

Proof: We establish the claim by proving that either $\mathrm{SW}(I, (y_1^*, y_1^*, y_3^*)) \geq \mathrm{SW}(I, \mathbf{y}^*)$ or $\mathrm{SW}(I, (y_1^*, y_3^*, y_3^*)) \geq \mathrm{SW}(I, \mathbf{y}^*)$.

Claim 1 implies that the set of agents $[n]$ can be partitioned into two sets $(S, \overline{S})$ such that $w_i(y_2)$ satisfies (1) for all i in S, and $w_i(y_2)$ satisfies (2) for all i in $\overline{S}$. Thus $\mathrm{SW}(y_2) = \sum_{i\in[n]} w_i(y_2) = \sum_{i\in S} w_i(y_2) + \sum_{i\in\overline{S}} w_i(y_2)$. By Lemma 1, there is a b in $\{y_1^*, y_3^*\}$ such that $\sum_{i\in S} w_i(b) \geq \sum_{i\in S} w_i(y_2)$ for all y_2 in $[y_1^*, y_3^*]$. For any i in $\overline{S}$, we deduce from (2) that $w_i(b) \geq w_i(y_2)$ for all y_2 in $[y_1^*, y_3^*]$. Therefore, $\mathrm{SW}(b) \geq \mathrm{SW}(y_2)$ for all y_2 in $[y_1^*, y_3^*]$. This completes our proof of Claim 2.

Having established Claim 2, we can assume without loss of generality that $y_2^* = y_3^*$. A simpler version of the arguments given in Claims 1 and 2 above can be used to prove that either $(0, y_2^*, y_2^*)$ or (y_2^*, y_2^*, y_2^*) is an efficient solution. If $(0, y_2^*, y_2^*)$ is efficient, then we can use a simpler version of the arguments in Claims 1 and 2 to prove that either $(0, 0, 0)$ or $(0, 1, 1)$ is efficient. If (y_2^*, y_2^*, y_2^*) is efficient, then by applying Lemma 2 with $k = 3$, we deduce that either $(0, 0, 0)$ or $(1, 1, 1)$ is efficient. Thus, there is a 0-1 efficient solution. The efficiency of Mechanism 2 follows. □

When $k = 2$ (resp., 1), we can add one (resp., two) dummy facilities and use Theorem 4 to establish that Mechanism 2 is efficient for $k = 2$ (resp., 1). Theorem 5 below provides a lower bound on the approximation ratio of any SGSP efficient mechanism; this result implies that Mechanism 2 is not SGSP.

Theorem 5. Let M be a single-facility SGSP α-efficient DOFLG mechanism for some positive constant α. Then $\alpha \geq 5/3$.

Proof. Let n be a large integer. We construct three $3n$-agent single-facility DOFLG instances (I, I), (I, I'), and (I, I''). In (I, I), (I, I'), and (I, I''), agent 1 is located at 0 and dislikes $\{F_1\}$, agent 2 is located at 1 and dislikes $\{F_1\}$, n agents are located at $1/2$ and dislike $\{F_1\}$, $n - 1$ agents forming a set U are located at 0 and dislike $\emptyset$, and $n - 1$ agents forming a set V are located at 1 and dislike $\emptyset$. In I, all agents report truthfully. In I' (resp. I''), all agents in U (resp., V) report $\{F_1\}$ and the remaining agents report truthfully.

Let the maximum social welfare for instances I' and I'' be OPT' and OPT'', respectively. It is easy to see that $\mathrm{OPT}' = \frac{5n}{2} - 1$ is achieved by building F_1 at 1 on I'. Likewise, $\mathrm{OPT}'' = \frac{5n}{2} - 1$ is achieved by building F_1 at 0 on I''. Let the social welfare achieved by mechanism M on I' (resp., I'') be ALG' (resp., ALG'').

Let M build F_1 at y (resp., y', y'') on I (resp., I', I''). We claim that $y = y' = y''$. To prove the claim, assume for the sake of contradiction that $y \neq y'$. We consider two cases. If $y < y'$, then agent 1 benefits by forming a coalition with V in (I, I'). Similarly, if $y > y'$, then agent 2 benefits by forming a coalition with U in (I, I'). Thus $y = y'$. Using a similar argument for (I, I''), we deduce that $y = y''$. Thus the claim holds. We now consider two cases.

Case 1: $y \leq 1/2$. Since $y' = y$, we have $\mathrm{ALG}' = \frac{3n}{2} - y$. Moreover, since $y \leq 1/2$, we have $\mathrm{ALG}' \leq 3n/2$. Using $\mathrm{OPT}' = \frac{5n}{2} - 1$ and $\mathrm{ALG}' \leq 3n/2$, we obtain

$$\alpha \geq \frac{\frac{5n}{2} - 1}{\frac{3n}{2}} = \frac{5}{3} - \frac{2}{3n}.$$

Case 2: $y \geq 1/2$. Using similar arguments as in Case 1, but now for y'', OPT'', and ALG'', we again find that $\alpha \geq \frac{5}{3} - \frac{2}{3n}$.

Thus, in all cases, α is at least $\frac{5}{3} - \frac{2}{3n}$. Since this bound approaches $\frac{5}{3}$ as n tends to infinity, the theorem follows. □

In view of Theorem 5, it is natural to try to determine the minimum value of α for which an SGSP α-efficient DOFLG mechanism exists. Below we present a 2-efficient SGSP mechanism for arbitrary k. Section 4.1.1 presents a 5/3-efficient SGSP mechanism for $k = 1$. For $k > 1$, it remains an interesting open problem to improve the approximation ratio of 2, or to establish a tighter lower bound for the approximation ratio.

Mechanism 3. Let $(n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance. Build all of the facilities at 0 if $\sum_{i\in[n]} x_i \geq \sum_{i\in[n]}(1 - x_i)$; otherwise, build all of the facilities at 1.

Theorem 6. Mechanism 3 is SGSP.

Proof. Reported dislikes do not affect the locations at which the facilities are built. Hence the theorem follows. □

Theorem 7. Mechanism 3 is 2-efficient.

Proof. Let $I = (n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance. Let ALG denote the social welfare obtained by Mechanism 3 on this instance, and let OPT denote the maximum possible social welfare on this instance. We need to prove that $2 \cdot \text{ALG} \geq \text{OPT}$.

Assume without loss of generality that Mechanism 3 builds all of the facilities at 0. (A symmetric argument can be used in the case where all facilities are built at 1.) Then the welfare of an agent i not in $\text{indiff}(I)$ is x_i and the welfare of an agent i in $\text{indiff}(I)$ is $\max(x_i, 1 - x_i) \geq x_i$. Thus $\text{ALG} \geq \sum_{i\in[n]} x_i$. As Mechanism 3 builds the facilities at 0 and not 1, we have $\sum_{i\in[n]} x_i \geq \sum_{i\in[n]}(1 - x_i)$, which implies that $\sum_{i\in[n]} x_i \geq n/2$. Combining the above two inequalities, we obtain $\text{ALG} \geq n/2$. Since no agent has welfare greater than 1, we have $n \geq \text{OPT}$. Thus $2 \cdot \text{ALG} \geq n \geq \text{OPT}$, as required. □

We now establish that the analysis of Theorem 7 is tight by exhibiting a two-facility DOFL instance on which Mechanism 3 achieves half of the optimal social welfare. For the reported DOFL instance $I = (2, 2, (0, 1), (\{F_1\}, \{F_2\}))$, it is easy to verify that the optimal social welfare is $\text{SW}(I, (1, 0)) = 2$, while the social welfare obtained by Mechanism 3 is $\text{SW}(I, (0, 0)) = 1$.

4.1.1 SGSP 5/3-efficient mechanism

In this section, we design an SGSP 5/3-efficient mechanism for the single-facility case. Throughout this section, we find it technically convenient to work over the interval $[-1, 1]$ instead of $[0, 1]$, and to allow the number of agents at a given location to be fractional. (We emphasize that the upper bound established in this section also holds for the special case in which the number of agents at any location is required to be an integer.) We begin by introducing some useful definitions; these definitions will only be used in the present section.

We define a distribution as a finite subset D of $[-1, 1] \times \mathbb{R}_{>0}$ where no two pairs in D share the same first component. For any distribution D, we define $D_{>0}$ as $\{(x, \gamma) \mid x > 0\}$. Related expressions such as $D_{=0}$ are defined similarly. For any distribution D, we define $\Gamma(D)$ as $\sum_{(x,\gamma)\in D} \gamma$ and $h(D)$ as $\sum_{(x,\gamma)\in D} \gamma x$.

We say that distribution D' is dominated by a distribution D if for any pair (x, γ') in D', there is a pair (x, γ) in D with $\gamma \geq \gamma'$. For any distributions D and D' such that D dominates D', and any y in $[-1, 1]$, we define $\Phi(D, D', y)$ as

$$\sum_{(x,\gamma)\in D} |y - x|(\gamma - \gamma') + (1 + |x|)\gamma'$$

where γ' denotes $\Gamma(D'_{=x})$.

Let us elaborate on the intended interpretation of the function $\Phi(D, D', y)$. The distribution D encodes the number of agents (which we allow to be fractional) at each location in $[-1, 1]$ with a nonzero number of agents: If the pair (x, γ) belongs to D, then there are $\gamma > 0$ agents at location x. The distribution D' encodes the preferences of the agents, in the following sense: If the pair (x, γ') belongs to D', then there is a pair of the form (x, γ) in D such that $\gamma' \leq \gamma$ (since D dominates D'), and we understand that γ' agents at x are indifferent to the facility and $\gamma - \gamma'$ agents at x dislike the facility. The location y represents the location of the facility. The value of the function $\Phi(D, D', y)$ represents the total welfare of the agents, where the welfare of an agent at location x is $|y - x|$ (i.e., the distance between the agent and the facility) if the agent dislikes the facility, and is $(1 + |x|)$ (i.e., the distance to the farthest endpoint of the interval $[-1, 1]$) otherwise.

As the reader may recognize, there is a close connection between the function $\Phi(D, D', y)$ used in the present section and the function $\mathrm{SW}(I, y)$ used elsewhere in the paper. Specifically, if we let I denote a DOFL instance, and we encode the locations and preferences of the agents in I using distributions D and D' as in the previous paragraph, then $\mathrm{SW}(I, y)$ is equal to $\Phi(D, D', y)$.

For any distributions D and D' such that D dominates D', any y in $\{-1, 0, 1\}$, and any y' in $[-1, 1]$, we define $\Psi(D, D', y, y')$ as

$$\frac{\Phi(D, D', y')}{\Phi(D, D', y)}.$$

For any distribution D and any location y in $\{-1, 0, 1\}$, we define $\beta(D, y)$ as the maximum, over all distributions D' dominated by D and all locations y' in $[-1, 1]$, of $\Psi(D, D', y, y')$. (Remark: If $D = \emptyset$, we consider the above ratio to be equal to 1.)

A DOFLG mechanism M maps any given distribution D to a location $M(D)$ in $[-1, 1]$. Observe that a DOFLG mechanism M is α-efficient if and only if $\beta(D, M(D)) \leq \alpha$ for all distributions D.

Mechanism 4. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance. We construct a corresponding distribution D as follows: For each location x with one or more agents, we include the pair (x, γ) in D where γ denotes the number of agents at x. We then build the facility F_1 at the lexicographically least location in $\arg\min_{y\in\{-1,0,1\}} \beta(D, y)$.

Mechanism 4 is SGSP because it disregards the reported aversion profile. In the remainder of this section, we establish two other key properties of Mechanism 4. First, in Theorem 8 below, we establish that Mechanism 4 admits a fast implementation. Second, in Theorem 9 below, we show that Mechanism 4 achieves an approximation ratio of 5/3. Given the 5/3 lower bound established in Theorem 5, this approximation ratio is optimal for any SGSP mechanism.

Lemma 3 below is a weighted generalization of Lemma 1, and can be justified in a similar manner. For the sake of completeness, and also to exercise some of the notations introduced above, we provide an alternate proof below.

Lemma 3. Let D be a distribution. Then

$$\max_{y\in[-1,1]} \Phi(D,\emptyset,y) = \max_{y\in\{-1,1\}} \Phi(D,\emptyset,y).$$

Proof. Let y belong to $[-1,1]$. We need to prove that $\Phi(D,\emptyset,y)$ is at most $\max(\Phi(D,\emptyset,-1),\Phi(D,\emptyset,1))$. We consider two cases.

Case 1: $\Gamma(D_{\le y}) \ge \Gamma(D)/2$. Observe that

$$\begin{aligned}\Phi(D_{\le y},\emptyset,1) - \Phi(D_{\le y},\emptyset,y) &= \sum_{(x,\gamma)\in D_{\le y}} [(1-x)-(y-x)]\gamma \\ &= (1-y)\Gamma(D_{\le y}),\end{aligned}$$

$\Phi(D_{>y},\emptyset,1) \ge 0$, and

$$\begin{aligned}\Phi(D_{>y},\emptyset,y) &= \sum_{(x,\gamma)\in D_{>y}} (x-y)\gamma \\ &\le (1-y)\Gamma(D_{>y}).\end{aligned}$$

Combining the above inequalities with $\Gamma(D_{\le y}) \ge \Gamma(D)/2 \ge \Gamma(D_{>y})$, we obtain

$$\begin{aligned}& \Phi(D,\emptyset,1) - \Phi(D,\emptyset,y) \\ = \; & \Phi(D_{\le y},\emptyset,1) - \Phi(D_{\le y},\emptyset,y) + \Phi(D_{>y},\emptyset,1) - \Phi(D_{>y},\emptyset,y) \\ \ge \; & (1-y)\Gamma(D_{\le y}) + 0 - (1-y)\Gamma(D_{>y}) \\ \ge \; & 0.\end{aligned}$$

Case 2: $\Gamma(D_{\ge y}) \ge \Gamma(D)/2$. Using an argument symmetric to that used in Case 1, we find that $\Phi(D,\emptyset,y) \le \Phi(D,\emptyset,-1)$. □

Lemma 4 below shows that given the locations and preferences of the agents, the optimal location for the facility is at one of the endpoints of the interval $[-1,1]$.

Lemma 4. Let D and D' be distributions such that D dominates D'. Then

$$\max_{y\in[-1,1]} \Phi(D,D',y) = \max_{y\in\{-1,1\}} \Phi(D,D',y).$$

Proof. Let y belong to $[-1,1]$ and let D^* denote the unique distribution such that $\Gamma(D^*_{=x}) = \Gamma(D_{=x}) - \Gamma(D'_{=x})$ for all x in $[-1,1]$. Observe that $\Phi(D,D',y)$ is equal to $\Phi(D^*,\emptyset,y) + \Phi(D',D',y)$. Since $\Phi(D',D',y)$ is equal to $\sum_{(x,\gamma)\in D'}(1+|x|)\gamma$, which is independent of y, and since Lemma 3 implies that $\Phi(D^*,\emptyset,y)$ is at most $\max_{y'\in\{-1,1\}} \Phi(D^*,\emptyset,y')$, we deduce that $\Phi(D,D',y)$ is at most $\max_{y'\in\{-1,1\}} \Phi(D,D',y')$. □

For any distributions D and D', we define $\min(D, D')$ as the unique distribution D^* such that

$$\Gamma(D^*_{=x}) = \min(\Gamma(D_{=x}), \Gamma(D'_{=x}))$$

for all x in $[-1, 1]$. Lemma 5 below gives a useful way to rewrite the expression $\Phi(D, D', y)$ for all y in $\{-1, 0, 1\}$.

Lemma 5. Let D and D' be distributions such that D dominates D'. Then $\Phi(D, D', -1)$ is equal to $\Gamma(D) + h(D) - 2h(\min(D', D_{<0}))$, $\Phi(D, D', 0)$ is equal to $\Gamma(D') - h(D_{<0}) + h(D_{>0})$, and $\Phi(D, D', 1)$ is equal to $\Gamma(D) - h(D) + 2h(\min(D', D_{>0}))$.

Proof. Let (x, γ) belong to D and let γ' denote $\Gamma(D'_{=x})$.

The contribution of the pair (x, γ) to $\Phi(D, D', -1)$ is $\gamma'(1-x) + (\gamma - \gamma')(1 + x) = \gamma + \gamma x - 2\gamma' x$ if x belongs to $D_{<0}$, and is $\gamma + \gamma x$ otherwise. The first claim of the lemma follows. A symmetric argument establishes the third claim.

The contribution of the pair (x, γ) to $\Phi(D, D', 0)$ is $\gamma'(1 - x) - (\gamma - \gamma')x = \gamma' - \gamma x$ if x belongs to $D_{<0}$, and is $\gamma'(1 + x) + (\gamma - \gamma')x = \gamma' + \gamma x$ otherwise. The second claim follows. □

Given the locations of the agents, but not their preferences, Lemma 6 below characterizes the best possible approximation ratio that can be guaranteed by locating the facility at -1 (resp., 1).

Lemma 6. Let D be a distribution and let y belong to $\{-1, 1\}$. Then

$$\beta(D, y) = \Psi(D, D', y, -y)$$

where D' denotes $D_{>0}$ (resp., $D_{<0}$) if y is equal to -1 (resp., 1).

Proof. By symmetry, it is sufficient to consider the case where y is equal to 1. Lemma 4 implies that $\beta(D, 1)$ is equal to the maximum, over all distributions D' dominated by D, of $\max_{y' \in \{-1,1\}} \Psi(D, D', 1, y')$. Lemma 5 implies that setting D' to $D_{<0}$ simultaneously maximizes $\Phi(D, D', -1)$ and minimizes $\Phi(D, D', 1)$. Thus, setting D' to $D_{<0}$ maximizes $\Psi(D, D', 1, -1)$. Moreover, it is easy to see that $\Psi(D, D_{<0}, 1, -1) \geq 1$, and that $\Psi(D, D', 1, 1) = 1$ for all D' dominated by D. The claim of the lemma follows. □

Given the locations of the agents, but not their preferences, Lemma 7 gives a useful way to rewrite the best possible approximation ratio that can be guaranteed by locating the facility in $\{-1, 1\}$.

Lemma 7. Let D be a distribution. Then

$$\min_{y \in \{-1,1\}} \beta(D, y) = \frac{\Gamma(D) - h(D_{<0}) + h(D_{>0})}{\Gamma(D) + |h(D)|}.$$

Proof. Follows straightforwardly from Lemmas 5 and 6. □

We will make repeated use of the following simple fact, so we state it explicitly.

Fact 1. Let $f(x)$ denote $\frac{x+c}{x+1}$ where c is a positive constant. Then $c > 1$ (resp., $c < 1$, $c = 1$) implies $f(x_0) > f(x_1)$ (resp., $f(x_0) < f(x_1)$, $f(x_0) = f(x_1)$) for all x_0 and x_1 such that $0 \leq x_0 < x_1$.

For any distribution D and any integer k in $\{0, \ldots, |D|\}$, we define prefix(D, k) (resp., suffix(D, k)) as the subset of D consisting of the k lexicographically smallest (resp., largest) pairs. Given the locations of the agents, but not their preferences, Lemma 8 below characterizes the best possible approximation ratio that can be guaranteed by locating the facility at 0.

Lemma 8. Let D be a distribution. Then $\beta(D, 0)$ is equal to

$$\max\left(\max_{k\in\{0,\ldots,|D_{<0}|\}} \Psi(D, \text{prefix}(D,k), 0, -1), \max_{k\in\{0,\ldots,|D_{>0}|\}} \Psi(D, \text{suffix}(D,k), 0, 1)\right).$$

Proof. Lemma 4 implies that $\beta(D, 0)$ is equal to the maximum, over all distributions D' dominated by D, of $\max_{y\in\{-1,1\}} \Psi(D, D', 0, y)$. Let D' be a distribution dominated by D that maximizes $\Psi(D, D', 0, -1)$, and let ξ denote $\Psi(D, D', 0, -1)$.

Claim 1: $D'_{\geq 0} = \emptyset$. Assume for the sake of contradiction that $D'_{\geq 0} \neq \emptyset$. Lemma 5 implies that $\Phi(D, D'_{<0}, -1) = \Phi(D, D', -1)$ and $\Phi(D, D'_{<0}, 0) < \Phi(D, D', 0)$. Hence $\Psi(D, D'_{<0}, 0, -1) > \xi$, a contradiction.

Claim 2: For any x and x' such that $-1 \leq x < x' < 0$ and $\Gamma(D'_{=x'}) > 0$, we have $\Gamma(D'_{=x}) = \Gamma(D_{=x})$. Assume for the sake of contradiction that $-1 \leq x < x' < 0$, $\Gamma(D'_{=x'}) > 0$, and $\Gamma(D'_{=x}) < \Gamma(D_{=x})$. Let δ denote $\min(\Gamma(D'_{=x'}), \Gamma(D_{=x}) - \Gamma(D'_{=x}))$. Thus $\delta > 0$. Let D'' denote the distribution

$$(D' \setminus \{(x, \Gamma(D'_{=x})), (x', \Gamma(D'_{=x'}))\}) \cup \{(x, \Gamma(D'_{=x}) + \delta), (x', \Gamma(D'_{=x'}) - \delta)\})$$

Note that D'' is dominated by D. Lemma 5 implies that $\Phi(D, D'', -1) > \Phi(D, D', -1)$ and $\Phi(D, D'', 0) = \Phi(D, D', 0)$. Hence $\Psi(D, D'', 0, -1) > \xi$, a contradiction.

Let k^* denote $|D'|$, which is at most $|D_{<0}|$ by Claim 1.

Claim 3: If $k^* > 0$ then

$$\max(\Psi(D, \text{prefix}(D, k^* - 1), 0, -1), \Psi(D, \text{prefix}(D, k^*), 0, -1)) = \xi.$$

Let (x, γ) denote the lexicographically greatest pair in prefix(D, k^*), let $D^{(0)}$ denote prefix$(D, k^* - 1)$, and for any t such that $0 < t \leq 1$, let $D^{(t)}$ denote $D^{(0)} + (x, t\gamma)$. For any t in $[0, 1]$, let $g(t)$ denote $\Psi(D, D^{(t)}, 0, -1)$. Using Fact 1, it is straightforward to prove that $g(t)$ is monotonic over the unit interval and hence $\max_{t\in[0,1]} g(t) = \max(g(0), g(1))$. Claim 3 follows.

Combining Claim 3 with the observation that $k^* = 0$ implies $D' = \text{prefix}(D, 0)$, we deduce that there is an integer k in $\{0, \ldots, |D_{<0}|\}$ such that $\Psi(D, \text{prefix}(D, k), 0, -1)) =$

ξ. In other words, there is an integer k in $\{0, \ldots, |D_{<0}|\}$ such that $\Psi(D, D', 0, -1))$ is maximized by setting D' to $\text{prefix}(D, k)$.

Using an entirely symmetric argument, we find that there is an integer k in $\{0, \ldots, |D_{>0}|\}$ such that $\Psi(D, D', 0, 1))$ is maximized by setting D' to $\text{suffix}(D, k)$. The claim of the lemma follows. $\square$

We are now ready to describe a fast implementation of Mechanism 4.

Theorem 8. There is an $O(|D| \log |D|)$-time implementation of Mechanism 4.

Proof. Let D be a given distribution. We begin by spending $O(|D| \log |D|)$ time to sort the pairs of D in lexicographic order. Using Lemma 6, we can then compute $\beta(D, -1)$ and $\beta(D, 1)$ in $O(|D|)$ time. Likewise, using Lemma 8, we can compute $\beta(D, 0)$ in $O(|D|)$ time. The theorem follows. $\square$

For any distribution D, let D_- (resp., D_+) denote $\text{prefix}(D, k)$ (resp., $\text{suffix}(D, k)$) where k is the least integer in $\{0, \ldots, |D_{<0}|\}$ (resp., $\{0, \ldots, |D_{>0}|\}$) maximizing the first (resp., second) max expression in the statement of Lemma 8.

We say that a distribution D is *trivial* if $D_{<0} = \emptyset$ or $D_{>0} = \emptyset$. We say that a distribution D is *special* if $|D_{<0}|$ and $|D_{>0}|$ each belong to $\{1, 2\}$. Lemma 9 below allows us to bound the performance of Mechanism 4 on any nontrivial distribution of agent locations in terms of its performance on any special distribution of agent locations.

Lemma 9. Let D be a nontrivial distribution. Then there exists a special distribution D' such that

$$\beta(D, y) = \beta(D', y)$$

for all locations y in $\{-1, 0, 1\}$.

Proof. We begin by introducing some useful notation. For any two distributions D and D', we write $D \sim D'$ to mean that $\Gamma(D) = \Gamma(D')$ and $h(D) = h(D')$, and we write $D \cong D'$ to mean that $D \sim D'$, $D_{<0} \sim D'_{<0}$, $D_{>0} \sim D'_{>0}$, $D_- \sim D'_-$, and $D_+ \sim D'_+$.

Using Lemmas 7 and 8, it is straightforward to prove that for any distributions D and D' such that $D \cong D'$, we have $\beta(D, y) = \beta(D', y)$ for all locations y in $\{-1, 0, 1\}$. Accordingly, it is sufficient to show how to map any given nontrivial distribution D to a special distribution D' such that $D \cong D'$.

Let D be a nontrivial distribution, let $D^{(-2)}$ denote D_-, let $D^{(-1)}$ denote $D_{<0} \setminus D_-$, let $D^{(0)}$ denote $D_{=0}$, let $D^{(1)}$ denote $D_{>0} \setminus D_+$, and let $D^{(2)}$ denote D_+. For any i in $\{-2, \ldots, 2\}$, let γ_i denote $\Gamma(D^{(i)})$, let Δ_i denote $h(D^{(i)})$, and let D' denote the distribution

$$\{(\Delta_i/\gamma_i, \gamma_i) \mid i \in \{-2, \ldots, 2\} \wedge \gamma_i > 0\}.$$

It is straightforward to argue that D' is special, $D' \sim D$, $D'_{<0} \sim D_{<0}$, and $D'_{>0} \sim D_{>0}$. To complete the proof, it remains to argue that $D'_- \sim D_-$ and $D'_+ \sim D_+$. Claims 1 and 2 below imply that $D'_- \sim D_-$. A symmetric argument establishes that $D'_+ \sim D_+$.

Claim 1: For any k in $\{0,\ldots,|D'_{<0}|\}$, there exists an ℓ in $\{0,\ldots,|D_{<0}|\}$ such that $\text{prefix}(D',k) \sim \text{prefix}(D,\ell)$; moreover, there exists a k in $\{0,\ldots,|D'_{<0}|\}$ such that $\text{prefix}(D',k) \sim D_-$. To prove Claim 1, we consider three cases.

Case 1: $D_- = \emptyset$. In this case, we have $|D'_{<0}| = 1$, $\text{prefix}(D',0) = \emptyset = \text{prefix}(D,0) = D_-$ and $\text{prefix}(D',1) \sim \text{prefix}(D,|D_{<0}|) = D_{<0}$.

Case 2: $D_- \neq \emptyset$ and $|D'_{<0}| = 1$. In this case, we have $\text{prefix}(D',0) = \emptyset = \text{prefix}(D,0)$ and $\text{prefix}(D',1) \sim \text{prefix}(D,|D_{<0}|) = D_{<0} = D_-$.

Case 3: $D_- \neq \emptyset$ and $|D'_{<0}| = 2$. In this case, we have $\text{prefix}(D',0) = \emptyset = \text{prefix}(D,0)$, $\text{prefix}(D',1) \sim \text{prefix}(D,|D_-|) = D_-$, and $\text{prefix}(D',2) \sim \text{prefix}(D,|D_{<0}|) = D_{<0}$.

Claim 2: For any k in $\{0,\ldots,|D'_{<0}|\}$ and any ℓ in $\{0,\ldots,|D_{<0}|\}$ such that $\text{prefix}(D',k) \sim \text{prefix}(D,\ell)$, we have

$$\Psi(D', \text{prefix}(D',k), 0, -1) = \Psi(D, \text{prefix}(D,\ell), -1, 0).$$

Claim 2 follows from Lemma 5 since $D' \sim D$, $D'_{<0} \sim D_{<0}$, $D'_{>0} \sim D_{>0}$, and $\text{prefix}(D',k) \sim \text{prefix}(D,\ell)$. □

When the agent locations are described by a special distribution D such that $h(D) > 0$, Lemma 10 below enables us to bound the performance of Mechanism 4 in terms of its performance on a special distribution D' such that either (1) $h(D') = 0$ or (2) $h(D') > 0$ and $1 = |D'_{>0}| < |D_{>0}| = 2$. A symmetric claim holds for the case where $h(D) < 0$.

The main idea underlying the proof of Lemma 10 is to create the desired distribution D' from the given distribution D by sliding the lexicographically least pair (x,γ) in $D_{>0}$ towards 0 (i.e., replacing it with a pair (x',γ) where $0 \leq x' < x$) until either 0 is reached or $h(D') = 0$. Using Lemmas 7 and 8, we are able to prove that the best possible approximation ratio that can be guaranteed by Mechanism 4 under distribution D' is higher than it is under distribution D.

Lemma 10. Let D be a special distribution such that $h(D) > 0$. Then there exists a special distribution D' such that

$$\min_{y\in\{-1,0,1\}} \beta(D',y) > \min_{y\in\{-1,0,1\}} \beta(D,y),$$

$0 \leq h(D') < h(D)$, and if $h(D') > 0$ then $1 = |D'_{>0}| < |D_{>0}| = 2$.

Proof. Let (x^-,γ^-) denote the lexicographically least pair in D, and let (x^+,γ^+) denote the lexicographically greatest pair in D.

Let (x^*,γ^*) denote the lexicographically least pair in $D_{>0}$. We remark that if $|D_{>0}| = 1$, then $(x^*,\gamma^*) = (x^+,\gamma^+)$.

We define x' as $\max(x^* - h(D)/\gamma^*, 0)$; thus $0 \leq x' < x^*$. We now construct the desired distribution D' as follows. If $x' > 0$, then we define D' as $D - (x^*,\gamma^*) + (x',\gamma^*)$. Otherwise, $x' = 0$ and we define D' as $D_{\neq 0} - (x^*,\gamma^*) + (0, \Gamma(D_{=0}) + \gamma^*)$. It is easy to see that the following conditions hold: D' is special; $0 \leq h(D') < h(D)$; if $h(D') > 0$ then $1 = |D'_{>0}| < |D_{>0}| = 2$; $\Gamma(D') = \Gamma(D)$; $D'_{<0} = D_{<0}$; $0 \leq h(D'_{>0}) - \gamma^* x' = h(D_{>0}) - \gamma^* x^*$.

Claim 1: $\min_{y\in\{-1,1\}} \beta(D',y) > \min_{y\in\{-1,1\}} \beta(D,y)$. Since $h(D) > 0$, Lemma 7 implies

$$\min_{y\in\{-1,1\}} \beta(D,y) = \frac{\Gamma(D) - h(D_{<0}) + h(D_{>0})}{\Gamma(D) + h(D_{<0}) + h(D_{>0})}.$$

Similarly, we have

$$\min_{y\in\{-1,1\}} \beta(D',y) = \frac{\Gamma(D') - h(D'_{<0}) + h(D'_{>0})}{\Gamma(D') + h(D'_{<0}) + h(D'_{>0})}.$$

Since $\Gamma(D') = \Gamma(D)$, $D'_{<0} = D_{<0}$, $0 \le x' < x^*$, $0 \le h(D'_{>0}) - \gamma^* x' = h(D_{>0}) - \gamma^* x^*$, $\gamma^* > 0$, and $\Gamma(D) - h(D_{<0}) > \Gamma(D) + h(D_{<0}) > 0$, Fact 1 implies that Claim 1 holds.

Let X denote $\max_{y\in\{-1,1\}} \Psi(D,\emptyset,0,y)$, let Y_{-1} denote $\Psi(D,D_{<0},0,-1)$, let Y_1 denote $\Psi(D,D_{>0},0,1)$, let Z_{-1} denote $\Psi(D,\{(x^-,\gamma^-)\},0,-1)$, let Z_1 denote $\Psi(D,\{(x^+,\gamma^+)\},0,1)$, and let X', Y'_{-1}, Y'_1, and Z'_{-1} be defined similarly, except in terms of D' instead of D. If $|D_{>0}| = 2$ then (x^+,γ^+) belongs to D', and we likewise define Z'_1 as $\Psi(D',\{(x^+,\gamma^+)\},0,1)$.

Since D is special, we deduce from Lemma 8 that

$$\beta(D,0) = \begin{cases} \max(X, Y_{-1}, Y_1, Z_{-1}) & \text{if } |D_{>0}| = 1 \\ \max(X, Y_{-1}, Y_1, Z_{-1}, Z_1) & \text{if } |D_{>0}| = 2, \end{cases}$$

and

$$\beta(D',0) = \begin{cases} \max(X', Y'_{-1}, Y'_1, Z'_{-1}) & \text{if } |D_{>0}| = 1 \\ \max(X', Y'_{-1}, Y'_1, Z'_{-1}, Z'_1) & \text{if } |D_{>0}| = 2. \end{cases}$$

Thus Claims 2 through 6 below imply that $\beta(D,0) \le \beta(D',0)$. Combining this inequality with Claim 1, we find that D' satisfies the requirements of the lemma. It remains only to prove Claims 2 through 6.

Claim 2: $X' > X$. Since $h(D) > 0$, Lemma 5 implies that

$$\begin{aligned} X &= \frac{\Gamma(D) + h(D)}{-h(D_{<0}) + h(D_{>0})} \\ &= \frac{\Gamma(D) + h(D_{<0}) + h(D_{>0})}{-h(D_{<0}) + h(D_{>0})}. \end{aligned}$$

Likewise, since $h(D') \ge 0$, Lemma 5 implies that

$$X' = \frac{\Gamma(D') + h(D'_{<0}) + h(D'_{>0})}{-h(D'_{<0}) + h(D'_{>0})}.$$

Since $\Gamma(D') = \Gamma(D)$, $D'_{<0} = D_{<0}$, $0 \le x' < x^*$, $0 \le h(D'_{>0}) - \gamma^* x' = h(D_{>0}) - \gamma^* x^*$, $\gamma^* > 0$, and $\Gamma(D) + h(D_{<0}) \ge \Gamma(D_{>0}) \ge h(D_{>0}) > -h(D_{<0}) > 0$, Fact 1 implies that Claim 2 holds.

Claim 3: $Y'_{-1} > Y_{-1}$. Lemma 5 implies that

$$Y_{-1} = \frac{\Gamma(D) - h(D_{<0}) + h(D_{>0})}{\Gamma(D_{<0}) - h(D_{<0}) + h(D_{>0})}$$

and

$$Y'_{-1} = \frac{\Gamma(D') - h(D'_{<0}) + h(D'_{>0})}{\Gamma(D'_{<0}) - h(D'_{<0}) + h(D'_{>0})}.$$

Since $\Gamma(D') = \Gamma(D)$, $D'_{<0} = D_{<0}$, $0 \le x' < x^*$, $0 \le h(D'_{>0}) - \gamma^* x' = h(D_{>0}) - \gamma^* x^*$, $\gamma^* > 0$, and $\Gamma(D) - h(D_{<0}) > \Gamma(D_{<0}) - h(D_{<0}) > 0$, Fact 1 implies that Claim 3 holds.

Claim 4: $Y'_1 > Y_1$. Lemma 5 implies that

$$Y_1 = \frac{\Gamma(D) - h(D_{<0}) + h(D_{>0})}{\Gamma(D_{>0}) - h(D_{<0}) + h(D_{>0})}$$

and

$$Y'_1 = \frac{\Gamma(D') - h(D'_{<0}) + h(D'_{>0})}{\Gamma(D'_{>0}) - h(D'_{<0}) + h(D'_{>0})}.$$

Let Y_1^* denote the intermediate expression

$$\frac{\Gamma(D') - h(D'_{<0}) + h(D'_{>0})}{\Gamma(D_{>0}) - h(D'_{<0}) + h(D'_{>0})}.$$

Since $\Gamma(D') = \Gamma(D)$, $D'_{<0} = D_{<0}$, $0 \le x' < x^*$, $0 \le h(D'_{>0}) - \gamma^* x' = h(D_{>0}) - \gamma^* x^*$, $\gamma^* > 0$, and $\Gamma(D) - h(D_{<0}) > \Gamma(D_{>0}) - h(D_{<0}) > 0$, Fact 1 implies that $Y_1^* > Y_1$. Since $\Gamma(D'_{>0}) \le \Gamma(D_{>0})$, we have $Y'_1 \ge Y_1^*$. Hence $Y'_1 > Y_1$.

Claim 5: $Z'_{-1} > Z_{-1}$. Observe that $\Gamma(D_{<0}) + h(D_{<0}) \ge \gamma^- + \gamma^- x^-$ and $\Gamma(D_{>0}) \ge h(D_{>0}) > -h(D_{<0})$. Thus we obtain the inequality $\Gamma(D) - h(D_{<0}) - \gamma^- x^- > \gamma^- - h(D_{<0})$, which will be used below. Lemma 5 implies that

$$\begin{aligned} Z_{-1} &= \frac{\Gamma(D) + h(D) - 2\gamma^- x^-}{\gamma^- - h(D_{<0}) + h(D_{>0})} \\ &= \frac{\Gamma(D) - h(D_{<0}) - 2\gamma^- x^- + h(D_{>0})}{\gamma^- - h(D_{<0}) + h(D_{>0})} \end{aligned}$$

and

$$Z'_{-1} = \frac{\Gamma(D') - h(D'_{<0}) - 2\gamma^- x^- + h(D'_{>0})}{\gamma^- - h(D'_{<0}) + h(D'_{>0})}.$$

Since $\Gamma(D') = \Gamma(D)$, $D'_{<0} = D_{<0}$, $0 \le x' < x^*$, $0 \le h(D'_{>0}) - \gamma^* x' = h(D_{>0}) - \gamma^* x^*$, $\gamma^* > 0$, and $\Gamma(D) - h(D_{<0}) - 2\gamma^- x^- > \Gamma(D) - h(D_{<0}) - \gamma^- x^- > \gamma^- - h(D_{<0}) > 0$, Fact 1 implies that Claim 5 holds.

Claim 6: If $|D_{>0}| = 2$ then $Z'_1 > Z_1$. Assume that $|D_{>0}| = 2$. Lemma 5 implies that $Z_1 = A/B$ where $A = \Gamma(D) - h(D) + 2\gamma^+ x^+ > 0$ and $B = \gamma^+ - h(D_{<0}) + h(D_{>0}) > 0$. Similarly, Lemma 5 implies that $Z'_1 = A'/B'$ where $A' = \Gamma(D') - h(D') + 2\gamma^+ x^+ > 0$ and $B' = \gamma^+ - h(D'_{<0}) + h(D'_{>0}) > 0$. Since $\Gamma(D') = \Gamma(D)$ and $h(D') < h(D)$, we have $A' > A$. Since $D'_{<0} = D_{<0}$ and $h(D'_{>0}) < h(D_{>0})$, we have $B' < B$. It follows that $Z'_1 > Z_1$. □

When the agent locations are described by a special distribution D such that $h(D) \neq 0$, Lemma 11 below enables us to bound the performance of Mechanism 4 in terms of its performance on a special distribution D' such that $h(D') = 0$.

Lemma 11. Let D be a special distribution such that $h(D) \neq 0$. Then there exists a special distribution D' such that

$$\min_{y \in \{-1,0,1\}} \beta(D', y) > \min_{y \in \{-1,0,1\}} \beta(D, y),$$

and $h(D') = 0$.

Proof. Let D be a special distribution such that $h(D) \neq 0$. If $h(D) > 0$, we can establish the claim using at most two applications of Lemma 10. If $h(D) < 0$, we proceed in the same fashion by appealing to a symmetric version of Lemma 10. □

When the agent locations are described by a special distribution D such that $h(D) = 0$, and locating the facility in $\{-1, 1\}$ does not guarantee an approximation ratio of at most 5/3, Lemma 12 below provides a useful lower bound on $h(D_{>0})$.

Lemma 12. Let D be a special distribution such that $h(D) = 0$ and

$$\min_{y \in \{-1,1\}} \beta(D, y) > \frac{5}{3}.$$

Then $h(D_{>0}) > \Gamma(D)/3$.

Proof. Since $h(D) = 0$, we have $h(D_{>0}) = -h(D_{<0})$. Thus Lemma 7 implies that

$$\min_{y \in \{-1,1\}} \beta(D, y) = 1 + \frac{2h(D_{>0})}{\Gamma(D)}.$$

The claim of the lemma follows. □

When the agent locations are described by a special distribution D such that $h(D) = 0$, and the inequality $h(D_{>0}) > \Gamma(D)/3$ appearing in the statement of Lemma 12 is satisfied, Lemma 13 below shows that locating the facility at 0 results in performance within a factor of 5/3 of that obtained by locating it at 1. A symmetric claim shows that locating the facility at 0 results in performance within a factor of 5/3 of that obtained by locating it at -1.

Lemma 13. Let D be a special distribution such that $h(D) = 0$ and $h(D_{>0}) > \Gamma(D)/3$. Then

$$\max_{k \in \{0,\ldots,|D_{>0}|\}} \Psi(D, \mathrm{suffix}(D, k), 0, 1) < \frac{5}{3}.$$

Proof. Let X denote $\Psi(D, \emptyset, 0, 1)$, let Y denote $\Psi(D, D_{>0}, 0, 1)$, let (x, γ) denote the lexicographically greatest pair in D, and let Z denote $\Psi(D, \{(x, \gamma)\}, 0, 1)$. It is sufficient to prove that $\max(X, Y, Z) < \frac{5}{3}$. The latter inequality is immediate from Claims 1 through 3 below. In the proofs of Claims 1 through 3, let Δ denote $h(D_{>0})$, which is equal to $-h(D_{<0})$ since $h(D) = 0$.

Claim 1: $X < \frac{3}{2}$. We have $X = \frac{\Gamma(D)}{2\Delta}$, and the claimed inequality follows since $\Delta > \Gamma(D)/3$.

Claim 2: $Y < \frac{5}{3}$. We have $Y = \frac{\Gamma(D)+2\Delta}{\Gamma(D_{>0})+2\Delta}$. Since $\Gamma(D_{>0}) \geq \Delta$, we have $Y \leq \frac{\Gamma(D)+2\Delta}{3\Delta}$. Since the latter bound is strictly decreasing in Δ and $\Delta > \Gamma(D)/3$, the desired inequality follows.

Claim 3: $Z < \frac{5}{3}$. We have $Z = \frac{\Gamma(D)+2\gamma x}{\gamma+2\Delta}$. Since $\gamma > 0$ and $\gamma \leq \Delta/x$, Fact 1 (viewing Z as a function of γ) implies that Z is at most

$$\max\left(\frac{\Gamma(D)}{2\Delta}, \frac{\Gamma(D)+2\Delta}{(2+\frac{1}{x})\Delta}\right).$$

Since $\Delta > \Gamma(D)/3$, the first argument above is less than $3/2$. Since $0 < x < 1$, the second argument above is less than $\frac{\Gamma(D)+2\Delta}{3\Delta}$, which is less than $5/3$. □

When the agent locations are described by a special distribution D such that $h(D) = 0$, and the inequality $h(D_{>0}) > \Gamma(D)/3$ appearing in the statement of Lemma 12 is satisfied, Lemma 14 below shows that locating the facility at 0 guarantees an approximation ratio less than $5/3$.

Lemma 14. Let D be a special distribution such that $h(D) = 0$ and $h(D_{>0}) > \Gamma(D)/3$. Then $\beta(D, 0) < \frac{5}{3}$.

Proof. Using an argument that is symmetric to the proof of Lemma 13, we obtain

$$\max_{k\in\{0,\ldots,|D_{<0}|\}} \Psi(D, \text{prefix}(D, k), 0, -1) < \frac{5}{3}.$$

Combining the above inequality with Lemma 13, the claim of the lemma follows by Lemma 8. □

We are now ready to bound the approximation ratio of Mechanism 4.

Theorem 9. Mechanism 4 achieves an approximation ratio of $5/3$.

Proof. Let D be a distribution. Given the definition of Mechanism 4, we need to prove that

$$\min_{y\in\{-1,0,1\}} \beta(D, y) \leq \frac{5}{3}.$$

If D is trivial, it is straightforward to argue that $\min_{y\in\{-1,0,1\}} \beta(D, y) = 1$. If D is nontrivial, the desired inequality follows from Lemmas 9, 11, 12, and 14. □

4.2 The cycle

Now we present a simple adaptation of Mechanism 3 for the case where the agents are located on a cycle.

Mechanism 5. Let $(n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance. Build all of the facilities at 0 if

$$\sum_{i\in[n]} \Delta(x_i, 0) \geq \sum_{i\in[n]} \Delta(x_i, 1/2);$$

otherwise, build all of the facilities at $1/2$.

As with Mechanism 3, reported dislikes do not affect the locations at which Mechanism 5 builds the facilities. Hence Mechanism 5 is SGSP.

Theorem 10. Mechanism 5 is SGSP.

Theorem 11. Mechanism 5 is 2-efficient.

Proof. We sketch a proof that is similar to our proof of Theorem 7. Let $I = (n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance. Let ALG denote the social welfare obtained by Mechanism 5 on this instance, and let OPT denote the maximum possible social welfare on this instance. We need to prove that $2 \cdot \text{ALG} \geq \text{OPT}$.

Assume without loss of generality that Mechanism 5 builds all of the facilities at 0. (A symmetric argument handles the case where all of the facilities are built at 1/2.) Using similar arguments as in the proof of Theorem 7, we obtain $\text{ALG} \geq \sum_{i\in[n]} \Delta(x_i, 0)$. As Mechanism 5 builds the facilities at 0 and not 1/2, we have $\sum_{i\in[n]} \Delta(x_i, 0) \geq \sum_{i\in[n]} \Delta(x_i, 1/2)$. We also have $\Delta(x_i, 0) + \Delta(x_i, 1/2) = 1/2$ for all agents i. Thus $\sum_{i\in[n]} \Delta(x_i, 0) \geq n/4$. Hence $\text{ALG} \geq n/4$. Since no agent has welfare greater than 1/2, we have $n/2 \geq \text{OPT}$. Thus $2 \cdot \text{ALG} \geq n/2 \geq \text{OPT}$, as required. □

4.3 The unit square

We now show how to adapt Mechanism 3 to the case where the agents are located in the unit square.

Mechanism 6. Let $(n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance. For each point p in $\{0, 1\}^2$, let d_p denote $\sum_{i\in[n]} \Delta(x_i, p)$. Let q be the point in $\{0, 1\}^2$ that maximizes d_q, breaking ties lexicographically. Build all of the facilities at q.

Mechanism 6 computes $4n$ Euclidean distances, and runs in $O(n)$ time. As in the case of Mechanism 3, reported dislikes do not affect the locations at which Mechanism 6 builds the facilities. Hence Mechanism 6 is SGSP.

Theorem 12. Mechanism 6 is SGSP.

Theorem 13. Mechanism 6 is 2-efficient.

Proof. We sketch a proof that is similar to our proof of Theorem 7. Let $I = (n, k, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance. Let ALG denote the social welfare obtained by Mechanism 6 on this instance, and let OPT denote the maximum possible social welfare on this instance. We need to prove that $2 \cdot \text{ALG} \geq \text{OPT}$.

Assume without loss of generality that Mechanism 6 builds all of the facilities at $(0, 0)$. (A symmetric argument handles the three remaining cases.) Using similar arguments as in the proof of Theorem 7, we obtain $\text{ALG} \geq \sum_{i\in[n]} \Delta(x_i, (0, 0))$. As Mechanism 6 builds the facilities at $(0, 0)$, we have

$$\sum_{i\in[n]} \Delta(x_i, (0, 0)) \geq \max_{p\in\{(0,1),(1,0),(1,1)\}} \sum_{i\in[n]} \Delta(x_i, p).$$

We also have $\Delta(x_i,(0,0))+\Delta(x_i,(0,1))+\Delta(x_i,(1,0))+\Delta(x_i,(1,1)) \geq 2\sqrt{2}$ for all agents i. Thus $\sum_{i\in[n]}\Delta(x_i,(0,0)) \geq n/\sqrt{2}$. Hence ALG $\geq n/\sqrt{2}$. Since no agent has welfare greater than $\sqrt{2}$, we have $\sqrt{2}n \geq$ OPT. Thus $2\cdot$ ALG $\geq \sqrt{2}n \geq$ OPT, as required. □

5 Egalitarian Mechanisms

We now design egalitarian mechanisms for DOFLG when the agents are located on an interval, cycle, or square.

In Definition 5 below, we introduce a simple way to convert a single-facility DOFLG mechanism into a DOFLG mechanism. For any DOFL instance $I = (n,k,\mathbf{x},\mathbf{a})$ and any j in $[k]$, let single(I,j) denote the single-facility DOFL instance $(n,1,\mathbf{x},\mathbf{a}')$ where a'_i is the singleton set containing the sole facility if i belongs to haters(I,j), and is $\emptyset$ otherwise.

Definition 5. For any single-facility DOFLG mechanism M, let Parallel(M) denote the DOFLG mechanism that takes as input a DOFL instance $I=(n,k,\mathbf{x},\mathbf{a})$ and outputs $\mathbf{y}=(y_1,\ldots,y_k)$ where y_j is the location at which M builds the facility on input single(I,j).

Lemmas 15 and 16 below reduce the task of designing a SP egalitarian DOFLG mechanism to the single-facility case.

Lemma 15. Let M be a SP single-facility DOFLG mechanism. Then Parallel(M) is a SP DOFLG mechanism.

Proof. Let (I,I') denote a DOFLG instance with $I=(n,k,\mathbf{x},\mathbf{a})$ and $I'=(n,k,\mathbf{x},\mathbf{a}')$, and let i be an agent such that $\mathbf{a}'=(\mathbf{a}_{-i},a'_i)$. Let $\mathbf{y}=(y_1,\ldots,y_k)$ (resp., $\mathbf{y}'=(y'_1,\ldots,y'_k)$) denote Parallel$(M)(I)$ (resp., Parallel$(M)(I')$). Since M is SP, we have $\Delta(x_i,y_j)\geq\Delta(x_i,y'_j)$ for each facility F_j in a_i. Thus $w(I,i,\mathbf{y})\geq w(I,i,\mathbf{y}')$, implying that agent i does not benefit by reporting a'_i instead of a_i. □

Lemma 16. Let M be an egalitarian single-facility DOFLG mechanism. Then Parallel(M) is an egalitarian DOFLG mechanism.

Proof. Let $I=(n,k,\mathbf{x},\mathbf{a})$ denote the reported DOFL instance. Let an optimal solution be $\mathbf{y}^*=(y^*_1,\ldots,y^*_k)$, and let the optimal (maximum) value of the minimum welfare be OPT $=$ MW$(I,\mathbf{y}^*)$. Assume that Parallel(M) builds the facilities at $\mathbf{y}'=(y'_1,\ldots,y'_k)$, resulting in minimum welfare ALG $=$ MW$(I,\mathbf{y}')$. For each facility F_j, we define OPT$_j$ (resp., ALG$_j$) as the distance from y^*_j (resp., y'_j) to the nearest agent in haters(I,j) (or ∞ if haters(I,j) is empty).

We have

$$\text{OPT} = \min\left(\min_j \text{OPT}_j, \min_{i\in\text{indiff}(I)} w(I,i,\mathbf{y}^*)\right)$$

and

$$\mathrm{ALG} = \min\left(\min_j \mathrm{ALG}_j, \min_{i \in \mathrm{indiff}(I)} w(I, i, \mathbf{y}')\right).$$

Since M is egalitarian, we have $\mathrm{OPT}_j = \mathrm{ALG}_j$ for all j. The welfare of the agents in $\mathrm{indiff}(I)$ does not depend on the locations of the facilities. Thus $\mathrm{ALG} = \mathrm{OPT}$, implying that $\mathrm{Parallel}(M)$ is egalitarian. □

5.1 The unit interval

We begin by describing a SP egalitarian mechanism for single-facility DOFLG when the agents are located in the unit interval.

Mechanism 7. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance and let H denote $\mathrm{haters}(I, 1)$. If H is empty, build F_1 at 0. Otherwise, let H contain ℓ agents $z_1, \ldots, z_\ell$ such that $x_{z_1} \leq x_{z_2} \leq \cdots \leq x_{z_\ell}$. Let $d_1 = x_{z_1}$ and $d_3 = 1 - x_{z_\ell}$. If $\ell = 1$, then build F_1 at 0 if $d_1 \geq d_3$, and at 1 otherwise. If $\ell > 1$, let m be the midpoint of the leftmost largest interval between consecutive agents in H. Formally, $m = (x_{z_s} + x_{z_{s+1}})/2$, where s is the smallest number in $[\ell - 1]$ such that $x_{z_{s+1}} - x_{z_s} = \max_{j \in [\ell-1]}(x_{z_{j+1}} - x_{z_j})$. Let $d_2 = m - x_{z_s}$. Then build facility F_1 at 0 if $d_1 \geq d_2$ and $d_1 \geq d_3$, at m if $d_2 \geq d_3$, and at 1 otherwise.

The following lemma shows that Mechanism 7 is SP. It is established by examining how the location of the facility changes when an agent misreports.

Lemma 17. Mechanism 7 is SP for single-facility DOFLG.

Proof. Let (I, I') denote a single-facility DOFLG instance with $I = (n, 1, \mathbf{x}, \mathbf{a})$ and $I' = (n, 1, \mathbf{x}, \mathbf{a}')$, and let i be an agent such that $\mathbf{a}' = (\mathbf{a}_{-i}, a_i')$. If F_1 does not belong to a_i, the welfare of agent i is independent of the location of F_1 and agent i does not benefit by reporting a_i'. Moreover, if F_1 belongs to $a_i \cap a_i'$, then the location of F_1 does not change by reporting a_i' instead of a_i, and again, agent i does not benefit by reporting a_i'. Accordingly, for the remainder of the proof, we assume that F_1 belongs to $a_i \setminus a_i'$.

Let y denote the location at which Mechanism 7 builds F_1 when agent i reports truthfully, and let H denote $\mathrm{haters}(I, 1)$. Note that Mechanism 7 does not build F_1 at the location of any agent in H, that is, $y \neq x_{i'}$ for all i' in H. Hence $y \neq x_i$. We can assume without loss of generality that $y < x_i$, since the case $y > x_i$ can be handled symmetrically. Let d_1, d_2, and d_3 be as defined in Mechanism 7 when all agents report truthfully. We consider two cases based on whether there is an agent in H between y and x_i.

Case 1: No agent in $H - i$ is located in $[y, x_i]$. We consider two cases.

Case 1.1: $y = 0$. Thus $d_1 = x_i$. When agent i reports a_i', F_1 is built at 0, which does not benefit agent i.

Case 1.2: $y \neq 0$. Thus $d_2 = x_i - y$, there is an agent i' in H at $y - d_2$, and there are no agents in H in $(y - d_2, y + d_2)$. We consider two cases.

Case 1.2.1: No agent in H is located to the right of x_i. Hence $x_i \geq 1 - d_2$. Thus when agent i reports a_i', F_1 is built at 1, which does not benefit agent i.

Case 1.2.2: There is an agent in H located to the right of x_i. Let i'' be the first agent to the right of x_i, breaking ties arbitrarily. Then $x_{i''} - x_i \leq 2d_2$. Thus when agent i reports a'_i, F_1 is built in $[y, x_i]$, which does not benefit agent i.

Case 2: There is an agent in $H - i$ in $[y, x_i]$. Let i' be the first agent to the right of y in $H - i$. Let d denote $d_1 = x_{i'}$ if $y = 0$, and $d_2 = x_{i'} - y$ otherwise. It follows that $x_i - y \geq d$. We consider two cases.

Case 2.1: No agent in H is located to the right of x_i. Hence $x_i \geq 1 - d$. Thus when agent i reports a'_i, F_1 is either built at y or at 1, neither of which benefits agent i.

Case 2.2: There is an agent in H located to the right of x_i. Let b be the first agent to the right of x_i, breaking ties arbitrarily. Let agent a be the agent located in $[0, x_i]$ that is closest to agent i, breaking ties arbitrarily. It follows that $x_i - x_a \leq 2d$ and $x_b - x_i \leq 2d$. When agent i reports a'_i, F_1 is built at y or in $[x_i - d, x_i + d]$, neither of which benefits agent i.

Thus agent i does not benefit by reporting a'_i. □

Lemma 18. Mechanism 7 is egalitarian for single-facility DOFLG.

Proof. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance, let H denote haters$(I, 1)$, let y^* denote an optimal location for the facility, let OPT denote $\text{MW}(I, y^*)$, let y' denote the location at which Mechanism 7 builds the facility, and let ALG denote $\text{MW}(I, y')$. Below we establish that ALG $\geq$ OPT. Since ALG $\leq$ OPT, we conclude that ALG $=$ OPT and hence that Mechanism 7 is egalitarian.

If H is empty, then trivially Mechanism 7 is egalitarian. For the remainder of the proof, assume that H is non-empty. We say that an agent in H is *tight* if it is as close to y^* as any other agent in H. Thus for any tight agent i, $\text{OPT} = |y^* - x_i|$. Similarly, ALG is the distance between y' and a closest agent in H. Let i be a tight agent, and consider the following three cases.

Case 1: $y^* = 0$. In this case, no agent in H is located in $[0, x_i)$. It follows that $d_1 = x_i = \text{OPT}$. Since $\text{ALG} \geq d_1$, we have $\text{ALG} \geq \text{OPT}$.

Case 2: $y^* = 1$. Symmetric to Case 1.

Case 3: $0 < y^* < 1$. Since $y^* = x_i$ implies $\text{OPT} = 0$, it is easy to see that $y^* \neq x_i$. We can assume without loss of generality that $x_i < y^*$, since the case $x_i > y^*$ can be handled symmetrically. Thus $\text{OPT} = y^* - x_i$ and no agent in H is located in $(x_i = y^* - \text{OPT}, y^* + \text{OPT})$. We consider two cases.

Case 3.1: There is no agent in H to the right of y^*. Thus $d_3 \geq \text{OPT}$. Since $\text{ALG} \geq d_3$, we have $\text{ALG} \geq \text{OPT}$.

Case 3.2: There is an agent in H to the right of y^*. Consider the leftmost such agent i'. Since $x_{i'} \geq y^* + \text{OPT}$, we have $d_2 \geq \text{OPT}$. Since $\text{ALG} \geq d_2$, we have $\text{ALG} \geq \text{OPT}$. □

We define Mechanism 8 as the DOFLG mechanism Parallel(M), where M denotes Mechanism 7. Using Lemmas 15 through 18, we immediately obtain Theorem 14 below.

Theorem 14. Mechanism 8 is SP and egalitarian.

Below we provide a lower bound on the approximation ratio of any WGSP egalitarian mechanism. Theorem 15 implies that Mechanism 8 is not WGSP.

Theorem 15. Let M be a WGSP α-egalitarian mechanism. Then α is $\Omega(\sqrt{n})$, where n denotes the number of agents.

Proof. Let q be a large even integer, let p denote q^2+1, and let U (resp., V) denote the set of all integers i such that $0 < i < q^2/2$ (resp., $q^2/2 < i < q^2$). We construct two $(p+3)$-agent two-facility DOFLG instances (I, I) and (I, I'). In both (I, I) and (I, I'), there is an agent located at i/q^2, called agent i, for each i in $U \cup V$, and there are two agents each at 0, 1/2, and 1. In I, each agent i in U dislikes $\{F_2\}$, each agent i in V dislikes $\{F_1\}$, one agent at 0 (resp., 1/2, 1) dislikes $\{F_1\}$, and the other agent at 0 (resp., 1/2, 1) dislikes $\{F_2\}$. In I', the agents i in $U \setminus \{1, \ldots, q-1\}$ have alternating reports: agent q reports $\{F_1\}$, agent $q+1$ reports $\{F_2\}$, agent $q+2$ reports $\{F_1\}$, and so on. Symmetrically, the agents i in $V \setminus \{q^2-q+1, \ldots, q^2-1\}$ have alternating reports: agent q^2-q reports $\{F_2\}$, agent q^2-q-1 reports $\{F_1\}$, agent q^2-q-2 reports $\{F_2\}$, and so on. All other agents in I' report truthfully.

Let the optimal minimum welfare for DOFL instance I (resp., I') be OPT (resp., OPT$'$). It is easy to see that OPT $= 1/4$ and OPT$' = \frac{1}{2q}$ (obtained by building the facilities at $(1/4, 3/4)$ and $(\frac{1}{2q}, 1-\frac{1}{2q})$, respectively). Let ALG (resp., ALG$'$) denote the minimum welfare achieved by M on instance I (resp., I'). Below we prove that either OPT/ALG $\geq q/4$ or OPT$'$/ALG$' \geq q/2$.

Let M build facilities at (y_1, y_2) (resp., (y'_1, y'_2)) on instance I (resp., I'). We consider two cases.

Case 1: $0 \leq y'_1 < 1/q$ and $1 - 1/q < y'_2 \leq 1$. Let S denote the set of agents who lie in I'. If $y'_1 < y_1$ and $y'_2 > y_2$, then all of the agents in S benefit by lying. Hence for M to be WGSP, either $y'_1 \geq y_1$ or $y'_2 \leq y_2$. Let us assume that $y'_1 \geq y_1$; the case where $y'_2 \leq y_2$ can be handled symmetrically. Since $y'_1 < 1/q$, we have $y_1 < 1/q$. Note that there is an agent at 0 who reports $\{F_1\}$. Thus ALG $\leq y_1 < 1/q$. Hence OPT/ALG $\geq q/4$.

Case 2: $y'_1 \geq 1/q$ or $y'_2 \leq 1 - 1/q$. If $y'_1 \geq 1/q$, then at least one agent within distance $1/q^2$ of y'_1 reported $\{F_1\}$ in I'. A similar observation holds for the case $y'_2 \leq 1 - 1/q$. Thus ALG$' \leq 1/q^2$. Hence OPT$'$/ALG$' \geq q/2$.

The preceding case analysis shows that $\alpha \geq q/4$. Since $q = \sqrt{p-1} = \sqrt{n-4}$, the theorem holds. □

The following variant of Mechanism 8 is SGSP. In this variant, we treat the reported dislikes of all agents as if they were $\{F_1\}$, and we use Mechanism 7 to determine where to build F_1. Then we build all of the remaining facilities at the same location as F_1. This mechanism is SGSP because it disregards the reported aversion profile. We claim that this mechanism is $2(n+1)$-egalitarian, where n denotes the number of agents. To prove this claim, we first observe that when Mechanism 7 is run as a subroutine within this mechanism, we have

$\max(d_1, 2d_2, d_3) \geq 1/(n+1)$. Thus the minimum welfare achieved by the mechanism is at least $\frac{1}{2(n+1)}$. Since the optimal minimum welfare is at most 1, the claim holds.

5.2 The cycle

In this section, we present egalitarian mechanisms for the case where the agents are located on the unit-circumference circle C. For any point u on C, we let $\widehat{u}$ denote the point antipodal to u. We begin by considering the natural adaptation of Mechanism 7 to a cycle.

Mechanism 9. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance and let H denote haters$(I, 1)$. If H is empty, then build facility F_1 at 0. If H has only one agent i, then build F_1 at $\widehat{x_i}$. Otherwise, build F_1 at the midpoint of the largest gap between any two consecutive agents in H. Formally, let H have ℓ agents $z_0, \ldots, z_{\ell-1}$ such that $x_{z_0} \leq x_{z_1} \leq \cdots \leq x_{z_{\ell-1}}$. Let $\oplus$ denote addition modulo ℓ. Build F_1 at the midpoint of x_{z_s} and $x_{z_{s\oplus 1}}$, where s is the smallest number in $\{0, \ldots \ell-1\}$ such that $\Delta(x_{z_{s\oplus 1}}, x_{z_s}) = \max_{j \in \{0, \ldots \ell-1\}} \Delta(x_{z_{j\oplus 1}}, x_{z_j})$.

Lemma 19. Mechanism 9 is SP for single-facility DOFLG.

Proof. Let (I, I') denote a single-facility DOFLG instance with $I = (n, 1, \mathbf{x}, \mathbf{a})$ and $I' = (n, 1, \mathbf{x}, \mathbf{a}')$, and let i be an agent such that $\mathbf{a}' = (\mathbf{a}_{-i}, a_i')$. As in the proof of Lemma 17, we can restrict our attention to the case where F_1 belongs to $a_i \setminus a_i'$.

Let y denote the location at which Mechanism 9 builds F_1 when agent i reports truthfully, and let H denote haters$(I, 1)$. Note that Mechanism 9 does not build F_1 at the location of any agent in H, that is, $y \neq x_{i'}$ for all i' in H. Hence $y \neq x_i$. Let the arc of C that goes clockwise from y to x_i be r_1 and let the arc of C that goes counterclockwise from y to x_i be r_2. Both arcs r_1 and r_2 include the endpoints y and x_i. We consider four cases.

Case 1: No agent in $H - i$ is on r_1 or r_2. Hence $H = \{i\}$. Thus $y = \widehat{x_i}$, and $\Delta(x_i, y) = 1/2$. When agent i reports a_i', F_1 is built at 0. Since $\Delta(x_i, 0) \leq 1/2$, reporting a_i' does not benefit agent i.

Case 2: No agent in $H - i$ is on r_1 and there is an agent in $H - i$ on r_2. Let i' be the closest agent to y in $H - i$ on r_2. Let d denote $\Delta(y, x_{i'})$. Thus y is the midpoint of the arc that runs clockwise from $x_{i'}$ to x_i. Hence $d = \Delta(x_i, y)$. Let i'' be the closest agent in $H - i$ in the clockwise direction from x_i. Thus $\Delta(x_{i''}, x_i) \leq 2d$. Since F_1 is built on r_1 when agent i reports a_i' and $\Delta(x_{i''}, x_i) \leq 2d$, reporting a_i' does not benefit agent i.

Case 3: No agent in $H - i$ is on r_2 and there is an agent in $H - i$ on r_1. Symmetric to Case 2.

Case 4: There is an agent in $H - i$ on r_1 and there is an agent in $H - i$ on r_2. Let the closest agent from y in $H - i$ on r_2 (resp., r_1) be a (resp., b), respectively. We have $\Delta(x_a, y) = \Delta(y, x_b)$. Let d denote $\Delta(x_a, y)$. Note that $\Delta(x_i, y) \geq d$. Let i' (resp., i'') be the first agent in $H - i$ encountered in the counterclockwise (resp., clockwise) direction from x_i. We have $\Delta(x_i, x_{i'}) \leq 2d$

and $\Delta(x_i, x_{i''}) \leq 2d$. Thus, when agent i reports a'_i, either F_1 is built at y or F_1 is built within distance d of x_i, neither of which benefits agent i.

Thus agent i does not benefit by reporting a'_i. □

Lemma 20. Mechanism 9 is egalitarian for single-facility DOFLG.

Proof. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance, let H denote haters$(I, 1)$, let y^* denote an optimal location for the facility, let OPT denote $\mathrm{MW}(I, y^*)$, let y' denote the location at which Mechanism 9 builds the facility, and let ALG denote $\mathrm{MW}(I, y')$. Below we establish that $\mathrm{ALG} \geq \mathrm{OPT}$. Since $\mathrm{ALG} \leq \mathrm{OPT}$, we conclude that $\mathrm{ALG} = \mathrm{OPT}$ and hence that Mechanism 9 is egalitarian.

If $|H| \leq 1$, it is easy to see that Mechanism 9 is egalitarian. For the remainder of the proof, we assume that $|H| \geq 2$. We say that an agent in H is *tight* if it is as close to y^* as any other agent in H. Thus for any tight agent i, $\mathrm{OPT} = \Delta(y^*, x_i)$.

Let i be a tight agent. Assume without loss of generality that in the shorter arc between x_i and y^*, x_i lies counterclockwise from y^*. Thus $\mathrm{OPT} = \Delta(x_i, y^*)$. Let i' be the closest agent in H in the clockwise direction from y^*. The definition of i' implies that $\Delta(x_{i'}, y^*) \geq \mathrm{OPT}$. Thus the length of the clockwise arc from x_i to $x_{i'}$ is at least $2 \cdot \mathrm{OPT}$. Since i and i' are consecutive agents in H and Mechanism 9 builds the facility at the midpoint of the largest gap between consecutive agents in H, we deduce that $\mathrm{ALG} \geq \mathrm{OPT}$. □

We define Mechanism 10 as the DOFLG mechanism Parallel(M), where M denotes Mechanism 9. Using Lemmas 15, 16, 19, and 20, we immediately obtain Theorem 16 below.

Theorem 16. Mechanism 10 is SP and egalitarian.

Theorem 17 below extends Theorem 15 to the case of the cycle. Theorem 17 implies that Mechanism 10 is not WGSP.

Theorem 17. Let M be a WGSP α-egalitarian mechanism. Then α is $\Omega(\sqrt{n})$, where n is the number of agents.

Proof. It is straightforward to verify that the construction used in the proof of Theorem 15 also works for the cycle and establishes the same lower bound. (We identify the point 1 with the point 0.) □

The following variant of Mechanism 10 is SGSP. As in the SGSP mechanism for the case when the agents are located in the unit interval, in this variant, we treat the reported dislikes of all agents as if they were $\{F_1\}$, and we use Mechanism 9 to determine where to build F_1. Then we build all of the remaining facilities at the same location as F_1. This mechanism is SGSP because it disregards the reported aversion profile. We claim that this mechanism is n-egalitarian, where n denotes the number of agents. To prove this claim, we first observe that the largest gap between two consecutive agents with reported dislikes $\{F_1\}$ is at least $\frac{1}{n}$. Thus the minimum welfare achieved by the mechanism is at least $\frac{1}{2n}$. Since the optimal minimum welfare is at most 1/2, the claim holds.

5.3 The unit square

In this section, we extend Mechanism 7 to a SP egalitarian mechanism for single-facility DOFLG when the agents are located in the unit square. Let S denote $[0,1]^2$, let B denote the boundary of S, and let $x_i = (a_i, b_i)$ denote the location of agent i. For convenience, we assume that all agents are located at distinct points; the results below generalize easily to instances where this assumption does not hold.

The analysis that we provide for our mechanism relies on results of Toussaint [24] concerning the largest empty circle with location constraints problem. An instance of the latter problem is given by a set of points in the plane. Toussaint makes the simplifying assumption that these points lie in general position in the sense that no three are collinear and no four are cocircular. In our application of Toussaint's work, the agent locations correspond to the set of input points. Accordingly, throughout this section, we assume that the agent locations are in general position.[2]

Mechanism 11. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance and let H denote haters$(I, 1)$. If H is empty, build F_1 at $(0, 0)$. Otherwise, construct the Voronoi diagram D associated with the locations of the agents in H. Let V denote the union of the following three sets of vertices: the vertices of D in the interior of S; the points of intersection between B and D; the four vertices of S. For each v in V, let d_v denote the minimum distance from v to any agent in H. Build F_1 at a vertex v maximizing d_v, breaking ties first by x-coordinate and then by y-coordinate.

Toussaint has presented an efficient $O(n \log n)$ algorithm to find the optimal v in Mechanism 11 [24]. The following lemma establishes that Mechanism 11 is egalitarian. The lemma is shown using a result of [24] concerning the largest empty circle with location constraints problem.

Lemma 21. Mechanism 11 is egalitarian for single-facility DOFLG.

Proof. Let $I = (n, 1, \mathbf{x}, \mathbf{a})$ denote the reported DOFL instance, let H denote haters$(I, 1)$, let y^* denote an optimal location for the facility, let OPT denote MW(I, y^*), let y' denote the location at which Mechanism 11 builds the facility, and let ALG denote MW(I, y'). Below we establish that ALG = OPT, which implies that Mechanism 11 is egalitarian.

If H is empty, then it is straightforward to prove that ALG = OPT. Otherwise, finding the optimal location at which to build facility F_1 is equivalent to finding the maximum-radius circle centered in the interior or on the boundary of S such that the interior of the circle has no points from $\{x_i \mid i \in H\}$.

[2] We suspect that Toussaint's results continue to hold when the points are not in general position. If so, we could drop our assumption that the agent locations are in general position.

This corresponds to the largest empty circle with location constraints problem. Toussaint shows (see Theorem 2 of [24]) that the optimal center for the circle is either a vertex of the Voronoi diagram of S, a point of intersection of D with B, or a vertex of S. Hence $\text{ALG} = \text{OPT}$. □

We use a case analysis to establish Lemma 22 below. The most interesting case deals with an agent i who dislikes F_1 but does not report it. In this case, the key insight is that when agent i misreports, facility F_1 is built either (1) at the same location as when agent i reports truthfully, or (2) inside or on the boundary of the Voronoi region that contains x_i when agent i reports truthfully.

Lemma 22. Mechanism 11 is SP for single-facility DOFLG.

Proof. Assume for the sake of contradiction that Mechanism 11 is not SP. Thus there exists a single-facility DOFLG instance (I, I') with $I = (n, 1, \mathbf{x}, \mathbf{a})$ and $I' = (n, 1, \mathbf{x}, \mathbf{a}')$, and an agent i with $\mathbf{a}' = (\mathbf{a}_{-i}, a_i')$ who benefits by reporting a_i'. Using the same arguments as in the proof of Lemma 17, we conclude that F_1 belongs to $a_i \setminus a_i'$.

Let y (resp., y') denote the location at which Mechanism 11 builds F_1 when agent i reports a_i (resp., a_i'), and let H denote $\text{haters}(I, 1)$. Note that Mechanism 11 does not build F_1 at the location of any agent in H, that is, $y \neq x_{i'}$ for all i' in H. Hence $y \neq x_i$. When all agents report truthfully, the Voronoi diagram partitions S into $|H|$ non-overlapping polygons, where each polygon contains one agent. Let P be the polygon that contains agent i when all agents report truthfully. When agent i reports a_i', the Voronoi diagram remains unchanged outside polygon P. It follows that when agent i reports a_i', facility F_1 is built either at y or at a point that belongs to P. If it is built at y, agent i does not benefit. Thus, for the remainder of the proof, we assume that y' belongs to P.

Let OPT (resp., OPT′) denote the closest distance of any agent in H (resp., $H - i$) to the point y (resp., y'). Let d and d' denote $\Delta(x_i, y)$ and $\Delta(x_i, y')$, respectively. Hence $d \geq \text{OPT}$. Since the distance from y to any agent in $H - i$ is at least OPT, we have $\text{OPT}' \geq \text{OPT}$.

Since agent i benefits by reporting a_i', we have $d' > d$. We begin by showing that $\text{OPT} = \text{OPT}'$. Suppose $\text{OPT} \neq \text{OPT}'$. Since $\text{OPT}' \geq \text{OPT}$, we have $\text{OPT}' > \text{OPT}$. Note that $\text{MW}(I, y') = \min(d', \text{OPT}')$. Since $d' > d$ and $\text{OPT}' > \text{OPT}$, we have $\text{MW}(I, y') > \min(d, \text{OPT})$. Since $d \geq \text{OPT}$, we have $\min(d, \text{OPT}) = \text{OPT}$. Moreover, $\text{OPT} = \text{MW}(I, y)$. Since $\text{MW}(I, y') > \min(d, \text{OPT})$ and $\min(d, \text{OPT}) = \text{MW}(I, y)$, we have $\text{MW}(I, y') > \text{MW}(I, y)$, a contradiction since Lemma 21 implies that Mechanism 11 is egalitarian. Thus $\text{OPT} = \text{OPT}'$.

Recall that y' belongs to P. Hence the closest agent in H to y' is agent i. Thus $d' \leq \text{OPT}'$. Since $\text{OPT} \leq d$, $d < d'$, and $d' \leq \text{OPT}'$, we obtain $\text{OPT} < \text{OPT}'$, which contradicts $\text{OPT} = \text{OPT}'$. Thus $d' \leq d$, and hence agent i does not benefit by reporting a_i'. □

We define Mechanism 12 as the DOFLG mechanism $\text{Parallel}(M)$, where M

denotes Mechanism 11. Using Lemmas 15, 16, 21, and 22, we immediately obtain Theorem 18 below.

Theorem 18. Mechanism 12 is SP and egalitarian.

6 Concluding Remarks

In this paper, we studied the obnoxious facility location game with dichotomous preferences. This game generalizes the obnoxious facility location game to more realistic scenarios. All of the mechanisms presented in this paper run in polynomial time, except that the running time of Mechanism 2 has exponential dependence on k (and polynomial dependence on n). We can extend the results of Section 4.3 (resp., Section 5.3) to obtain an analogue of Theorems 12 and 13 (resp., Theorem 18) that holds for an arbitrary rectangle (resp., convex polygon). We showed that Mechanism 2 is WGSP for all k and is efficient for $k \leq 3$. Properties 1 and 2 in the proof of our associated theorem, Theorem 4, do not hold for $k > 3$. Nevertheless, we conjecture that Mechanism 2 is efficient for all k. It remains an interesting open problem to reduce the gap between the $\Omega(\sqrt{n})$ and $O(n)$ bounds for the approximation ratio α of WGSP α-egalitarian mechanisms.

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