A Direct Proof of Hosoi's Theorem (Extended Abstract)

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The propositional logic of here-and-there, denoted by HT, is the superintuitionistic logic characterized by linearly ordered Kripke frames with two worlds (the "here" world and the "there" world). It was introduced by Arend Heyting [Heyting, 1930] as a tool for proving the independence of the law of the excluded middle. Recent interest in HT is related to its role in the theory of logic programming [Lifschitz *et al.*, 2001].

Toshio Umezawa [Umezawa, 1959] observed that the axiom schema

$$F \vee (F \to G) \vee \neg G \tag{1}$$

is sound in HT, and Tsutomu Hosoi [Hosoi, 1966] proved that HT can be axiomatized by adding this axiom schema to intuitionistic logic. Hosoi's proof consists of two parts: he gives another, more complicated axiomatization of HT, and then shows that its additional axiom schemas are intuitionistically derivable from (1).

We outline here a direct proof of the completeness of this axiomatization, which is similar to the well-known completeness proof for classical propositional logic due to László Kalmár [Kalmár, 1936].

Propositional Kripke models with two worlds will be represented by pairs $\langle I, J\rangle$ of sets of atoms such that $I \subseteq J$: the set of atoms that are true "here", and the set of atoms that are true "there".

For any such pair $\langle I, J\rangle$, let M_{IJ} be the set

$$I \cup \{\neg\neg p \mid p \in J\} \cup \{\neg p \mid p \in \sigma \setminus J\} \cup \{p \to q \mid p, q \in J \setminus I\}.$$

Lemma. *For any formula F and any pair $\langle I, J\rangle$,*

(i) if $\langle I, J\rangle \models F$ then F is intuitionistically derivable from M_{IJ};

(ii) if $\langle I, J\rangle \not\models F$ but $\langle J, J\rangle \models F$ then for every atom q in $J \setminus I$, $F \leftrightarrow q$ is intuitionistically derivable from M_{IJ};

(iii) if $\langle J, J\rangle \not\models F$ then $\neg F$ is intuitionistically derivable from M_{IJ}.

Hosoi's Theorem. *If F is satisfied in every linearly ordered model with two worlds then F is intuitionistically derivable from (1).*

To prove the theorem, note that the disjunction of the formulas

$$\bigwedge_{F \in M_{IJ}} F$$

over all pairs $\langle I, J \rangle$ is intuitionistically derivable from (1). (Apply distributivity to the conjunction of formulas (1) for all literals F, G.) The theorem is immediate from this fact and part (i) of the lemma.

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References

[Heyting, 1930] Arend Heyting. Die formalen Regeln der intuitionistischen Logik. *Sitzungsberichte der Preussischen Akademie von Wissenschaften. Physikalisch-mathematische Klasse*, pages 42–56, 1930.

[Hosoi, 1966] T. Hosoi. The axiomatization of the intermediate propositional systems S_n of Gödel. *Journal of the Faculty of Science of the University of Tokyo*, 13:183–187, 1966.

[Kalmár, 1936] László Kalmár. Zurückführung des Entscheidungsproblems auf den Fall von Formeln mit einer einzigen, bindren, Funktionsvariablen. *Compositio Mathematica*, 4:137–144, 1936.

[Lifschitz *et al.*, 2001] Vladimir Lifschitz, David Pearce, and Agustin Valverde. Strongly equivalent logic programs. *ACM Transactions on Computational Logic*, 2:526–541, 2001.

[Umezawa, 1959] Toshio Umezawa. On intermediate many-valued logics. *Journal of the Mathematical Society of Japan*, 11(2):116–128, 1959.