On Equivalent Transformations of Infinitary Formulas under the Stable Model Semantics

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Abstract. It has been known for a long time that intuitionistically equivalent formulas have the same stable models. We extend this theorem to propositional formulas with infinitely long conjunctions and disjunctions and show how to apply this generalization to proving properties of aggregates in answer set programming.

1 Introduction

This note is about the extension of the stable model semantics to infinitary propositional formulas defined by Truszczynski [9]. That extension, introduced originally as a tool for proving a theorem about the logic FO(ID), has been used also to prove a new generalization of Fages' theorem [4].

One of the reasons why stable models of infinitary formulas are important is that they are closely related to aggregates in answer set programming (ASP). The semantics of aggregates proposed by Ferraris [1, Section 4.1] treats a ground aggregate as shorthand for a propositional formula. An aggregate with variables has to be grounded before that semantics can be applied to it. For instance, to explain the precise meaning of the expression $1\{p(X)\}$ ("there exists at least one object with the property p") in the body of an ASP rule we first rewrite it as

$$1\{p(t_1), \dots, p(t_n)\},$$

where $t_1, \dots, t_n$ are all ground terms in the language of the program, and then turn it into the propositional formula

$$p(t_1) \vee \dots \vee p(t_n). \tag{1}$$

But this description of the meaning of $1\{p(X)\}$ implicitly assumes that the Herbrand universe of the program is finite. If the program contains function symbols then an infinite disjunction has to be used instead of (1).[3]

[3] There is nothing exotic or noncomputable about ASP programs containing both aggregates and function symbols. For instance, the program

$$p(f(a))$$
$$q \leftarrow 1\{p(X)\}$$

Our goal here is to develop methods for proving that pairs F, G of infinitary formulas have the same stable models. From the results of Pearce [7] and Ferraris [1] we know that in the case of grounded logic programs in the sense of Gelfond and Lifschitz [2] and, more generally, finite propositional formulas it is sufficient to check that the equivalence $F \leftrightarrow G$ is provable intuitionistically. Some extensions of intuitionistic propositional logic, including the logic of here-and-there, can be used as well. In this note we extend these results to deductive systems of infinitary propositional logic.

This goal is closely related to the idea of strong equivalence [5]. The provability of $F \leftrightarrow G$ in the deductive systems of infinitary logic described below guarantees not only that F and G have the same stable models, but also that for any set $\mathcal{H}$ of infinitary formulas, $\mathcal{H} \cup \{F\}$ and $\mathcal{H} \cup \{G\}$ have the same stable models.

We review the stable model semantics of infinitary propositional formulas in Section 2. Then we define a basic infinitary system of natural deduction, similar to propositional intuitionistic logic (Section 3), and study its properties (Section 4). The main theorem is stated and proved in Section 5, and applied to examples involving aggregates in Section 6. A useful extension of the basic system is discussed in Section 7.

2 Stable Models of Infinitary Propositional Formulas

The definitions of infinitary formulas and their stable models given below are equivalent to the definitions proposed by Truszczynski [9].

Let σ be a propositional signature, that is, a set of propositional atoms. The sets $\mathcal{F}_0^\sigma$, $\mathcal{F}_1^\sigma$, ... are defined as follows:

- $\mathcal{F}_0^\sigma = \sigma \cup \{\bot\}$,
- $\mathcal{F}_{i+1}^\sigma$ is obtained from $\mathcal{F}_i^\sigma$ by adding expressions $\mathcal{H}^\vee$ and $\mathcal{H}^\wedge$ for all subsets $\mathcal{H}$ of $\mathcal{F}_i^\sigma$, and expressions $F \rightarrow G$ for all $F, G \in \mathcal{F}_i^\sigma$.

The elements of $\bigcup_{i=0}^\infty \mathcal{F}_i^\sigma$ are called *(infinitary) formulas* over σ.[4]

has simple intuitive meaning, and its stable model $\{p(f(a)), q\}$ can be computed by existing solvers.

References to grounding in other theories of aggregates suffer from the same problem. For instance, the definition of a ground instance of a rule in Section 2.2 of the ASP Core document (`https://www.mat.unical.it/aspcomp2013/files/ASP-CORE-2.0.pdf`, Version 2.02) talks about replacing the expression $\{e_1; \dots; e_n\}$ in a rule with a set denoted by $\text{inst}(\{e_1; ...; e_n\})$. But that set can be infinite.

[4] This definition differs from the syntax introduced in early work on infinitary propositional formulas [8, 3] in several ways. It treats the collection $\mathcal{H}$ of conjunctive or disjunctive terms as a set, rather than a family indexed by ordinals. Thus there is no order among conjunctive or disjunctive terms in this framework, and there can be no repetitions among them. More importantly, there is no restriction here on the cardinality of the set of conjunctive or disjunctive terms. On the other hand, in the

Negation and equivalence will be understood as abbreviations: $\neg F$ stands for $F \to \bot$, and $F \leftrightarrow G$ stands for $(F \to G) \wedge (G \to F)$.

We will write $\{F, G\}^\wedge$ as $F \wedge G$, and $\{F, G\}^\vee$ as $F \vee G$. This convention allows us to view finite propositional formulas over σ as a special case of infinitary formulas.

Subsets of a signature σ will be also called its *interpretations*. The satisfaction relation between an interpretation I and a formula F is defined as follows:

- $I \not\models \bot$.
- For every $p \in \sigma$, $I \models p$ if $p \in I$.
- $I \models \mathcal{H}^\vee$ if there is a formula $F \in \mathcal{H}$ such that $I \models F$.
- $I \models \mathcal{H}^\wedge$ if for every formula $F \in \mathcal{H}$, $I \models F$.
- $I \models F \to G$ if $I \not\models F$ or $I \models G$.

We say that I satisfies a set $\mathcal{H}$ of formulas if I satisfies all elements of $\mathcal{H}$. Two sets of formulas are *equivalent* to each other if they are satisfied by the same interpretations. A formula F is *tautological* if it is satisfied by all interpretations.

The *reduct* F^I of a formula F with respect to an interpretation I is defined as follows:

- $\bot^I = \bot$.
- For $p \in \sigma$, $p^I = \bot$ if $I \not\models p$; otherwise $p^I = p$.
- $(\mathcal{H}^\wedge)^I = \{G^I \mid G \in \mathcal{H}\}^\wedge$.
- $(\mathcal{H}^\vee)^I = \{G^I \mid G \in \mathcal{H}\}^\vee$.
- $(G \to H)^I = \bot$ if $I \not\models G \to H$; otherwise $(G \to H)^I = G^I \to H^I$.

The *reduct* $\mathcal{H}^I$ of a set $\mathcal{H}$ of formulas is the set consisting of the reducts of the elements of $\mathcal{H}$. An interpretation I is a *stable model* of a set $\mathcal{H}$ of formulas if it is minimal w.r.t. set inclusion among the interpretations satisfying $\mathcal{H}^I$; a stable model of a formula F is a stable model of singleton $\{F\}$. This is a straightforward extension of the definition of a stable model due to Ferraris [1] to infinitary formulas.

It is easy to see that $I \models F^I$ iff $I \models F$. It follows that every stable model of $\mathcal{H}$ satisfies $\mathcal{H}$.

3 Basic Infinitary System of Natural Deduction

Inference rules of the deductive system described below are similar to the standard natural deduction rules of propositional logic (see, for instance, [6, Section 1.2.1]).[5]

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hierarchy $\mathcal{F}_i^\sigma$ of sets of formulas, i is a natural number; transfinite levels are not allowed.

[5] The conjunction introduction rule in that system is

$$\frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow G}{\Gamma, \Delta \Rightarrow F \wedge G};$$

In this system, derivable objects are *(infinitary) sequents*—expressions of the form $\Gamma \Rightarrow F$, where F is an infinitary formula, and Γ is a *finite* set of infinitary formulas ("F under assumptions Γ"). To simplify notation, we will write Γ as a list. We will identify a sequent of the form $\Rightarrow F$ with the formula F.

There is one axiom schema $F \Rightarrow F$. The inference rules are the introduction and elimination rules for the propositional connectives

$$(\wedge I)\ \frac{\Gamma \Rightarrow H \quad \text{for all } H \in \mathcal{H}}{\Gamma \Rightarrow \mathcal{H}^{\wedge}} \qquad (\wedge E)\ \frac{\Gamma \Rightarrow \mathcal{H}^{\wedge}}{\Gamma \Rightarrow H} \quad (H \in \mathcal{H})$$

$$(\vee I)\ \frac{\Gamma \Rightarrow H}{\Gamma \Rightarrow \mathcal{H}^{\vee}} \quad (H \in \mathcal{H}) \qquad (\vee E)\ \frac{\Gamma \Rightarrow \mathcal{H}^{\vee} \qquad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathcal{H}}{\Gamma, \Delta \Rightarrow F}$$

$$(\rightarrow I)\ \frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \rightarrow G} \qquad (\rightarrow E)\ \frac{\Gamma \Rightarrow F \quad \Delta \Rightarrow F \rightarrow G}{\Gamma, \Delta \Rightarrow G}$$

and the contradiction and weakening rules

$$(C)\ \frac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow F}$$

$$(W)\ \frac{\Gamma \Rightarrow F}{\Gamma, \Delta \Rightarrow F}.$$

(Note that we did not include the law of the excluded middle in the set of axioms, so that this deductive system is similar to intuitionistic, rather than classical, propositional logic.)

The set of *theorems of the basic system* is the smallest set of sequents that includes the axioms of the system and is closed under the application of its inference rules. We say that formulas F and G are *equivalent in the basic system* if $F \leftrightarrow G$ is a theorem of the basic system. The reason why we are interested in this relation is that formulas equivalent in the basic system have the same stable models, as discussed in Section 5 below.

Example 1. Consider a formula of the form

$$F_0 \wedge \{F_i \rightarrow F_{i+1} \mid i \geq 0\}^{\wedge}$$

or, in more compact notation,

$$F_0 \wedge \bigwedge_{i \geq 0} (F_i \rightarrow F_{i+1}). \qquad (2)$$

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the corresponding infinitary rule, presented in this note, is similar to the more restrictive version:

$$\frac{\Gamma \Rightarrow F \quad \Gamma \Rightarrow G}{\Gamma \Rightarrow F \wedge G}.$$

In the presence of the weakening rule (W), the two versions are equivalent to each other. The situation with disjunction elimination is similar.

Let us check that it is equivalent in the basic system to the formula $\bigwedge_{i\geq 0} F_i$. The sequent

$$F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1}) \Rightarrow\; F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1})$$

belongs to the set of theorems of the basic system. Consequently so do the sequents

$$F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1}) \;\Rightarrow\; F_0$$

and

$$F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1}) \Rightarrow\; F_j \to F_{j+1}$$

for all $j \geq 0$. Consequently the sequents

$$F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1}) \Rightarrow\; F_j$$

for all $j \geq 0$ belong to the set of theorems as well (by induction on j). Consequently so does the sequent

$$F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1}) \;\Rightarrow\; \bigwedge_{i\geq 0} F_i.$$

A similar argument (except that induction is not needed) shows that the sequent

$$\bigwedge_{i\geq 0} F_i \;\Rightarrow\; F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1})$$

is a theorem of the basic system also. Consequently so is the sequent

$$\Rightarrow\; F_0 \wedge \bigwedge_{i\geq 0}(F_i \to F_{i+1}) \;\leftrightarrow\; \bigwedge_{i\geq 0} F_i.$$

This argument could be expressed more concisely, without explicit references to the set of theorems of the basic system, as follows. Assume (2). Then F_0 and, for every $i \geq 0$, $F_i \to F_{i+1}$. Then, by induction, F_i for every i. And so forth. This style of presentation is used in the next example.

Example 2. Let $\{F_\alpha\}_{\alpha\in A}$ be a family of formulas from some $\mathcal{F}_i^\sigma$, and let G be a formula. We show that

$$\left(\bigvee_{\alpha\in A} F_\alpha\right) \to G \tag{3}$$

is equivalent in the basic system to the formula

$$\bigwedge_{\alpha\in A}(F_\alpha \to G). \tag{4}$$

Left-to-right: assume (3) and F_α. Then $\bigvee_{\alpha\in A} F_\alpha$, and consequently G. Thus we established $F_\alpha \rightarrow G$ under assumption (3) alone for every α, and consequently established (4) under this assumption as well. Right-to-left: assume (4) and $\bigvee_{\alpha\in A} F_\alpha$, and consider the cases corresponding to the disjunctive terms of this disjunction. Assume F_α. From (4), $F_\alpha \rightarrow G$, and consequently G. Thus we established G in each case, so that (3) follows from (4) alone.

4 Properties of the Basic System

The following property of the basic system is easy to verify.

Proposition 1. *If a sequent consisting of finite formulas is intuitionistically provable then it is a theorem of the basic system.*

When we want to prove that every theorem of the basic system has a certain property P, it is clearly sufficient to check that every axiom has the property P, and that the set of sequents that have the property P is closed under the application of the inference rules. In this way we can establish, in particular, the following fact:

Proposition 2. *For any theorem $\Gamma \Rightarrow F$ of the basic system, the formula $\Gamma^\wedge \rightarrow F$ is tautological.*

Let σ and σ' be disjoint signatures. In this section, a *substitution* is an arbitrary function from σ' to $\mathcal{F}_i^\sigma$, where i is a nonnegative integer. For any substitution α and any formula F over the signature $\sigma \cup \sigma'$, F^α stands for the formula over σ formed as follows:

- If $F \in \sigma$ then $F^\alpha = F$.
- If $F \in \sigma'$ then $F^\alpha = \alpha(F)$.
- If F is $\mathcal{H}^\wedge$ then $F^\alpha = \{G^\alpha | G \in \mathcal{H}\}^\wedge$.
- If F is $\mathcal{H}^\vee$ then $F^\alpha = \{G^\alpha | G \in \mathcal{H}\}^\vee$.
- If F is $G \rightarrow H$ then $F^\alpha = G^\alpha \rightarrow H^\alpha$.

It is easy to see that if $F \in \mathcal{F}_j^{\sigma\cup\sigma'}$ then $F^\alpha \in \mathcal{F}_{i+j}^\sigma$. Formulas of the form F^α will be called *instances* of F.

Proposition 3. *If F is a theorem of the basic system then every instance of F is a theorem of the basic system also.*

Proof. The notation F^α extends to sequents in a natural way. The property "S^α is a theorem of the basic system" holds for every axiom S of the basic system, and it is preserved by all inference rules.

We will refer to Proposition 3 as the substitution property of the basic system.

Example 3. We will show that for any formulas F, G, the formula $\neg(F \vee G)$ is equivalent to $\neg F \wedge \neg G$ in the basic system. Note first that the formula

$$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q \tag{5}$$

is intuitionistically provable. By Proposition 1, it follows that it is a theorem of the basic system. The equivalence

$$\neg(F \vee G) \leftrightarrow \neg F \wedge \neg G$$

is an instance of (5): take $\alpha(p) = F$, $\alpha(q) = G$. By the substitution property, it follows that it is a theorem of the basic system as well.

Proposition 4. *For any substitutions α, β, the implication*

$$\bigwedge_{p \in \sigma'} (\alpha(p) \leftrightarrow \beta(p)) \rightarrow (F^\alpha \leftrightarrow F^\beta)$$

is a theorem of the basic system.

Proof. The proof is by induction on j such that $F \in \mathcal{F}_j^{\sigma \cup \sigma'}$, and it considers several cases, depending on the syntactic form of F. Assume, for instance, that F is $\mathcal{H}^\vee$. Then

$$F^\alpha = \{G^\alpha | G \in \mathcal{H}\}^\vee, \quad F^\beta = \left\{G^\beta | G \in \mathcal{H}\right\}^\vee.$$

By the induction hypothesis, for each G in $\mathcal{H}$, the implication

$$\bigwedge_{p \in \sigma'} (\alpha(p) \leftrightarrow \beta(p)) \rightarrow (G^\alpha \leftrightarrow G^\beta) \tag{6}$$

is a theorem of the basic system. We need to show that

$$\bigwedge_{p \in \sigma'} (\alpha(p) \leftrightarrow \beta(p)) \rightarrow \left(\{G^\alpha | G \in \mathcal{H}\}^\vee \leftrightarrow \{G^\beta | G \in \mathcal{H}\}^\vee\right)$$

is a theorem of the basic system also. Assume

$$\bigwedge_{p \in \sigma'} (\alpha(p) \leftrightarrow \beta(p)) \tag{7}$$

and $\{G^\alpha | G \in \mathcal{H}\}^\vee$, and consider the cases corresponding to the terms of this disjunction. Assume G^α. Then, by (6) and (7), G^β. We can conclude $\left\{G^\beta | G \in \mathcal{H}\right\}^\vee$, that is, F^β. So we established the implication $F^\alpha \rightarrow F^\beta$. The implication in the other direction is proved in a similar way.

Corollary. *If for every atom p, $\alpha(p)$ is equivalent to $\beta(p)$ in the basic system then F^α is equivalent to F^β in the basic system.*

We will refer to this corollary as the replacement property of the basic system.

Example 4. The formula

$$\bigwedge_{k \geq 1} (p_k \rightarrow \neg p_k) \rightarrow p_0 \tag{8}$$

is equivalent to

$$\bigwedge_{k\geq 1} \neg p_k \;\to\; p_0 \tag{9}$$

in the basic system, because (9) can be obtained from (8) by replacing $p_k \to \neg p_k$ with the intuitionistically equivalent $\neg p_k$. More formally, let $\sigma' = \{q_k \mid k \geq 1\}$, and let F be $\bigwedge_{k\geq 1} q_k \;\to\; p_0$. For the substitutions

$$\alpha(q_k) = p_k \to \neg p_k, \quad \beta(q_k) = \neg p_k,$$

F^α is (8), and F^β is (9). By the replacement property, (8) is equivalent to (9).

5 Relation of the Basic System to Stable Models

Main Theorem. *For any set $\mathcal{H}$ of formulas,*

(a) if a formula F is provable in the basic system then $\mathcal{H} \cup \{F\}$ has the same stable models as $\mathcal{H}$;
(b) if F is equivalent to G in the basic system then $\mathcal{H} \cup \{F\}$ and $\mathcal{H} \cup \{G\}$ have the same stable models.

Lemma 1. *For any formula F and interpretation I, if I does not satisfy F then $F^I \;\Rightarrow\; \bot$ is a theorem of the basic system.*

The proof is straightforward by induction on i such that $F \in \mathcal{F}_i^\sigma$.

By Γ^I we denote the set $\{G^I \mid G \in \Gamma\}$; $(\Gamma \Rightarrow F)^I$ stands for $\Gamma^I \Rightarrow F^I$.

Lemma 2. *For any sequent S and any interpretation I, if S is a theorem of the basic system then so is S^I.*

Proof. Consider the property of sequents: "S^I is a theorem of the basic system." To prove the lemma, it suffices to show that all theorems of the basic system have that property. It is clear that the reduct of every axiom of the basic system is a theorem (of the basic system). Verifying that the set of sequents with that property is closed under inference rules follows the same pattern for all inference rules but those involving implication. Consider, for instance, disjunction elimination:

$$\frac{\Gamma \Rightarrow \mathcal{H}^\vee \qquad \Delta, H \Rightarrow F \quad \text{for all } H \in \mathcal{H}}{\Gamma, \Delta \Rightarrow F} \tag{10}$$

and assume that the reducts of all sequents that are premises of that rule are theorems. Because $(\mathcal{H}^\vee)^I$ is $(\mathcal{H}^I)^\vee$, all premises of the disjunction elimination rule:

$$\frac{\Gamma^I \Rightarrow (\mathcal{H}^I)^\vee \qquad \Delta^I, H^I \Rightarrow F^I \quad \text{for all } H \in \mathcal{H}}{\Gamma^I, \Delta^I \Rightarrow F^I}$$

are theorems. Therefore, so is the sequent $\Gamma^I, \Delta^I \Rightarrow F^I$ and consequently, also the sequent $(\Gamma, \Delta \Rightarrow F)^I$.

Consider now the implication introduction rule:

$$\frac{\Gamma, F \Rightarrow G}{\Gamma \Rightarrow F \to G}$$

and assume that the reduct $(\Gamma, F \Rightarrow G)^I$ is a theorem. To show that $(\Gamma \Rightarrow F \to G)^I$ is a theorem it suffices to show that $\Gamma^I \Rightarrow (F \to G)^I$ is a theorem.

Case 1: I satisfies Γ. Since the sequent $(\Gamma, F \Rightarrow G)^I$ is a theorem, so is the sequent $\Gamma^I, F^I \Rightarrow G^I$. Thus, $\Gamma^I \Rightarrow F^I \to G^I$ is a theorem and so, $(\Gamma^I)^\wedge \to (F^I \to G^I)$ is tautological. Since I satisfies Γ, the comment at the end of Section 2 implies that I satisfies Γ^I. Consequently, I satisfies $F^I \to G^I$ and, by the same comment again, also $F \to G$. It follows that $(F \to G)^I$ is $F^I \to G^I$. Since the sequent $(\Gamma, F \Rightarrow G)^I$ or, equivalently, the sequent $\Gamma^I, F^I \Rightarrow G^I$ is a theorem, applying the rule

$$\frac{\Gamma^I, F^I \Rightarrow G^I}{\Gamma^I \Rightarrow F^I \to G^I}$$

we obtain that $\Gamma^I \Rightarrow F^I \to G^I$ is a theorem. Thus, $\Gamma^I \Rightarrow (F \to G)^I$, is a theorem, too.

Case 2: I does not satisfy Γ. Then I does not satisfy one of the elements H of Γ. By Lemma 1, $H^I \Rightarrow \bot$ is a theorem, and $\Gamma^I \Rightarrow (F \to G)^I$ can be derived from $H^I \Rightarrow \bot$ by rules (C) and (W). Thus, it is a theorem.

Next, consider the implication elimination rule:

$$\frac{\Gamma \Rightarrow F \qquad \Delta \Rightarrow F \to G}{\Gamma, \Delta \Rightarrow G}$$

and assume that the sequents $(\Gamma \Rightarrow F)^I$ and $(\Delta \Rightarrow F \to G)^I$ are theorems. We will show that $(\Gamma, \Delta \Rightarrow G)^I$ or, equivalently, $\Gamma^I, \Delta^I \Rightarrow G^I$ is a theorem, too.

Case 1: I satisfies $F \to G$. Then $(F \to G)^I$ is $F^I \to G^I$. Thus, the sequents $\Gamma^I \Rightarrow F^I$ and $\Delta^I \Rightarrow F^I \to G^I$ are theorems, and the claim follows by applying the rule

$$\frac{\Gamma^I \Rightarrow F^I \qquad \Delta^I \Rightarrow F^I \to G^I}{\Gamma^I, \Delta^I \Rightarrow F^I}.$$

Case 2: I does not satisfy $F \to G$. Then $(F \to G)^I$ is $\bot$ and so, $\Delta^I \Rightarrow \bot$ is a theorem. Moreover, $\Gamma^I, \Delta^I \Rightarrow F^I$ can be derived from $\Delta^I \Rightarrow \bot$ by rules (C) and (W). Thus, $\Gamma^I, \Delta^I \Rightarrow F^I$ is a theorem, too.

Proof of the Theorem. (a) Assume that F is provable in the basic system. By the lemma, for any interpretation I, F^I is provable in the basic system, and consequently is tautological, by Proposition 2. It follows that $\mathcal{H}^I$ and $(\mathcal{H} \cup F)^I$ are satisfied by the same interpretations.

(b) Assume that F is equivalent to G in the basic system. The formula $F \leftrightarrow G$ is tautological, so that for any interpretation I, $(F \leftrightarrow G)^I$ is $F^I \leftrightarrow G^I$. By the lemma, this equivalence is a theorem of the basic system, so that it is tautological as well. We showed that for any interpretation I, F^I is equivalent to G^I. It follows that $(\mathcal{H} \cup F)^I$ and $(\mathcal{H} \cup G)^I$ are satisfied by the same interpretations.

6 Examples Involving Aggregates

As discussed in the introduction, infinitary formulas can be used to precisely define the semantics of aggregates in ASP when the Herbrand universe is infinite. In this section, we give two examples demonstrating how the theory described above can be applied to prove equivalences between programs involving aggregates.

Example 5. Intuitively, the rule

$$q(X) \leftarrow 1\{p(X,Y)\} \tag{11}$$

has the same meaning as the rule

$$q(X) \leftarrow p(X,Y). \tag{12}$$

To make this claim precise, consider first the result of grounding rule (11) under the assumption that the Herbrand universe C is finite. In accordance with standard practice in ASP, we treat variable X as global and Y as local. Then the result of grounding (11) is the set of ground rules

$$q(a) \leftarrow 1\{p(a,b) \mid b \in C\}$$

for all $a \in C$. In the spirit of the semantics for aggregates proposed by Ferraris [1, Section 4.1] these rules have the same meaning as the propositional formulas

$$\left(\bigvee_{b\in C} p(a,b)\right) \rightarrow q(a). \tag{13}$$

Likewise, rule (12) can be viewed as shorthand for the set of formulas

$$p(a,b) \rightarrow q(a) \tag{14}$$

for all $a, b \in C$. It easy to see that these sets of formulas are intuitionistically equivalent.

How can we lift the assumption that the Herbrand universe is finite? We can treat (13) as an infinitary formula, and show that the conjunction of formulas (13) is equivalent to the conjunction of formulas (14) in the basic system. The fact that the conjunction of formulas (14) for all $b \in C$ is equivalent to (13) in the basic system follows from Example 2 (Section 3).

Example 6. Intuitively,

$$q(X) \leftarrow 2\{p(X,Y)\} \tag{15}$$

has the same meaning as the rule

$$q(X) \leftarrow p(X,Y1),\ p(X,Y2),\ Y1 \neq Y2. \tag{16}$$

To make this claim precise, consider the infinitary formulas corresponding to (15):

$$\left(\bigvee_{b\in C} p(a,b) \wedge \bigwedge_{b\in C}\left(p(a,b) \to \bigvee_{\substack{c\in C\\ c\neq b}} p(a,c)\right)\right) \to q(a) \tag{17}$$

($a \in C$); see [1, Section 4.1] for details on representing aggregates with propositional formulas. The formulas corresponding to (16) are

$$(p(a,b) \wedge p(a,c)) \to q(a) \tag{18}$$

($a, b, c \in C,\ \ b \neq c$). We will show that the conjunction of formulas (17) is equivalent to the conjunction of formulas (18) in the basic system.

It is sufficient to check that for every $a \in C$, (17) is equivalent to the conjunction of formulas (18) over all $b, c \in C$ such that $b \neq c$. By Example 2, this conjunction is intuitionistically equivalent to

$$\left(\bigvee_{\substack{b,c\in C\\ b\neq c}} (p(a,b) \wedge p(a,c))\right) \to q(a). \tag{19}$$

By the replacement property of infinitary formulas, it suffices to check that the antecedents of (17) and (19) are equivalent to each other.

Left-to-right: assume

$$\bigvee_{b\in C} p(a,b) \wedge \bigwedge_{b\in C}\left(p(a,b) \to \bigvee_{\substack{c\in C\\ c\neq b}} p(a,c)\right). \tag{20}$$

Then $\bigvee_{b\in C} p(a,b)$. We will reason by cases, with one case corresponding to each possible value b_0 of b. Case $p(a,b_0)$: by the second conjunctive term of (20),

$$p(a,b_0) \to \bigvee_{\substack{c\in C\\ c\neq b_0}} p(a,c).$$

Then the consequent of this implication follows. Again we will reason by cases, with one case for each value c_0 of c where $c_0 \neq b_0$. Case $p(a,c_0)$: then $p(a,b_0) \wedge p(a,c_0)$. Consequently

$$\bigvee_{\substack{b,c\in C\\ b\neq c}} p(a,b) \wedge p(a,c). \tag{21}$$

Right-to-left: assume (21). We reason by cases, with one case for each pair $b_0,\ c_0$, where $b_0 \neq c_0$. Case $p(a,b_0) \wedge p(a,c_0)$: from $p(a,b_0)$ we derive the first conjunctive term of (20); from $p(a,c_0)$ we derive

$$\bigvee_{\substack{c\in C,\\ c\neq b}} p(a,c),$$

and consequently the implication

$$p(a,b) \rightarrow \bigvee_{\substack{c\in C\\ c\neq b}} p(a,c).$$

The conjunction of these implications for all $b \in C$ is the second conjunctive term of (20).

7 The Extended System of Natural Deduction

In this section we show that the assertion of the main theorem will remain true if we extend the basic system by the axiom schema

$$\bigvee_{\mathcal{I}\subseteq\mathcal{H}} \left(\neg \bigvee_{F\in\mathcal{H}\setminus\mathcal{I}} F \wedge \neg\neg \bigwedge_{F\in\mathcal{I}} F \right), \tag{22}$$

where $\mathcal{H}$ is an arbitrary subset of one of the sets $\mathcal{F}_i$ (see Section 2). This is a generalization of the weak law of the excluded middle

$$\neg F \vee \neg\neg F \tag{23}$$

to sets of infinitary formulas. (Formula (23) is equivalent in the basic system to the special case of (22) corresponding to $\mathcal{H} = \{F\}$.)

In the extended system we can derive the infinitary version of the "difficult part" of De Morgan's laws

$$\neg \bigwedge_{F\in\mathcal{H}} F \rightarrow \bigvee_{F\in\mathcal{H}} \neg F. \tag{24}$$

Indeed, consider the case corresponding to one of the disjunctive terms $D_{\mathcal{I}}$ of (22). If $\mathcal{I} = \mathcal{H}$ then the second conjunctive term of $D_{\mathcal{I}}$ contradicts the antecedent of (24). Otherwise take any F from $\mathcal{H} \setminus \mathcal{I}$. Assume F. Then $\bigvee_{F\in\mathcal{H}\setminus\mathcal{I}} F$, which contradicts the first conjunctive term of $D_{\mathcal{I}}$. Therefore $\neg F$, which implies the consequent of (24).

It is easy to check that the properties of the basic system proved in Section 4 hold for the extended system as well.

To show that the assertion of the main theorem applies to the extended system we need to prove the following modification of Lemma 2: *For any sequent S and any interpretation I, if S is a theorem of the extended system then so is S^I.* This fact is established by the same argument as in the proof of Lemma 2, augmented by the following assertion.

Lemma 3. *For any formula S of the form (22) and any interpretation I, S^I is provable in the basic system.*

Proof. Clearly, S^I is

$$\bigvee_{\mathcal{I}\subseteq\mathcal{H}}\left(\left(\neg\bigvee_{F\in\mathcal{H}\setminus\mathcal{I}}F\right)^I\wedge\left(\neg\neg\bigwedge_{F\in\mathcal{I}}F\right)^I\right). \tag{25}$$

Consider the disjunctive term of (25) with $\mathcal{I}=\{F\in\mathcal{H}\mid I\models F\}$. It can be written as

$$\left(\neg\bigvee_{I\not\models F}F\right)^I\wedge\left(\neg\neg\bigwedge_{I\models F}F\right)^I. \tag{26}$$

The first conjunctive term of (26) is $(\bigvee_{I\not\models F}F\to\bot)^I$, which can be rewritten as

$$\left(\bigvee_{I\not\models F}F\right)^I\to\bot. \tag{27}$$

From Lemma 1, applied to $\bigvee_{I\not\models F}F$ as F, it follows that (27) is provable in the basic system. On the other hand, the second conjunctive term of (26) is

$$\left(\left(\bigwedge_{I\models F}F\to\bot\right)\to\bot\right)^I.$$

It can be rewritten as $\bot\to\bot$ and therefore is provable in the basic system as well.

Example 7. Intuitively, the cardinality constraint $\{p(X)\}0$ ("the set of true atoms with form $p(X)$ has cardinality at most 0") has the same meaning as the conditional literal $\bot:p(X)$ ("for all X, $p(X)$ is false"). If we represent this conditional literal by the infinitary formula

$$\bigwedge_{a\in C}\neg p(a) \tag{28}$$

then this claim can be made precise by showing that (28) is equivalent in the extended system to the infinitary formula corresponding to $\{p(X)\}0$ in the sense of [1]:

$$\bigwedge_{\substack{A\subseteq C\\ A\neq\emptyset}}\left(\bigwedge_{a\in A}p(a)\to\bigvee_{a\in C\setminus A}p(a)\right) \tag{29}$$

(where C is the Herbrand universe).

It is easy to derive (29) from (28) in the basic system. The derivation of (28) from (29) will use the following instance of axiom schema (22):

$$\bigvee_{A\subseteq C}\left(\neg\bigvee_{a\in C\setminus A}p(a)\wedge\neg\neg\bigwedge_{a\in A}p(a)\right). \tag{30}$$

We will reason by cases, with one case corresponding to each disjunctive term D_A in (30). In the case that A is empty, (28) follows from the first conjunctive term of D_A by De Morgan's law. Otherwise, assume $\bigwedge_{a\in A} p(a)$. Then by (29), $\bigvee_{a\in C\setminus A} p(a)$, which contradicts the first conjunctive term of D_A. We conclude $\neg \bigwedge_{a\in A} p(a)$, which contradicts the second conjunctive term of D_A. So the assumptions D_A and (29) are contradictory. Consequently, they imply (28).

8 Future Work

Two finite propositional formulas are strongly equivalent if and only if they are equivalent in the logic of here-and-there [1, Proposition 2]. The results of this note are similar to the if part of that theorem; we don't know how to extend the only if part to infinitary formulas. It appears that axioms or inference rules not included in the extended system may be required, and identifying them is a topic for future work.

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