FLOATING POINT

SYSTEMS I

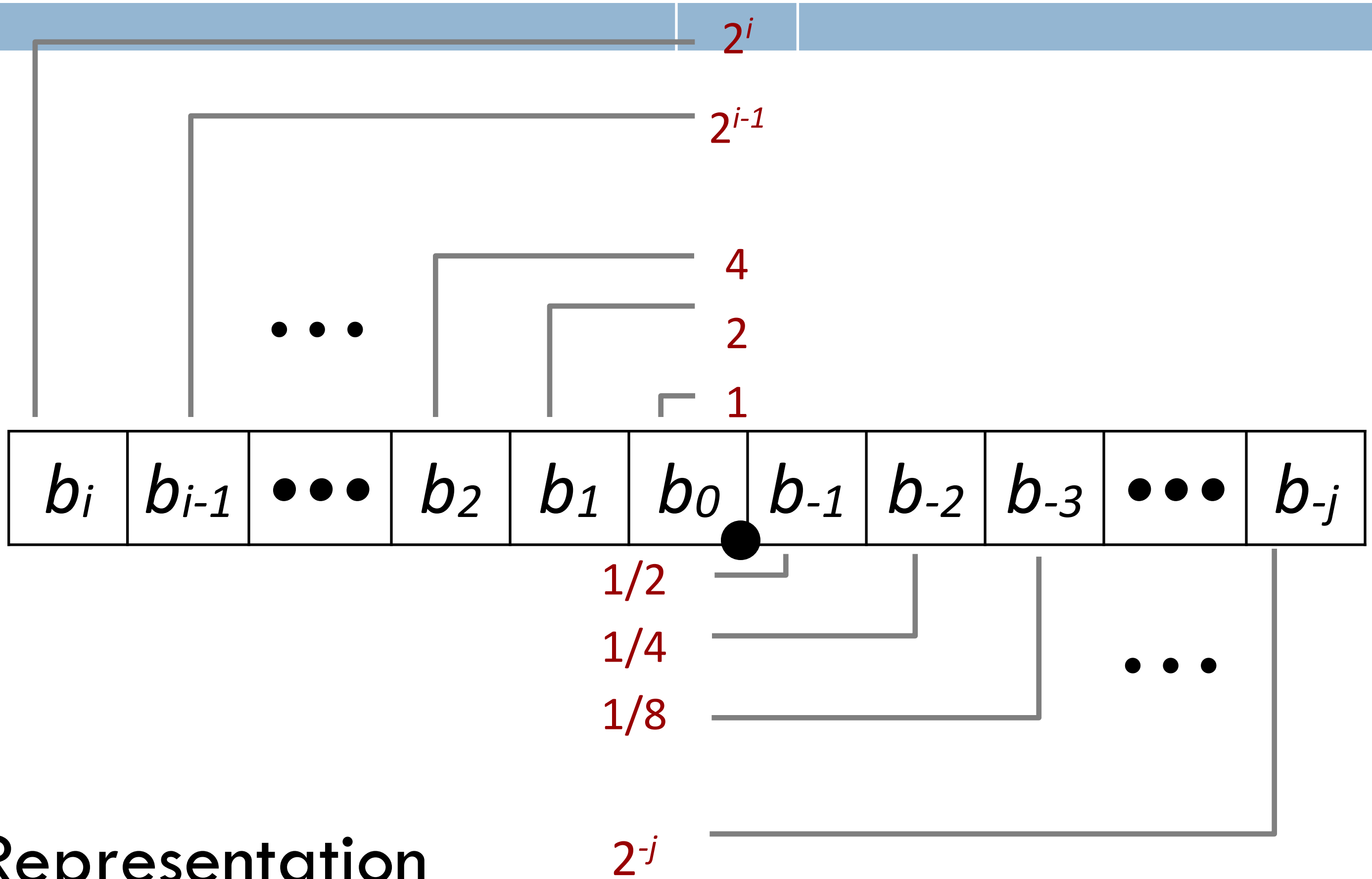
Today: Floating Point

- Background: Fractional binary numbers
- IEEE floating point standard: Definition
- Example and properties
- Rounding, addition, multiplication
- Floating point in C
- Summary

Fractional binary numbers

- What is 1011.101_2?

Fractional Binary Numbers



- Representation
  - Bits to right of "binary point" represent fractional powers of 2
  - Represents rational number: $\sum_{k=-j}^{i} b_k \times 2^k$

Fractional Binary Numbers: Examples

- **Value** **Representation**

Value	Representation
5 3/4	101.11_2
2 7/8	10.111_2
1 7/16	1.0111_2
63/64	0.111111_2

- **Observations**
  - Divide by 2 by shifting right
  - Multiply by 2 by shifting left
  - Numbers of form $0.111111\ldots_2$ are just below 1.0
    - $1/2 + 1/4 + 1/8 + \ldots + 1/2^i + \ldots \rightarrow 1.0$
    - Use notation $1.0 - \varepsilon$

Representable Numbers

- Limitation
  - Can only exactly represent numbers of the form $x/2^k$
  - Other rational numbers have repeating bit representations

- Value Representation
  - 1/3 $0.0101010101[01]\ldots_2$
  - 1/5 $0.001100110011[0011]\ldots_2$
  - 1/10 $0.0001100110011[0011]\ldots_2$

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IEEE Floating Point

- IEEE Standard 754
  - Established in 1985 as uniform standard for floating point arithmetic
    - Before that, many idiosyncratic formats
  - Supported by all major CPUs

- Driven by numerical concerns
  - Nice standards for rounding, overflow, underflow
  - Hard to make fast in hardware
    - Numerical analysts predominated over hardware designers in defining standard

Floating Point Representation

- Numerical Form:

$$(-1)^s \; M \;\; 2^E$$

  - **Sign bit** ***s*** determines whether number is negative or positive
  - **Significand** ***M*** normally a fractional value in range [1.0,2.0).
  - **Exponent** ***E*** weights value by power of two

- Encoding
  - MSB S is sign bit ***s***
  - exp field encodes ***E*** (but is not equal to E)
  - frac field encodes ***M*** (but is not equal to M)

s	exp	frac

Precisions

- Single precision: 32 bits

s	exp	frac
1	8-bits	23-bits

- Double precision: 64 bits

s	exp	frac
1	11-bits	52-bits

- Extended precision: 80 bits (Intel only)

s	exp	frac
1	15-bits	63 or 64-bits

Normalized Values

- Condition: exp ≠ 000…0 and exp ≠ 111…1

- Exponent coded as ***biased*** value: ***E* = *Exp* – *Bias***
  - *Exp*: unsigned value exp
  - *Bias* = 2^{k-1} - 1, where *k* is number of exponent bits
    - Single precision: 127 (Exp: 1…254, E: -126…127)
    - Double precision: 1023 (Exp: 1…2046, E: -1022…1023)

- Significand coded with implied leading 1: ***M*** = $1.xxx\ldots x_2$
  - xxx…x: bits of frac
  - Minimum when 000…0 (*M* = 1.0)
  - Maximum when 111…1 (*M* = 2.0 – ε)
  - Get extra leading bit for "free"

Normalized Encoding Example

- Value: `Float F = 15213.0;`
  - $15213_{10} = 11101101101101_2$
    $= 1.1101101101101_2 \times 2^{13}$

- Significand
  $M = 1.\underline{1101101101101}_2$
  $\texttt{frac} = \underline{1101101101101}0000000000_2$

- Exponent
  $E = 13$
  $Bias = 127$
  $Exp = 140 = 10001100_2$

- Result:

s	exp	frac
0	10001100	11011011011010000000000

Denormalized Values

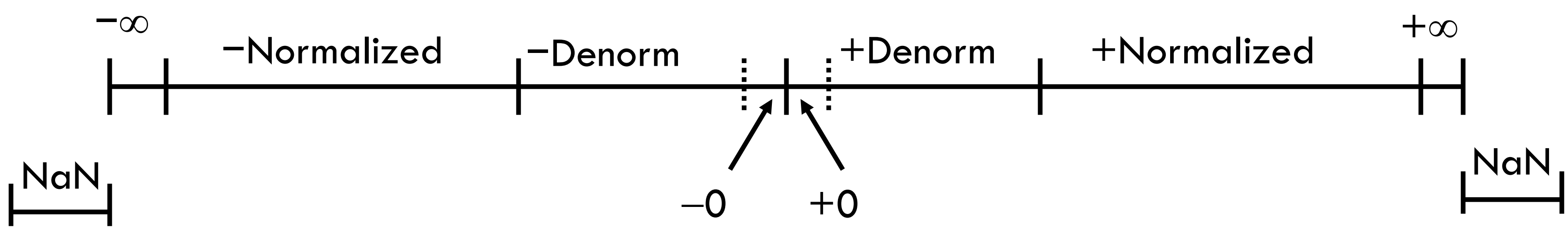
- Condition: exp = 000…0

- Exponent value: $E = -Bias + 1$ (instead of $E = 0 - Bias$)
- Significand coded with implied leading 0: $M = 0.xxx\ldots x_2$
  - `xxx…x`: bits of `frac`
- Cases
  - **`exp`** = 000…0, **`frac`** = 000…0
    - Represents zero value
    - Note distinct values: +0 and –0 (why?)
  - **`exp`** = 000…0, **`frac`** ≠ 000…0
    - Numbers very close to 0.0
    - Lose precision as get smaller
    - Equispaced

Special Values

- Condition: **`exp`** = 111…1
- Case: **`exp`** = 111…1, **`frac`** = 000…0
  - Represents value ∞ (infinity)
  - Operation that overflows
  - Both positive and negative
  - E.g., $1.0/0.0 = -1.0/-0.0 = +\infty$, $1.0/-0.0 = -\infty$
- Case: **`exp`** = 111…1, **`frac`** $\neq$ 000…0
  - Not-a-Number (NaN)
  - Represents case when no numeric value can be determined
  - E.g., sqrt(–1), $\infty - \infty$, $\infty \times 0$

Visualization: Floating Point Encodings



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Tiny Floating Point Example

s	exp	frac
1	4-bits	3-bits

- 8-bit Floating Point Representation
  - the sign bit is in the most significant bit
  - the next four bits are the exponent, with a bias of 7
  - the last three bits are the **`frac`**

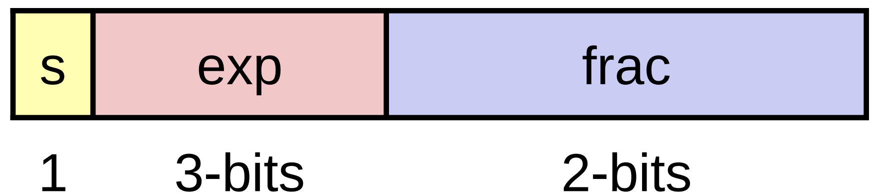
- Same general form as IEEE Format
  - normalized, denormalized
  - representation of 0, NaN, infinity

Dynamic Range (Positive Only)

	s	exp	frac	E	Value	
	0	0000	000	-6	0	
	0	0000	001	-6	1/8*1/64 = 1/512	closest to zero
Denormalized numbers	0	0000	010	-6	2/8*1/64 = 2/512	
	…					
	0	0000	110	-6	6/8*1/64 = 6/512	
	0	0000	111	-6	7/8*1/64 = 7/512	largest denorm
	0	0001	000	-6	8/8*1/64 = 8/512	smallest norm
	0	0001	001	-6	9/8*1/64 = 9/512	
	…					
	0	0110	110	-1	14/8*1/2 = 14/16	
	0	0110	111	-1	15/8*1/2 = 15/16	closest to 1 below
Normalized numbers	0	0111	000	0	8/8*1 = 1	
	0	0111	001	0	9/8*1 = 9/8	closest to 1 above
	0	0111	010	0	10/8*1 = 10/8	
	…					
	0	1110	110	7	14/8*128 = 224	
	0	1110	111	7	15/8*128 = 240	largest norm
	0	1111	000	n/a	inf	

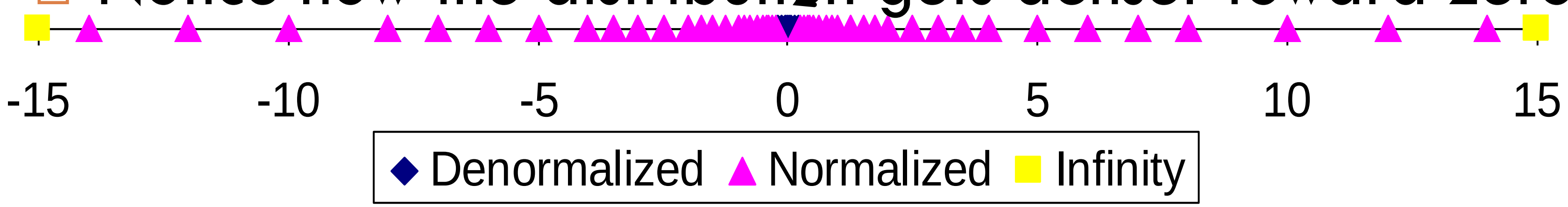
Distribution of Values

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 23-1-1 = 3



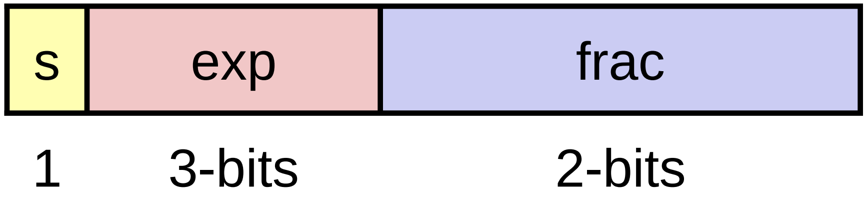
8 values

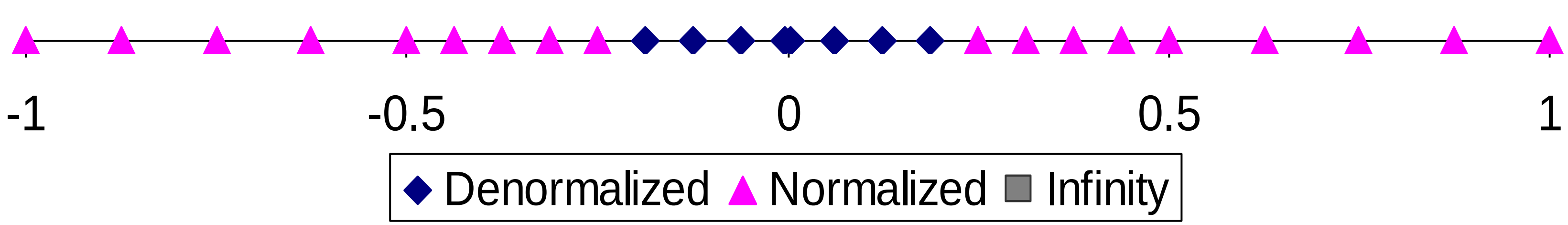
- Notice how the distribution gets denser toward zero.



Distribution of Values (close-up view)

- 6-bit IEEE-like format
  - e = 3 exponent bits
  - f = 2 fraction bits
  - Bias is 3





Interesting Numbers

`{single,double}`

Description	*exp*	*frac*	*Numeric Value*
Zero	00…00	00…00	0.0
Smallest Pos. Denorm.	00…00	00…01	$2^{-\{23,52\}} \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.4×10^{-45}			
▪ Double ≈ 4.9×10^{-324}			
Largest Denormalized	00…00	11…11	$(1.0 - \varepsilon) \times 2^{-\{126,1022\}}$
▪ Single ≈ 1.18×10^{-38}			
▪ Double ≈ 2.2×10^{-308}			
Smallest Pos. Normalized	00…01	00…00	$1.0 \times 2^{-\{126,1022\}}$
▪ Just larger than largest denormalized			
One	01…11	00…00	1.0
Largest Normalized	11…10	11…11	$(2.0 - \varepsilon) \times 2^{\{127,1023\}}$
▪ Single ≈ 3.4×10^{38}			
▪ Double ≈ 1.8×10^{308}			

Special Properties of Encoding

- FP Zero Same as Integer Zero
  - All bits = 0
- Can (Almost) Use Unsigned Integer Comparison
  - Must first compare sign bits
  - Must consider $-0 = 0$
  - NaNs problematic
    - Will be greater than any other values
    - What should comparison yield?
  - Otherwise OK
    - Denorm vs. normalized
    - Normalized vs. infinity

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Floating Point Operations: Basic Idea

- `x +f y = Round(x + y)`

- `x ×f y = Round(x × y)`

- Basic idea
  - First compute exact result
  - Make it fit into desired precision
    - Possibly overflow if exponent too large
    - Possibly round to fit into `frac`

Rounding

- Rounding Modes (illustrate with $ rounding)

	$1.40	$1.60	$1.50	$2.50	–$1.50
Towards zero	$1	$1	$1	$2	–$1
Round down (−∞)	$1	$1	$1	$2	–$2
Round up (+∞)	$2	$2	$2	$3	–$1
Nearest Even (default)	$1	$2	$2	$2	–$2

- What are the advantages of the modes?

Closer Look at Round-To-Even

- Default Rounding Mode
  - Hard to get any other kind without dropping into assembly
  - All others are statistically biased
    - Sum of set of positive numbers will consistently be over- or under-estimated

- Applying to Other Decimal Places / Bit Positions
  - When exactly halfway between two possible values
    - Round so that least significant digit is even
  - E.g., round to nearest hundredth

1.2349999	1.23	(Less than half way)
1.2350001	1.24	(Greater than half way)
1.2350000	1.24	(Half way—round up)
1.2450000	1.24	(Half way—round down)

Rounding Binary Numbers

- Binary Fractional Numbers
  - "Even" when least significant bit is 0
  - "Half way" when bits to right of rounding position = $100\ldots_2$

- Examples
  - Round to nearest 1/4 (2 bits right of binary point)

Value	Binary	Rounded	Action	Rounded Value
2 3/32	10.00011_2	10.00_2	(<1/2—down)	2
2 3/16	10.00110_2	10.01_2	(>1/2—up)	2 1/4
2 7/8	10.11100_2	11.00_2	(1/2—up)	3
2 5/8	10.10100_2	10.10_2	(1/2—down)	2 1/2

FP Multiplication

- $(-1)^{s1}$ $\boldsymbol{M1}$ $2^{\boldsymbol{E1}}$ x $(-1)^{\boldsymbol{s2}}$ $\boldsymbol{M2}$ $2^{\boldsymbol{E2}}$
- Exact Result: $(-1)^{s}$ $\boldsymbol{M}$ $2^{\boldsymbol{E}}$
  - Sign *s*: *s1* ^ *s2*
  - Significand *M*: *M1* x *M2*
  - Exponent *E*: *E1* + *E2*

- Fixing
  - If $M \geq 2$, shift *M* right, increment *E*
  - If *E* out of range, overflow
  - Round *M* to fit **`frac`** precision

- Implementation
  - Biggest chore is multiplying significands

Mathematical Properties of FP Add

- Compare to those of Abelian Group
  - Closed under addition? ***Yes***
    - But may generate infinity or NaN
  - Commutative? ***Yes***
  - Associative? ***No***
    - Overflow and inexactness of rounding
  - 0 is additive identity? ***Yes***
  - Every element has additive inverse ***Almost***
    - Except for infinities & NaNs
- Monotonicity
  - $a \geq b \Rightarrow a+c \geq b+c$? ***Almost***
    - Except for infinities & NaNs

Mathematical Properties of FP Mult

- Compare to Commutative Ring
  - Closed under multiplication? **Yes**
    - But may generate infinity or NaN
  - Multiplication Commutative? **Yes**
  - Multiplication is Associative? **No**
    - Possibility of overflow, inexactness of rounding
  - 1 is multiplicative identity? **Yes**
  - Multiplication distributes over addition? **No**
    - Possibility of overflow, inexactness of rounding

- Monotonicity
  - $a \geq b$ & $c \geq 0 \Rightarrow a * c \geq b * c$? **Almost**
    - Except for infinities & NaNs

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Floating Point in C

- C Guarantees Two Levels
  - **`float`** single precision
  - **`double`** double precision
- Conversions/Casting
  - Casting between **`int`**, **`float`**, and **`double`** changes bit representation
  - **`double`/`float`** → **`int`**
    - Truncates fractional part
    - Like rounding toward zero
    - Not defined when out of range or NaN: Generally sets to TMin
  - **`int`** → **`double`**
    - Exact conversion, as long as **`int`** has ≤ 53 bit word size
  - **`int`** → **`float`**
    - Will round according to rounding mode

Floating Point Puzzles

- For each of the following C expressions, either:
  - Argue that it is true for all argument values
  - Explain why not true

```
int x = …;
float f = …;
double d = …;
```

Assume neither **d** nor **f** is NaN

- x == (int)(float) x
- x == (int)(double) x
- f == (float)(double) f
- d == (float) d
- f == -(-f);
- 2/3 == 2/3.0
- d < 0.0 ⇒ ((d*2) < 0.0)
- d > f ⇒ -f > -d
- d * d >= 0.0
- (d+f)-d == f

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Summary

- IEEE Floating Point has clear mathematical properties
- Represents numbers of form M x 2^E
- One can reason about operations independent of implementation
  - As if computed with perfect precision and then rounded
- Not the same as real arithmetic
  - Violates associativity/distributivity
  - Makes life difficult for compilers & serious numerical applications programmers