On two types of infinite sets of infinite sequences.

by the Tuesday Afternoon Club.

In the following x will stand for the infinite sequence

$$(x(0), x(1), x(2), \dots)$$

where the x(i) are taken from a finite alphabet of at least two characters, say $\{0, 1\}$.

Let us consider the sets of sequences S1 and S2, defined as the solution sets of the equations P1(x) and P2(x), respectively, with

$$P1(x): \quad (\underline{E} \text{ i: } i \ge 0: (\underline{A} \text{ j: } 0 \le j < i: x(j) = 0) \text{ and } (\underline{A} \text{ j: } i \le j: x(j) = 1))$$

$$P2(x): \quad (\underline{A} \text{ i: } i \geq 0: x(i) = 0 \text{ or } (\underline{A} \text{ j: } j \geq i: x(j) = 1)) \quad .$$

Because $P1(x) \Rightarrow P2(x)$, S1 is a subset of 52, even a proper subset: there exists one solution of P2(x) and non P1(x), viz. the sequence with x(i) = 0 for all $i \geq 0$.

The set 52 has the property that for any non-member of 52 --i.e. any solution of <u>non</u> P2(x)-- non-membership can be established on account of an initial segment of it. As a matter of fact, in this particular case only two elements suffice for this evidence, as follows from <u>non</u> P2(x):

$$(\underline{E} \text{ i: } i \ge 0: x(i) \ne 0 \text{ and } (\underline{E} \text{ j: } j \ge i: x(j) \ne 1))$$
.

The set S1 doesn't have this property because there exists a non-member --viz. x(i) = 0 for all $i \ge 0$ -- with the property that any initial segment of it is also the initial segment of a member of S1. We call S2 "closed" and S1 "non-closed", that is:

"The set S is closed" means "any non-member of S has an initial segment that is not the initial segment of any member of S".

"The set S is non-closed" means "there exists a non-member of S such that any initial segment of it is also the initial segment of some member of S ".

from the above it follows

- 1) that a set of sequences is either closed or non-closed,
- 2) that any finite set --in particular the empty set-- is closed,
- 3) that the (uncountably infinite) universe of all possible sequences is closed.

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The notion of a closed set is of significance in connection with nondeterministic infinite computations. Consider the set of possible output sequences y corresponding to

"initialize; i:= 0; (1)

do true
$$\rightarrow$$
 compute; print(y(i)); i:= i + 1 od

Here "initialize" and "compute" stand for terminating computations not affecting the value of i . If "initialize" and "compute" stand for deterministic computations, the set of possible output sequences consists of a single
element; if "initialize" and (in particular) "compute" are non-deterministic,
the corresponding set of possible output sequences can be infinite.

"Program (1) is a continuous machine" means "in program (1) "initialize" and "compute" are both of bounded non-determinacy".

We can now prove the following

<u>Theorem 1.</u> The possible output sequences of program (1) form a closed set <u>or</u> program (1) is not a continuous machine.

<u>Proof.</u> Let 5 be the set of possible output sequences of program (1).

Either 5 is closed —in which case Theorem 1 holds—— or 5 is non-closed.

In the latter case, let x be a non-member of 5 such that each initial segment of x is also an initial segment of some member of 5. Consider now the following program:

"initialize; i:= 0; (2)

do "y(0),...,y(i-1) is an initial segment of
$$x$$
" \rightarrow

compute; print(y(i)); i:= i + 1

od; print(i)" .

Non-termination of (2) is excluded because x is not a member of S.

Because <u>each</u> initial segment of x is also the initial segment of some member of S, the final value of i is unbounded. Hence, program (2) is a (weakly) terminating program of unbounded non-determinacy; hence "initialize" or "compute" is of unbounded non-determinacy, i.e. program (1) is not a continuous machine. (End of Proof.)

<u>Theorem 2</u>. A set of sequences is non-closed <u>or</u> is the set of possible output sequences of some continuous machine.

<u>Proof.</u> Let S be the set of sequences. Either set S is non-closed --in which case Theorem 2 holds-- or it is closed. In the latter case consider program (1) with "initialize" deterministic and "compute" only constrained by the requirement that the next y(i) leads to an initial segment of some element of S. Because the alphabet is finite, "compute" need not be of unbounded non-determinacy. By virtue of its construction program (1) may then generate <u>any</u> member of S, and, S being closed, it cannot generate any non-member of S. (End of Proof.)

Having identified closed sets of possible output sequences with continuous machines, we propose to ignore for the time being non-deterministic infinite computations to which non-closed sets correspond.

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