D.A.Turner's reply.

In reply to EWD759, D.A.Turner sent me the following proof by returning mail.

"Thank you for your "somewhat open letter", which arrived yesterday. You pose several task — the order in which I have decided to tackle them is to establish first the precise formal relationship between f and g.

First some notation. I shall call the elements of a list f, f_0 f_1 f_2 etc. and I shall use the notation

$$[h(i)]_{i=A}^{B}$$

for the list $[h(A), h(A+1), \ldots, h(B)]$.

Now I define a function "upto"

$$\text{upto } i\ f = \text{ least } j \geq 0 \text{ such that } f_j > i \qquad (\text{upto}.0)$$

The claim to be established is that

$$g = [\text{upto } i\ f]_{i=0}^{\infty} \qquad (\text{Theorem } 1)$$

From upto.0 we deduce the following two propositions, which could be considered a SASL definition of upto

$$a > i \;\vdash\; \text{upto } i\ (a ; f) = 0 \qquad \text{(upto.1)}$$
$$a \leq i \;\vdash\; \text{upto } i\ (a ; f) = 1 + \text{upto } i\ f \qquad \text{(upto.2)}$$

We have also your definition of the function "k"

$$p \leq y \;\vdash\; k\ x\ y\ (p : q) = k\ (x+1)\ y\ q \qquad \text{(k.1)}$$
$$p > y \;\vdash\; k\ x\ y\ (p : q) = x : k\ x\ (y+1)(p : q) \qquad \text{(k.2)}$$

From these four propositions we shall deduce the following generalization of Theorem 1.

<u>Theorem 0</u>. $k\ x\ y\ f = [x + \text{upto } i\ f]_{i=y}^{\infty}$

<u>Proof</u> by structural induction on f, which is an infinite list of integers.

<u>case</u> Ω_L (Note — we need to distinguish between Ω_L, the undefined element in the space to which f belongs, and Ω_I, the undefined integer. The relationship between them is $\Omega_L = [\Omega_I]_0^{\infty}$)

$k\ x\ y\ \Omega_L = \Omega_L$ from k.1, k.2 by case exhaustion whereas

$$[x + \text{upto } i\ \Omega_L]_{i=y}^{\infty}$$
$$= [x + \Omega_I]_{i=y}^{\infty} \qquad \text{from upto.1, upto.2}$$
$$= [\Omega_I]_{i=y}^{\infty} \qquad \text{properties of } \Omega$$
$$= \Omega_L \qquad \text{as required}$$

<u>case</u> $p:f$

$$
\begin{aligned}
& k\ x\ y\ (p:f) \\
&= [x]^{p-y} \mathbin{+\!\!+} k\ x\ p\ (p:f) && \text{by repeated appl. of } k.2 \\
&= [x]^{p-y} \mathbin{+\!\!+} k\ (x+1)\ p\ f && \text{by } k.1 \\
&= [x]^{p-y} \mathbin{+\!\!+} [(x+1) + \text{upto } i\ f]_{i=p}^{\infty} && \underline{\text{ex hyp}}. \\
&= [x]^{p-y} \mathbin{+\!\!+} [x + (1 + \text{upto } i\ f)]_{i=p}^{\infty} && \text{properties of } + \\
&= [x]^{p-y} \mathbin{+\!\!+} [x + \text{upto } i\ (p:f)]_{i=p}^{\infty} && \text{by upto.2} \\
&= [x + \text{upto } i\ (p:f)]_{i=y}^{\infty} && \text{by upto.1 and rearranging}
\end{aligned}
$$

<u>QED Theorem 0</u>.

Whence, since $g = k\ 0\ 0\ f$, we have immediately

<u>Theorem 1</u> $\quad g = [\text{upto } i\ f]_{i=0}^{\infty}$.

Also you asked me to establish that g is A) ascending and B) unbounded, given appropriate assumptions about f. This now follows easily from the above. (Relaxing the level of formality somewhat) we have:

A) From upto.0 it follows (by transitivity of "$>$") that "upto i f" is an ascending function of i. Therefore, whatever the nature of f, g is ascending.

B) Let us define "f is unbounded" to mean: "for any $N \geq 0$, there is a $j \geq 0$ such that $f_j > N$".

Assume f is unbounded (f ascending <u>not relevant</u>).
Then, from upto.0,

upto i f is defined for all $i \geq 0$

Given any $N \geq 0$, define $j = \max\{f_0, \ldots, f_N\}$
then upto j $f = g_j$ exists, and by construction $> N$
So g too is unbounded."

* * *

So far Turner's reply. I like Turner's proof, and in view of the fact that Turner answered me by returning mail it would be misplaced to complain too much about the fact that, in "<u>case</u> $p{:}f$" the definitely less interesting case $p < y$ hasn't been dealt with explicitly.

I am slightly uneasy in "<u>case</u> Ω_L" — particularly after the parenthetical remark explaining the difference between Ω_L and Ω_I — about the justification "from k.1, k.2 by case exhaustion". My uneasiness is certainly caused by lack of familiarity how to deal with Ω. Take

$$\text{funny } (p{:}q) = \begin{array}{l} \textbf{if } p \geq 10 \rightarrow 1 : \text{funny } q \\ \;[\!]\; p < 10 \rightarrow 1 : \text{funny } q \\ \textbf{fi} \end{array}$$

Is funny $\Omega_L = \Omega_L$? Or is funny Ω_L = ones

(with ones = 1: ones)? I expect the first answer, though I would prefer the second one, if I am giving full weight to the remark (in [1], pg 37)

"The first point to be made is that in reasoning about SASL programs, Ω can be treated just like any other value as regards being substitutable in equations."

Perhaps I have failed to fathom the complete depth of the constraint "as regards being substitutable in equations".

The correspondence was triggered by remarks in [1] such as the recommendation of applicative programming (pg. 14):

"The proofs (like the programs themselves) are very much shorter than the proofs of the corresponding imperative programs."

I had my doubts, which have not been dispelled by the comparison of Turner's proof with the one given in EWD758.

[1] Turner, D.A. "Program Proving and Applicative Languages", August 1980

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