A sequence with $|x_n| = x_{n-1} + x_{n+1}$ has period 9

Recently I heard the theorem that any (in both directions) infinite sequence of real numbers x_n such that for all n

$$|x_n| = x_{n-1} + x_{n+1} \qquad (0)$$

has a period of length 9. Here is my proof.

From (0) we conclude (i) that the sequence contains a nonnegative element, (ii) that one of its neighbours is nonnegative, and (iii) that at least one of the two elements adjacent to a pair of nonnegative neighbours is nonnegative. More precisely:

p		the sequence contains in some direc-
p+r	($\geqslant 0$)	tion a triple of adjacent elements
r	($\geqslant 0$)	of the form (p, p+r, r) with
-p	($\leqslant 0$)	$0 \leqslant r \leqslant p$. To the left we have extended
p-r	($\geqslant 0$)	the sequence with another 8 elements.
2·p-r	($\geqslant 0$)	From (0) we further conclude that
p	($\geqslant 0$)	the whole sequence is determined
r-p	($\leqslant 0$)	by a pair of adjacent values;
-r	($\leqslant 0$)	hence, the repetition of the pair
p	($\geqslant 0$)	(p, p+r) at distance 9 proves
p+r		the theorem. [The above deserves recording for its lack of case analyses.]

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