Partitioning predicates and punctual functions (see EWD955)

Let f be a function from structures to structures, both on the same space. We then have Leibniz's Rule, i.e.

(0) $\quad (\underline{A}\, X,Y :: [X=Y] \Rightarrow [f.X = f.Y])$.

We may have the stronger

(1) $\quad (\underline{A}\, X,Y :: [X=Y \;\Rightarrow\; f.X = f.Y])$,

a state of affairs we characterize by saying "f is a punctual function". In this note we present another characterization of punctual dependence, viz. distribution over partitioning.

In what follows

- green and yellow are some space-independent types (like boolean, real, integer, etc.)
- c is a green variable
- B is a function from a green argument to a boolean structure and it is partitioning, i.e.

$$[(\underline{N}c : B.c : true) = 1]$$

- R is a function from a green argument to a yellow structure

We now consider the equation in the unknown yellow structure Z

(2) $\quad Z : [(\underline{Q}c : B.c : Z = R.c)]$

in which, B being partitioning, it does not matter whether we read $\underline{A}$ or $\underline{E}$ for $\underline{Q}$ — see EWD955 — .

In order to show that the solution of (2), if existing, is unique, we first prove

<u>Lemma 0</u> For partinioning $\mathcal{B}$ and boolean structure X we have

$$[(\underline{Q}c: \mathcal{B}.c: X)] \equiv [X]$$

<u>Proof</u>

$$\begin{array}{ll}
 & [(\underline{Q}c: \mathcal{B}.c: X)] \\
= & \{\text{EWD955}\} \\
 & [(\underline{E}c: \mathcal{B}.c: X)] \\
= & \{\text{pred. calc}\} \\
 & [(\underline{E}c: \mathcal{B}.c: true) \wedge X] \\
= & \{\text{rel. between } \underline{E} \text{ and } \underline{N}\} \\
 & [(\underline{N}c: \mathcal{B}.c: true) \geq 1 \wedge X] \\
= & \{\mathcal{B} \text{ is partitioning}\} \\
 & [X]
\end{array}$$

(End of Proof.)

We now observe

$$\begin{array}{ll}
 & [(\underline{Q}c: \mathcal{B}.c: U = \mathcal{R}.c)] \wedge [(\underline{Q}c: \mathcal{B}.c: V = \mathcal{R}.c)] \\
= & \{\text{EWD955}\} \\
 & [(\underline{A}c: \mathcal{B}.c: U = \mathcal{R}.c)] \wedge [(\underline{A}c: \mathcal{B}.c: V = \mathcal{R}.c)] \\
= & \{\text{pred. calc.}\} \\
 & [(\underline{A}c: \mathcal{B}.c: U = \mathcal{R}.c \wedge V = \mathcal{R}.c)] \\
\Rightarrow & \{\text{pred. calc.}\} \\
 & [(\underline{A}c: \mathcal{B}.c: U = V)] \\
= & \{\text{Lemma 0}\} \\
 & [U = V]
\end{array}$$

from which the uniqueness follows.

But does the solution of (2) always exist? It does if "yellow" means "boolean"; then we can write it down in closed form, viz.

(3) $$(\underline{Q}c: \mathcal{B}.c: \mathcal{R}.c)$$.

<u>Proof</u>

$$[(\underline{Q}c: B.c: (\underline{Q}d: B.d: R.d) \Rightarrow R.c)] \qquad (x)$$
$$= \{\text{EWD 955}\}$$
$$[(\underline{A}c: B.c: (\underline{Q}d: B.d: R.d) \Rightarrow R.c)]$$
$$= \{\text{pred. calc}\}$$
$$[(\underline{Q}d: B.d: R.d) \Rightarrow (\underline{A}c: B.c: R.c)]$$
$$= \{\text{EWD 955}\}$$
$$\text{true}$$

$$[(\underline{Q}c: B.c: (\underline{Q}d: B.d: R.d) \Leftarrow R.c)] \qquad (xx)$$
$$= \{\text{EWD955}\}$$
$$[(\underline{E}c: B.c: (\underline{Q}d: B.d: R.d) \Leftarrow R.c)]$$
$$= \{\text{pred. calc}\}$$
$$[(\underline{Q}d: B.d: R.d) \Leftarrow (\underline{A}c: B.c: R.c)]$$
$$= \{\text{EWD 955}\}$$
$$\text{true}$$

From (x) and (xx) follows that (3) solves (2).
(End of Proof.)

For "yellow" another than boolean, we postulate that (2) has a solution. As $\underline{Q}$ stands traditionally for $\underline{A}$ or $\underline{E}$, we invent a new notation; in view of the close connection with the case expression, we write the solution of (2) as

(4) $\qquad (\,[\!]\,c: B.c: R.c)$

<u>Note</u> For "yellow" meaning "real" or "integer", $[\!]$ could stand for <u>MIN</u>, <u>MAX</u>, <u>S</u>(um), or <u>P</u>(roduct). For these operations the analogue of the one-point rule holds. (End of Note.)

Now we can formulate

<u>Theorem</u> "function f is punctual" $\equiv$

"$[f.([\!]c: B.c: R.c) = ([\!]c: B.c: f.(R.c))]$
for all R and partitioning B " .

Proof

LHS $\Rightarrow$ RHS

With $[Z = ([\!]c: B.c: R.c)]$, we derive for any R and B

$$
\begin{aligned}
& \text{RHS} \\
= & \ \{\text{def. of } Z\} \\
& [f.Z = ([\!]c: B.c: f.(R.c))] \\
= & \ \{\text{def. of } [\!], (2) \text{ with } Z, R.c := f.Z, f.(R.c)\} \\
& [(\underset{\sim}{Q}c: B.c: f.Z = f.(R.c))] \qquad (***) \\
\Leftarrow & \ \{\text{LHS, i.e. } (1) \text{ with } X, Y := Z, R.c\} \\
& [(\underset{\sim}{Q}c: B.c: Z = R.c)] \\
= & \ \{\text{def. of } Z \text{ and } [\!], (2)\} \\
& \text{true} \quad .
\end{aligned}
$$

RHS $\Rightarrow$ LHS

For arbitrary X and Y, we have to show, using RHS,

$$[X = Y \Rightarrow f.X = f.Y] \quad .$$

In view of (***) we shall derive this by proper instantiation of RHS, viz.

- c ranges over the two boolean structures $X=Y$ and $X \neq Y$
- B is the identity function; in view of $[(X=Y) \not\equiv (X \neq Y)]$, B is partitioning
- $[R.(X=Y) = X]$ and $[R.(X \neq Y) = Y]$

With Z, as above: $[Z = ([\!]c: B.c: R.c)]$, Z is with this instantiation given as the unique solution of (see (2)), with $\underset{\sim}{Q} := \underset{\sim}{E}$,

$2: [(X = Y \wedge Z = X) \vee (X \neq Y \wedge Z = Y)]$

of which the —only!— solution Z is given by

$$[Z = Y]$$

After this groundwork we proceed

$$
\begin{array}{ll}
& \text{RHS} \\
= & \{\text{see } (***)\text{, and } \underline{Q} := \underline{A}\} \\
& [(\underline{A}c : B.c : f.Z = f.(R.c))] \\
\Rightarrow & \{\text{with the given instantiation, } c := (X = Y)\} \\
& [X = Y \Rightarrow f.Z = f.(R.(X = Y))] \\
= & \{[Z = Y] \text{ and } [R.(X = Y) = X] \text{ and } (0)\} \\
& [X = Y \Rightarrow f.Y = f.X]
\end{array}
$$

(End of Proof.)

Acknowledgement The above was written with A.J.M. van Gasteren looking over my shoulder.

Austin, 14 May 1986

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