Relational Calculus according to ATAC

This is my personal summary of the relational calculus as it emerged at the ATAC session of 23.9.1986. Major credit is due to He Jifeng (in absentia), Tony Hoare and Wim H. Hesselink; no one but me is to be blamed for shortcomings in this text.

To begin with we introduce a prefix operator $\backsim$ — read "tilde" — . It has the same — i.e. maximum — binding power as $\neg$; unary prefix operators are right associative (as usual).

Axiom 0 $\quad [X \Rightarrow \backsim Y] \equiv [Y \Rightarrow \backsim X]$.

Theorem 0 $\quad [\backsim false \equiv true]$.

Proof 0 $\quad$ true

$\quad = \{$predicate calculus$\}$

$\quad [false \Rightarrow \backsim true]$

$\quad = \{$Axm. 0 with $X, Y := false, true\}$

$\quad [true \Rightarrow \backsim false]$

$\quad = \{$predicate calculus$\}$

$\quad [\backsim false \equiv true]$.

(End of Proof 0.)

Axiom 1 $\quad [X \Leftarrow \backsim Y] \equiv [Y \Leftarrow \backsim X]$

Theorem 1 $\quad [\backsim true \equiv false]$

Proof 1 Similar to Proof 0. (End of Proof 1.)

<u>Remark</u> Replacing $\sim$ by $\neg$ in the above axioms yields theorems from the predicate calculus, but please note that the negation satisfies the stronger

$$[X \Rightarrow \neg Y \;\equiv\; Y \Rightarrow \neg X] \qquad \text{and}$$

$$[X \Leftarrow \neg Y \;\equiv\; Y \Leftarrow \neg X] \;.$$

(End of Remark)

<u>Theorem 2</u> $[X \equiv \sim Y] \equiv [Y \equiv \sim X]$.

<u>Proof 2</u>

$$
\begin{array}{ll}
 & [X \equiv \sim Y] \\
= & \{\text{predicate calculus}\} \\
 & [X \Rightarrow \sim Y] \wedge [X \Leftarrow \sim Y] \\
= & \{\text{Axms. 0 and 1}\} \\
 & [Y \Rightarrow \sim X] \wedge [Y \Leftarrow \sim X] \\
= & \{\text{predicate calculus}\} \\
 & [Y \equiv \sim X] \quad .
\end{array}
$$

(End of Proof 2.)

<u>Theorem 3</u> $[X \equiv \sim\sim X]$

<u>Proof 3</u>

$$
\begin{array}{ll}
 & [X \equiv \sim\sim X] \\
= & \{\text{Thm. 2 with } Y := \sim X\} \\
 & [\sim X \equiv \sim X] \\
= & \{\text{predicate calculus}\} \\
 & \mathit{true} \quad .
\end{array}
$$

(End of Proof 3.)

Next we introduce a binary infix operator, called "composition" and denoted by ; . Its binding power is less than that of the unary operators and higher than the binary logical operators.

<u>Axiom 2</u> $[X;(Y;Z) \equiv (X;Y);Z]$.

In view of the associativity of composition we feel from now on free (but not obliged) to omit parenthesis pairs from continued compositions.

The next axiom introduces a constant, denoted by D and called "diversity".

<u>Axiom 3</u> $[X;Y \Rightarrow D] \equiv [X \Rightarrow \sim Y]$.

<u>Theorem 4</u> $[X0;\ldots;X1;Y0;\ldots;Y1 \Rightarrow D] \equiv$
$[Y0;\ldots;Y1;X0;\ldots;X1 \Rightarrow D]$,

i.e. with a continued composition everywhere implying D, we are free to rotate the arguments of the continued composition: only their cyclic order matters.

<u>Proof 4</u> $[X0;\ldots;X1;Y0;\ldots;Y1 \Rightarrow D]$
$=$ { Axiom 2 : we are free to place parentheses }
$[(X0;\ldots;X1);(Y0;\ldots;Y1) \Rightarrow D]$
$=$ { Axiom 3 }
$[(X0;\ldots;X1) \Rightarrow \sim(Y0;\ldots;Y1)]$
$=$ { Axiom 0 }
$[(Y0;\ldots;Y1) \Rightarrow \sim(X0;\ldots;X1)]$
$=$ { Axioms 3 and 2 }
$[Y0;\ldots;Y1;X0;\ldots;X1 \Rightarrow D]$.

(End of Proof 4)

<u>Theorem 5a</u> $[X \equiv X;\sim D]$

<u>Proof 5a</u>

$$\begin{array}{cl} & [Z \Rightarrow \sim X] \\ = & \{\text{Axiom 3 with } X,Y := Z,X\} \\ & [Z;X \Rightarrow D] \\ = & \{\text{Thm 3 with } X := D\} \\ & [Z;X \Rightarrow \sim\sim D] \\ = & \{\text{Axiom 3 with } X,Y := (Z;X), \sim D\} \\ & [(Z;X);\sim D \Rightarrow D] \\ = & \{\text{Axiom 2}\} \\ & [Z;(X;\sim D) \Rightarrow D] \\ = & \{\text{Axiom 3 with } X,Y := Z,(X;\sim D)\} \\ & [Z \Rightarrow \sim(X;\sim D)] \end{array}$$

Since the equivalence between the extreme lines of the above holds for any Z and X, we conclude for any X

$$\begin{array}{cl} & \text{true} \\ = & \{\text{above observation}\} \\ & [\sim X \equiv \sim(X;\sim D)] \\ \Rightarrow & \{\text{Leibniz}\} \\ & [\sim\sim X \equiv \sim\sim(X;\sim D)] \\ = & \{\text{Theorem 3 twice}\} \\ & [X \equiv (X;\sim D)] \end{array}$$

(End of Proof 5a)

<u>Remark</u> We could similarly show that $\sim D$ is also the left-identity element of composition, but this conclusion is postponed. (End of Remark.)

The next (and final?) axiom connects $\neg$ and $\sim$ with composition.

<u>Axiom 4</u> $[\sim(\sim X;\sim Y) \equiv \neg(\neg Y; \neg X)]$.

<u>Theorem 6a</u> $[X \equiv \neg D; X]$

<u>Proof 6a</u>

$$
\begin{array}{cl}
 & true \\
= & \{\text{Axiom 4 with } X,Y := \neg X, D\} \\
 & [\sim(\sim\neg X;\sim D) \equiv \neg(\neg D; \neg\neg X)] \\
= & \{\text{Thm 5a ; predicate calculus}\} \\
 & [\sim\sim\neg X \equiv \neg(\neg D; X)] \\
= & \{\text{Thm 3}\} \\
 & [\neg X \equiv \neg(\neg D; X)] \\
= & \{\text{predicate calculus}\} \\
 & [X \equiv \neg D; X] \quad .
\end{array}
$$

(End of Proof 6a)

<u>Remark</u> We could similarly have shown that $\neg D$ is a right identity element of composition, but this conclusion now follows in a moment. (End of Remark)

<u>Theorem 7</u> $[\sim D \equiv \neg D]$

<u>Proof 7</u>

$$
\begin{array}{cl}
 & \sim D \\
= & \{\text{Thm 6a with } X := \sim D \\
 & \neg D; \sim D \\
= & \{\text{Thm 5a with } X := \neg D\} \\
 & \neg D \quad .
\end{array}
$$

(End of Proof 7)

Combining Theorems 5a, 6a and 7, we conclude

<u>Theorem 5</u> $[X \equiv X;\sim D]$ and $[X \equiv \sim D; X]$.

<u>Theorem 6</u> $[X \equiv \neg D ; X]$ and $[X \equiv X ; \neg D]$.

And now we are ready to show that our two unary operators commute:

<u>Theorem 7</u> $[\neg \backsim X \equiv \backsim \neg X]$

<u>Proof 7</u> $[\neg \backsim X \Rightarrow H]$

$=$ {predicate calculus}

$[\neg H \Rightarrow \backsim X]$

$=$ {Axiom 3 with $X, Y := \neg H, X$}

$[\neg H ; X \Rightarrow D]$

$=$ {predicate calculus}

$[\neg H ; \neg\neg X \Rightarrow D]$

$=$ {predicate calculus}

$[\neg D \Rightarrow \neg(\neg H ; \neg\neg X)]$

$=$ {Axiom 4 with $X, Y := \neg X, H$}

$[\neg D \Rightarrow \backsim(\backsim\neg X ; \backsim H)]$

$=$ {Axiom 3 with $X, Y := \neg D, (\backsim\neg X ; \backsim H)$}

$[\neg D ; \backsim\neg X ; \backsim H \Rightarrow D]$

$=$ {Theorem 6}

$[\backsim\neg X ; \backsim H \Rightarrow D]$

$=$ {Axiom 3 with $X, Y := \backsim\neg X, \backsim H$}

$[\backsim\neg X \Rightarrow \backsim\backsim H]$

$=$ {Theorem 3}

$[\backsim\neg X \Rightarrow H]$

From first and last line we conclude Theorem 7, as H was arbitrary.

(End of Proof 7.)

<u>Theorem 8</u> Composition is universally disjunctive in each of its arguments, i.e.

(i) $[(\underline{E}i :: X.i) ; Y \equiv (\underline{E}i :: X.i ; Y)]$ and

(ii) $[X ; (\underline{E}i :: Y.i) \equiv (\underline{E}i :: X ; Y.i)]$

To prove (i) we observe for any H

$$
\begin{array}{ll}
 & [(\underline{E}i :: X.i) ; Y \Rightarrow \sim H] \\
= & \{\text{Axiom } 3\} \\
 & [(\underline{E}i :: X.i) ; Y ; H \Rightarrow D] \qquad\qquad * \\
= & \{\text{Axiom } 3\} \\
 & [(\underline{E}i :: X.i) \Rightarrow \sim(Y ; H)] \\
= & \{\text{quasi-distribution consequent}\} \\
 & [(\underline{A}i :: X.i \Rightarrow \sim(Y ; H))] \\
= & \{\text{Axiom } 3\} \\
 & [(\underline{A}i :: X.i ; Y ; H \Rightarrow D)] \qquad\qquad ** \\
= & \{\text{Axiom } 3\} \\
 & [(\underline{A}i :: X.i ; Y \Rightarrow \sim H)] \\
= & \{\text{quasidistribution consequent}\} \\
 & [(\underline{E}i :: X.i ; Y) \Rightarrow \sim H]
\end{array}
$$

and since $\sim H$ is arbitrary, (i) has been proved.

To prove (ii) we observe

$$
\begin{array}{ll}
 & [X ; (\underline{E}i :: Y.i) \Rightarrow \sim H] \\
= & \{\text{Axiom } 3\} \\
 & [X ; (\underline{E}i :: Y.i) ; H \Rightarrow D] \\
= & \{\text{Theorem } 4\} \\
 & [(\underline{E}i :: Y.i) ; H ; X \Rightarrow D] \\
= & \{\text{as above from } * \text{ to } ** \text{ with } X, Y, H := Y, H, X\}
\end{array}
$$

$$
\begin{aligned}
& [(\underline{A}i :: Y.i \,;\, H \,;\, X \Rightarrow D)] \\
= \quad & \{\text{Theorem 4}\} \\
& [(\underline{A}i :: X ; Y.i \,;\, H \Rightarrow D)] \\
= \quad & \{\text{Axiom 3}\} \\
& [(\underline{A}i :: X ; Y.i \Rightarrow \sim H)] \\
= \quad & \{\text{quasidistribution}\} \\
& [(\underline{E}i :: X ; Y.i) \Rightarrow \sim H]
\end{aligned}
$$

and since $\sim H$ is arbitrary, (ii) has been proved.

(End of Proof 8)

* * *

Inspired by Theorem 7 we introduce a last unary operator, called "the converse"; we shall tentatively denote it by ϕ .

Axiom 5 $\quad [\phi X \equiv \neg \sim X]$

In view of Theorem 3 $- [X \equiv \sim\sim X] -$, of Theorem 7 $- [\neg \sim X \equiv \sim \neg X] -$ and of $[X \equiv \neg\neg X]$ we have

Theorem 9 Of the three operators $\neg$, $\sim$, and ϕ
(i) each is the functional composition of the other two
(ii) any pair commutes
(iii) each is its own inverse

Proof 9 is left to the reader.

Theorem 10 $[\phi X \Rightarrow Y] \equiv [X \Rightarrow \phi Y]$

Proof 10

$$
\begin{aligned}
& [\phi X \Rightarrow Y] \\
= & \quad \{\text{Theorem 9}\} \\
& [\backsim \neg X \Rightarrow Y] \\
= & \quad \{\text{Axiom 1}\} \\
& [\backsim Y \Rightarrow \neg X] \\
= & \quad \{\text{predicate calculus}\} \\
& [X \Rightarrow \neg \backsim Y] \\
= & \quad \{\text{Theorem 9}\} \\
& [X \Rightarrow \phi Y] \quad .
\end{aligned}
$$

(End of Proof 10)

Theorem 11 Operator ϕ is universally junctive.

Proof 11 We shall first prove that ϕ is universally conjunctive by observing

$$
\begin{aligned}
& [H \Rightarrow \phi(\underline{A}i :: X.i)] \\
= & \quad \{\text{Theorem 10}\} \\
& [\phi H \Rightarrow (\underline{A}i :: X.i)] \\
= & \quad \{\text{predicate calculus}\} \\
& [(\underline{A}i :: \phi H \Rightarrow X.i)] \\
= & \quad \{\text{predicate calculus}\} \\
& (\underline{A}i :: [\phi H \Rightarrow X.i]) \\
= & \quad \{\text{Theorem 10}\} \\
& (\underline{A}i :: [H \Rightarrow \phi X.i]) \\
= & \quad \{\text{predicate calculus}\} \\
& [(\underline{A}i :: H \Rightarrow \phi X.i)] \\
= & \quad \{\text{predicate calculus}\} \\
& [H \Rightarrow (\underline{A}i :: \phi X.i)]
\end{aligned}
$$

and hence $[\phi(\underline{A}i :: X.i) \equiv (\underline{A}i :: \phi X.i)]$ has been established.

Universal disjunctivity is now proved by observing

$$
\begin{array}{ll}
 & \phi(\underline{E}i :: X.i) \\
= & \{\text{Theorem 9, predicate calculus}\} \\
 & \neg\phi\,(\underline{A}i :: \neg X.i) \\
= & \{\phi \text{ universally conjunctive}\} \\
 & \neg\,(\underline{A}i :: \phi\neg X.i) \\
= & \{\text{predicate calculus, Theorem 9}\} \\
 & (\underline{E}i :: \phi X.i)
\end{array}
$$

(End of Proof 10).

Because —Theorem 9— ϕ commutes with $\neg$, and equivalence and implication can be expressed in $\wedge$, $\vee$ and $\neg$, can we take the converse of arbitray boolean expressions by replacing all atoms by their converses.

So much for my summary of yesterday afternoon.

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