The majority vote among three

Our purpose is to prove the equivalence of

(0) $(X \wedge Y) \vee (Y \wedge Z) \vee (Z \wedge X)$ and

(1) $(X \vee Y) \wedge (Y \vee Z) \wedge (Z \vee X)$.

To save work, $(?,!)$ is a permutation of $(\vee, \wedge)$; note that these operators satisfy

(2) $[X?Y \equiv X \equiv Y \equiv X!Y]$.

We observe for any A, B, C

$$
\begin{array}{ll}
 & A ? B ? C \\
= & \quad \{ (2) \text{ with } X,Y := A, (B?C) \} \\
 & A \equiv B ? C \equiv A!(B?C) \\
= & \quad \{ (2) \text{ with } X,Y := B, C \} \\
 & A \equiv B \equiv C \equiv B!C \equiv A!(B \equiv C \equiv B!C) \\
= & \quad \{ ! \text{ distributes over } \equiv\equiv \} \\
 & A \equiv B \equiv C \equiv A!B \equiv B!C \equiv C!A \equiv A!B!C!
\end{array}
$$

Applying the equivalence of first and last line with $A,B,C := X!Y, Y!Z, Z!X$, we find

(3) $[\,(X!Y)\ ?\ (Y!Z)\ ?\ (Z!X) \equiv$
$\qquad X!Y \equiv Y!Z \equiv Z!X\,]$

as the idempotence of ! reduces the four last terms of the last line all to $X!Y!Z$.

We now observe

$$
\begin{array}{ll}
 & (X \wedge Y) \vee (Y \wedge Z) \vee (Z \wedge X) \\
= & \{ (3) \text{ with } !, ? := \wedge, \vee \} \\
 & X \wedge Y \equiv Y \wedge Z \equiv Z \wedge X \\
= & \{ \text{Golden Rule thrice and } [X \equiv Y \equiv Y \equiv Z \equiv Z \equiv X] \} \\
 & X \vee Y \equiv Y \vee Z \equiv Z \vee X \\
= & \{ (3) \text{ with } !, ? := \vee, \wedge \} \\
 & (X \vee Y) \wedge (Y \vee Z) \wedge (Z \vee X)
\end{array}
$$

In passing we found two other expressions for the majority vote among three, viz. the continued equivalence of the pairwise disjunctions or of the pairwise conjunctions.

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prof. dr. Edsger W. Dijkstra
Department of Computer Sciences
The University of Texas at Austin
Austin, TX 78712-1188
USA