"Less than" in terms of "at most"

If, a in the case of comparing reals, $\leq$ is a total order, $<$ is most simply defined by

$$(0) \qquad x<y \;\equiv\; \neg(y \leq x) \quad .$$

The "transitivity" rule

$$(1) \qquad a \leq b \wedge b < c \;\Rightarrow\; a < c$$

is then a direct consequence of the transitivity of $\leq$:

$$
\begin{array}{ll}
 & a \leq b \wedge b < c \Rightarrow a < c \\
= & \quad \{ (0), \text{ twice} \} \\
 & a \leq b \wedge \neg(c \leq b) \Rightarrow \neg(c \leq a) \\
= & \quad \{ \text{shunting, twice} \} \\
 & c \leq a \wedge a \leq b \Rightarrow c \leq b \\
= & \quad \{ \leq \text{ transitive} \} \\
 & \text{true} \quad .
\end{array}
$$

If $\leq$ is not a total order, this does not work, and I used to use the definition

$$(2) \qquad x<y \;\equiv\; x \leq y \wedge x \neq y \quad .$$

The proof of (1) is then as follows:

$$
\begin{array}{ll}
 & a \leq b \wedge b < c \Rightarrow a < c \\
= & \quad \{ (2), \text{ twice} \} \\
 & a \leq b \wedge b \leq c \wedge b \neq c \Rightarrow a \leq c \wedge a \neq c \\
= & \quad \{ \leq \text{ is transitive} \}
\end{array}
$$

$$
\begin{array}{ll}
 & a \leq b \wedge b \leq c \wedge b \neq c \Rightarrow a \neq c \\
= & \quad \{\text{shunting, twice}\} \\
 & a \leq b \wedge b \leq c \wedge a = c \Rightarrow b = c \\
= & \quad \{\text{Leibniz}\} \\
 & c \leq b \wedge b \leq c \wedge a = c \Rightarrow b = c \\
= & \quad \{\leq \text{ reflexive}\} \\
 & \text{true}
\end{array}
$$

In [GS94] — "A logical approach to discrete math" by Gries & Schneider — I found the much nicer definition:

(3) $\quad x < y \;\equiv\; x \leq y \wedge \neg(y \leq x)$.

The proof of (1) is now very clean:

$$
\begin{array}{ll}
 & a \leq b \wedge b < c \Rightarrow a < c \\
= & \quad \{(3), \text{twice}\} \\
 & a \leq b \wedge b \leq c \wedge \neg(c \leq b) \Rightarrow a \leq c \wedge \neg(c \leq a) \\
\Leftarrow & \quad \{\text{pred. calc.}\} \\
 & (a \leq b \wedge b \leq c \Rightarrow a \leq c) \;\wedge \\
 & (c \leq a \wedge a \leq b \Rightarrow c \leq b) \\
= & \quad \{\leq \text{ transitive, twice}\} \\
 & \text{true.}
\end{array}
$$

The moral of the story is that the conjunct $x \neq y$ in (2) is awkward: it forces in general, in one way or another, an appeal to Leibniz, and in this special case also an appeal to the reflexivity of $\leq$. Definition (3) is much nicer because it uses $\leqslant$ only.

In retrospect I am amazed that I did not come up with (3) myself, for it is a straightforward strengthening of (0), with which I grew up.

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