A result of rabbit removal

In this note, all variables are of the same type.

In "Introduction to Logic and to the Methodology of Deductive Sciences" by Alfred Tarski, I found the following theorem — my formulation —

Let $\bullet$ be an associative, symmetric infix operator such that

(0) equation x: $a = b \bullet x$ is solvable for all a, b ; then

(1) $x \bullet y = x \bullet z \Rightarrow y = z$.

In comparison, Tarski calls his calculational proof "considerably more involved"; it was in any case too complicated to be remembered (at least: by me), it also seemed to me to be more complicated than necessary. (This last judgement was, of necessity, very tentative, because without detailed analysis it was impossible to see how much of the complexity belonged to the argument proper, and how much of it was the result of the style of presentation deemed acceptable in the 30s. Of course we cannot blame Tarski for having done most of his work before Wim Feijen developed his proof format.)

Under the circumstances there was only one action possible for me, viz. to ignore what I had seen and could not remember, and to try to design a proof myself. (Here, the ambitious reader is invited to stop reading and to try to prove the theorem himself.)

* * *

Since the consequent of our demonstrandum (1) is the equality $y = z$, the suggested structure of our proof could be

"We observe for any x, y, z

$$\begin{array}{l} y \\ = \quad \{\text{antecedent } x \cdot y = x \cdot z\} \\ z \end{array}$$

" ,

but this sketch contains little more than a proposal how to let the proof structure reflect that our demonstrandum has the shape of an implication.

In the one and only step of the above proof sketch, an equality has to be justified by an equality, hence an application of Leibniz's principle is dictated! Our next proof sketch is therefore:

"We observe for any x, y, z

$$
\begin{array}{ll}
 & y \\
= & \{\text{definition of } f\} \\
 & f.(x \bullet y) \\
= & \{\text{antecedent } x \bullet y = x \bullet z\} \\
 & f.(x \bullet z) \\
= & \{\text{definition of } f\} \\
 & z
\end{array}
$$

" .

<u>Remark</u> As far as the use of the antecedent via Leibniz's principle is concerned, one could consider the alternative

$$
\begin{array}{ll}
 & y \\
= & \{\text{definition of } f\} \\
 & f.(x \bullet z) \\
= & \{\text{antecedent } x \bullet y = x \bullet z\} \\
 & f.(x \bullet y) \\
= & \{\text{definition of } f\} \\
 & z
\end{array}
$$

,

but that does not work, and we shall later see why. (End of Remark.)

Our next task is to design a suitable f, i.e. to propose how to rewrite y as an expression that contains $x \bullet y$ as subexpression. Since $\bullet$ is the only operator defined on our type, we have little freedom: (0) with $a, b := y, x \cdot y$ allows us to postulate that p satisfies

(2) $\quad y = x \cdot y \cdot p \quad ,$

and now our proof can start as follows:

$$
\begin{array}{ll}
 & y \\
= & \quad \{(2)\} \\
 & x \cdot y \cdot p \\
= & \quad \{\text{antecedent } x \cdot y = x \cdot z\} \\
 & x \cdot z \cdot p
\end{array}
$$

Our remaining problem is how to transform $x \cdot z \cdot p$ into z, and we have little choice but to use the same (2), which has just been used for (x,p)-introduction, a second time, but now for (x,p)-removal. But for this removal to work, we need a temporary reïntroduction of y. For rewriting z so that it contains y as subexpression, the simplest solution, allowed by (0) with $a, b := z, y$, is to postulate that q satisfies

(3) $\quad z = y \cdot q \quad .$

And now we can give the complete proof. For any x, y, z, we postulate, as permitted by (0), that p satisfies (2) and q satisfies (3), and subsequently we observe

	Δy	Δz	Δp	Δq
y				
$= \{(2)\}$			$+1$	
$x \cdot y \cdot p$				
$= \{x \cdot y = x \cdot z\}$	-1	$+1$		
$x \cdot z \cdot p$				
$= \{(3)\}$	$+1$	-1		$+1$
$x \cdot y \cdot q \cdot p$				
$= \{(2)\}$			-1	
$y \cdot q$				
$= \{(3)\}$	-1	$+1$		-1
z				

To the right of the proof we have recorded the changes in the number of occurrences of variables y, z, p, q. Because the net effect of the appeals to (2) and (3) is $\Delta p = 0 \wedge \Delta q = 0$, it yields $\Delta y = 0 \wedge \Delta z = 0$ as well. The total effect of the proof (whose steps transform y into z) has to be $\Delta y = -1 \wedge \Delta z = +1$, and this is the change provided by the use of the antecedent via Leibniz's principle. This counting argument explains why the alternative proof sketch in the Remark does not work: it works in the wrong direction, viz. $\Delta y = +1 \wedge \Delta z = -1$.

* * *

To put the above proof into perspective, let me render —reduced to its essentials— Tarski's proof.

For any x,y,z, we postulate, as permitted by (0), that u,v,w satisfy,

$$
\begin{array}{ll}
(4) & y = y \cdot u \\
(5) & z = y \cdot v \\
(6) & u = x \cdot w
\end{array}
$$

and subsequently observe

$$
\begin{array}{ll}
 & y \\
= & \{(4)\} \\
 & y \cdot u \\
= & \{(6)\} \\
 & y \cdot x \cdot w \\
= & \{x \cdot y = x \cdot z\} \\
 & z \cdot x \cdot w \\
= & \{(6)\} \\
 & z \cdot u \\
= & \{(5)\} \\
 & y \cdot v \cdot u \\
= & \{(4)\} \\
 & y \cdot v \\
= & \{(5)\} \\
 & z
\end{array}
$$

Tarski's proof is 2 steps longer than ours because of the occurrence of u, which has to be introduced, and removed later. In fact, our proof can be obtained from Tarski's by eliminating u from (4)&(6). It is not quite clear why u was introduced in the first place; could it have happened because the heuristics remained completely implicit?

* * *

The structure of Tarski's proof was further obscured by his insistence of fully parenthesizing his expressions: symmetry and associativity thus add another 7 steps to the proof, which I consider a high price for so little.

About 20 pages later, he proves another theorem — again my formulation —

Let $\bullet$ be an infix operator that is symmetric and associative wherever defined, and such that

(0) equation x: $a = b \bullet x$ is solvable for all a, b ; then

(7) $x \bullet y$ is defined for all x, y .

The plan of the proof is to construct, as permitted by (0), for given x, y, a quantity z which, by virtue of its construction, can be transformed into $x \bullet y$. For reasons of heuristics, we start at the other end and ask ourselves what we have to introduce that will enable us to transform $x \bullet y$ into z .

Starting from

(α) $\qquad x \bullet y$

we have to do 3 things: remove x , remove y , and introduce z . We can achieve 2 of them by defining — as (0) allows us to do — z for suitably chosen w by

(8) z: $\quad y = w \bullet z \quad$;

then (α) can be rewritten as

(β) $\quad x \bullet w \bullet z \quad$.

The remaining task consist of 2 things, viz. removal of x and removal of w. Remembering (i) that, as yet, w is undefined, and (ii) that (β) provides the environment in which to exploit that $\bullet$ is associative, we can effectuate both removals by defining — as (0) allows us to do — w for suitably chosen u by

(9) w: $\quad u = x \bullet w \quad$;

then (β) can be rewritten as

(γ) $\quad u \bullet z \quad$.

The remaining task is to chose u in such a way that we can remove it. For <u>given</u> z, (0) would allow us to define u by

$$z = z \bullet u$$

and — thanks to the symmetry of $\bullet$ — such a u would do the job, but via (8) and (9), our z is functionally dependent on u. Having to define u independently of z, we generalize our last relation and define —as (0) allows us to do— u for suitably chosen c by

(10) u: $\quad c = c \bullet u$,

and hope that we can find a c that does the job.

Appealing to (10) requires in our rewriting exercise the temporary introduction of c into the intermediate result. Also, for the very last step of the transformation, we need an expression that can be equated to z. These two considerations suggest a next —and last— appeal to (0): we define v by

(11) v: $\quad z = c \bullet v$.

Now the transformation continues:

(γ)
$$\begin{array}{ll} & u \bullet z \\ = & \quad \{(11)\} \\ & u \bullet c \bullet v \\ = & \quad \{(10)\} \\ & c \bullet v \\ = & \quad \{(11)\} \\ & z \end{array}$$

and we are done. For c, any element will do.

In summary: choose a c , define <u>in order</u> u by (10), w by (9), z by (8), v by (11) and observe

$$
\begin{array}{ll}
 & z \\
= \{(11)\} & c \cdot v \\
= \{(10)\} & u \cdot c \cdot v \\
= \{(11)\} & u \cdot z \\
= \{(9)\} & x \cdot w \cdot z \\
= \{(8)\} & x \cdot y
\end{array}
$$

Tarski's presentation of this proof is obscured by the fact that without any comment he has made the arbitrary choice $c = y$.

<u>Ref.</u> Tarski, Alfred, "Introduction to Logic and to the Methodology of Deductive Sciences", Dover Publications Inc., New York, 1995

Austin, 11 June 1995

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