Courtesy Rajeev Joshi

Denoting for any relation universal quantification over its two arguments by enclosing it within a pair of square brackets, we can formulate the

Theorem For relations Q,R of the same type we have

$$\langle\forall a,b,A,B: aQb \Rightarrow AQB: aRb \Rightarrow ARB\rangle \equiv$$
$$[\neg R] \vee [Q \equiv R] \vee [R]$$

Proof We observe for any Q,R

$$\begin{array}{ll}
& \langle\forall a,b,A,B: aQb \Rightarrow AQB: aRb \Rightarrow ARB\rangle \\
\equiv & \{[x \Rightarrow y \equiv \neg x \vee y]\text{; range split; trading}\} \\
& \langle\forall a,b,A,B:: aQb \vee \neg aRb \vee ARB\rangle \;\wedge \\
& \langle\forall a,b,A,B:: \neg AQB \vee \neg aRb \vee ARB\rangle \\
\equiv & \{\text{nesting; } \vee \text{ distributes over } \forall\text{; } [\,] \text{ convention:} \\
& \quad \langle\forall a,b:: aQb\rangle \equiv [Q]\rangle\text{, etc.}\} \\
& ([Q \vee \neg R] \vee [R]) \wedge ([\neg R] \vee [\neg Q \vee R]) \\
\equiv & \{\wedge \text{ distributes over } \vee\} \\
& ([Q \vee \neg R] \wedge [\neg R]) \vee ([Q \vee \neg R] \wedge [\neg Q \vee R]) \;\vee \\
& ([R] \wedge [\neg R]) \vee ([R] \wedge [\neg Q \vee R]) \\
\equiv & \{\text{absorption, mutual implication, pred. calc.}\} \\
& [\neg R] \vee [Q \equiv R] \vee [R]
\end{array}$$

(End of Proof.)

* * *

The above proof (or something very close to it) emerged when Rajeev Joshi decided to design a calculational proof for the theorem of EWD1245, which the above yields with $Q := J$. The first calculation did not use Q or J, but expressed the range as "$a = b \Rightarrow A = B$"; it was only after "$=$" had been replaced by "J" that it became patently obvious that <u>none</u> of J's properties had been used and that, therefore, J could be generalized to any relation. It was a modest example of how notation can stimulate abstraction.

Reference

EWD1245 A kind of converse of Leibniz's Principle (17 September 1996).

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