SDS 321 Homework 2

? 1/2 = 50%

1. A family has two children (not twins). Assume that each child is equally likely to be left-handed or right-handed.

   Define the following events:

   S: family has at least one Right-handed child named Sam

   R: child is right-handed

   L: child is left-handed

   a) What is the sample space for the family's two children with respect to handedness? $\Omega = \{(R,R), (R,L), (L,R), (L,L)\}$

   b) What is the probability that both children are right-handed? Write the probability statement, and calculate your answer. $P(R_1 \cap R_2) = \boxed{1/4} = \frac{1}{2} \cdot \frac{1}{2} = P(R)P(R)$ 0.25

   c) If we know that at least one of the children is right-handed, what is the probability that both are right-handed? Write the probability statement, and calculate your answer.

   $P(\text{at least one R}) = P(R,R) + P(R,L) + P(L,R) = 1/4 + 1/4 + 1/4 = 3/4 = 0.75$

   $P(R_1 \cap R_2 \mid \text{least one R}) = \frac{P(R \cap R)}{P(\text{least 1 R})} = \frac{0.25}{0.75} = \boxed{0.33 \text{ or } 1/3}$ Answer for 1c.

   d) If we know that the older child is right-handed, what is the probability that both are right-handed? Write the probability statement, and calculate your answer. $P(R_1 \cap R_2 \mid \text{older child is R}) = \frac{P(R,R)}{P(\text{older child R})} = \frac{0.25}{0.25 + 0.25} = \frac{0.25}{0.5} = \boxed{0.5}$

   e) When asked if he has at least one right-handed child named Sam, the father replies, "Yes." What is the probability that both children are right-handed, if we know that the family has at least one right-handed child named Sam? Assume the following:

   - If a child is right-handed, their name will be Sam with probability α.
   - It's possible for both children to be named Sam.
   - If the child is left-handed, their name cannot be Sam.

   Write the probability statement and show your steps as you calculate your answer. Note that your answer will include α in it.

   Hint: Bayes Theorem might be useful.

   Subcases for $P(S)$

   $P(\text{Both named Sam}) = \frac{a^2}{4}$

   $P(\text{One named Sam, one not}) = \frac{a(1-a)}{2}$

   $P(1^{st} \text{ named Sam, } 2^{nd} \text{ Left handed}) = \frac{1}{2} \cdot a \cdot \frac{1}{2}$

   $P(2^{nd} \text{ named Sam, } 1^{st} \text{ Left handed}) = \frac{1}{2} \cdot \frac{1}{2} a$ Logically equiv.

   $P(\text{Right and Sam, other left}) = 2\left(\frac{1}{2} \cdot a \cdot \frac{1}{2}\right) = \frac{a}{2}$

   $P(\text{First child Sam, not Sam}) = \frac{1}{2} a \cdot \frac{1}{2}(1-a)$

   $P(\text{Second child Sam, not Sam}) = \frac{1}{2}(1-a) \cdot \frac{1}{2} a$ Logically equiv.

   $P(R \cap R) = 1/4$

   $P(S) = ?$

   Bayes' Theorem

   $P(R_1 \cap R_2 \mid S) = \frac{P(R_1 \cap R_2 \cap S)}{P(S)} = \frac{P(R \cap R)P(S)}{P(S)}$

   $P(\text{Both right \& Sam}) = \left(\frac{1}{2}\alpha\right)\left(\frac{1}{2}\alpha\right) = \frac{1}{4}(\alpha^2)$

   $P(\text{Either one named Sam, not named}) = 2\left(\frac{1}{2} a \cdot \frac{1}{2}(1-a)\right) = \frac{a(1-a)}{2}$

   Subcases for $P(R \cap R \cap S)$

   Application of Bayes'

   $$\frac{\frac{a^2}{4} + \frac{a(1-a)}{2}}{\frac{a^2}{4} + \frac{a(1-a)}{2} + \frac{a}{2}}$$

   Subcases for $P(R \cap R)P(S)$

   Subcases for $P(S)$

   Simplify! (Go to next page)

   $$P(R \cap R \mid S) = \boxed{\frac{a-2}{a-4}}$$ Reference the next page for the work.

2. A, B and C are events such that P(A) = 0.1, P(B) = 0.3, and P(C) = 0.2. Find P(AUBUC) under each of the following assumptions:

   a) If A, B, and C are mutually exclusive. $P(A \cup B \cup C) = P(A) + P(B) + P(C) = 0.1 + 0.3 + 0.2 = \boxed{0.6} = P(A \cup B \cup C)$

   b) If A, B, and C are independent. $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$
   $= 0.1 + 0.3 + 0.2 - (0.1 \cdot 0.3) - (0.1 \cdot 0.2) - (0.3 \cdot 0.2) + (0.1 \cdot 0.3 \cdot 0.2)$
   $P(A \cup B \cup C) = \boxed{0.496}$

   Joint

   $P(A_1 \cap A_2) = 1/4$

   $P(A_1 \cap A_3) = 1/4$

   $P(A_2 \cap A_3) = 1/4$

   A = 1 2 3

   110

   101

   011

   000

   $\frac{2}{4} = P(A_1)$

   $\frac{2}{4} = P(A_2)$

   $\frac{2}{4} = P(A_3)$

3. Consider an urn containing four balls, numbered 110, 101, 011 and 000, from which one ball is drawn at random. For k = 1, 2, 3 let A_k be the event of drawing a ball with 1 in the k^{th} position. Show your work in answering the following:

   a) Are A_1, A_2, and A_3 pairwise independent? Yes.

   b) Are A_1, A_2, and A_3 mutually independent? No.

   3 conditions

   $P(A_1 \cap A_2) = P(A_1)P(A_2) \rightarrow \frac{1}{4} = \left(\frac{2}{4}\right)\left(\frac{2}{4}\right) = \frac{4}{16} = \frac{1}{4}$ ✓

   $P(A_1 \cap A_3) = P(A_1)P(A_3) \rightarrow \frac{1}{4} = \left(\frac{2}{4}\right)\left(\frac{2}{4}\right) = 4/16 = \frac{1}{4}$ ✓

   $P(A_2 \cap A_3) = P(A_2)P(A_3) \rightarrow \frac{1}{4} = \left(\frac{2}{4}\right)\left(\frac{2}{4}\right) = 4/16 = 1/4$ ✓

   These condition are satisfied for the 3 events to be pairwise independent.

   x Condition 4: $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

   These events are not mutually independent per condition 4. $0 = \left(\frac{2}{4}\right)\left(\frac{2}{4}\right)\left(\frac{2}{4}\right)$

   $\therefore 0 \neq \frac{8}{64} = 0.125$

for Question 1e continues!

$$\frac{\frac{a^2}{4}+\frac{a(1-a)}{2}}{\frac{a^2}{4}+\frac{a(1-a)}{2}+\frac{a}{2}} \xrightarrow{\text{SIMP.}} \frac{\frac{a^2}{4}+\frac{2a(1-a)}{4}}{\frac{a^2}{4}+\frac{a(1-a)}{2}+\frac{a}{2}} = \frac{\frac{a^2+2a(1-a)}{4}}{\frac{a^2}{4}+\frac{a(1-a)}{2}+\frac{a}{2}} = \frac{\frac{a^2+2a-2a^2}{4}}{\frac{a^2}{4}+\frac{a(1-a)}{2}+\frac{a}{2}} = \frac{\frac{2a-a^2}{4}}{\left(\frac{a^2}{4}+\frac{a(1-a)+a}{2}\right)\cdot\frac{2}{2}} = \frac{\frac{2a-a^2}{4}}{\underbrace{\frac{a^2}{4}+\frac{2a(1-a)+2a}{2}}_{\frac{a^2+2a-2a^2+2a}{4}=\frac{4a-a^2}{4}}} \quad \frac{\frac{2a-a^2}{4}}{\frac{4a-a^2}{4}}$$

Nextline

$$\frac{\frac{2a-a^2}{4}}{\frac{4a-a^2}{4}} = \frac{2a-a^2}{\cancel{4}}\cdot\frac{\cancel{4}}{4a-a^2} = \frac{-a^2+2a}{-a^2+4a} = \frac{\cancel{a}(-a+2)}{\cancel{a}(-a+4)} = \frac{(-a+2)}{(-a+4)} \xrightarrow{\text{Fac} \cdot -1} \frac{\cancel{-1}^{\,1}(a-2)}{\cancel{-1}(a-4)}$$

$$\to P(R\cap R|S) = \frac{(a-2)}{(a-4)}$$

$P(H) = 0.03$ | $P(T|H) = 0.93$

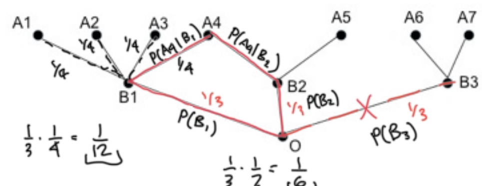
$P(H^c) = 0.97$ | $P(T^c|H) = 0.07 = 1 - 0.93$

$P(T|H^c) = 0.10 = 1 - 0.90$

$P(T^c|H^c) = 0.90$

4. In a large population, 3% of the people are heroin users. A new drug test correctly identifies users 93% of the time and correctly identifies nonusers 90% of the time. Answer the following, showing all work for full credit. Write the probability statement for parts (b) through (e) and show your steps as you calculate your answer.
   a) Define events. Draw the probability tree for this scenario. Label the outcomes and indicate all the probabilities on the tree.
   b) What is the probability that a person who does not use heroin in this population tests positive? $P(T|H^c) = \boxed{0.10} = 1 - 0.90$
   c) What is the probability that a randomly chosen person from this population is a heroin user and tests positive? $P(H \cap T) = P(H) \cdot P(T|H) = 0.03 \cdot 0.93 = \boxed{0.0279}$
   d) What is the probability that a randomly chosen person from this population tests positive? $P(T) = P(T|H)P(H) + P(T|H^c)P(H^c) = (0.93 \cdot 0.03) + (0.10 \cdot 0.97) = \boxed{0.1249}$
   e) If a person tests positive for heroin, what is the probability that he/she is a heroin user? $P(H|T) = \frac{P(H \cap T)}{P(T)} = \frac{0.0279}{0.1249} = \boxed{0.2233}$

5. A man starts at the point O on the map below. He chooses a path at random and follows it to point B1, B2, or B3. From that point, he chooses a new path at random and follows it to one of the points A1 – A7. Suppose the man arrives at point A4, but it is not known which route he took, what is the probability that he passed through each of the three points: B1, B2, B3? Calculate each one.



$P(B_1) = 1/3$ | $P(A_4|B_1) = \frac{1}{4}$

$P(B_2) = 1/3$ | $P(A_4|B_2) = 1/2$

$P(B_3) = 1/3$ | $P(A_4|B_3) = 0 \rightarrow$ Not possible

$P(A_4) = \underbrace{\cancel{P(B_1 \cap A_4)}}_{1/12} + \underbrace{\cancel{P(B_2 \cap A_4)}}_{1/6} + \underbrace{\cancel{P(B_3 \cap A_4)}}_{0}$

$= \frac{1}{12} + \frac{2}{12}$

$= \frac{3}{12} = \boxed{1/4}$

$P(B_1 \cap A_4) = P(B_1) \cdot P(A_4|B_1) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

$P(B_2 \cap A_4) = P(B_2) \cdot P(A_4|B_2) = 1/3 \cdot \frac{1}{2} = 1/6$

$P(B_3 \cap A_4) = P(B_3) \cdot P(A_4|B_3) = 1/3 \cdot 0 = 0$

$P(B_1|A_4) = \frac{P(B_1 \cap A_4)}{P(A_4)} = \frac{\frac{1}{12}}{\frac{1}{4}} = \frac{1}{12} \cdot \frac{4}{1} = \boxed{\frac{1}{3}}$, he took B_1

$P(B_2|A_4) = \frac{P(B_2 \cap A_4)}{P(A_4)} = \frac{1/6}{1/4} = \frac{1}{6} \cdot 4 = \boxed{\frac{2}{3}}$, he took B_2

$P(B_3|A_4) = \frac{P(B_3 \cap A_4)}{P(A_4)} = \frac{0}{1/4} = \boxed{0}$, he took B_3

Probability tree for Q4a.