SDS 321

Lecture 4

Bayes Rule

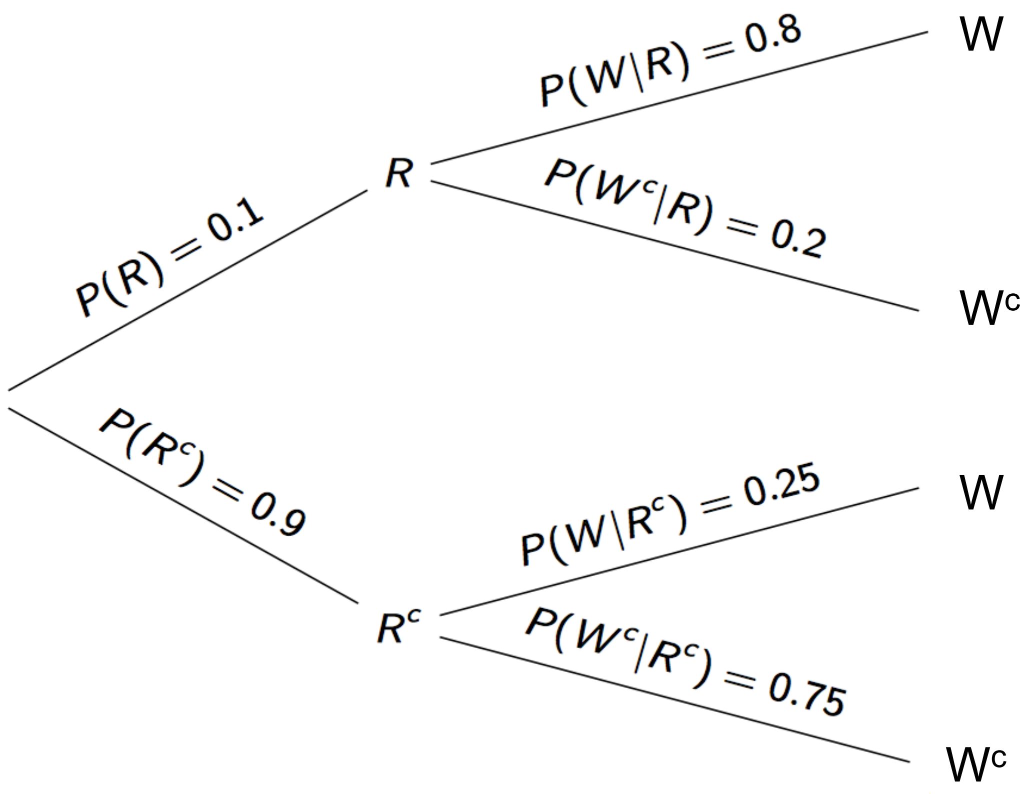
Bayes Rule

Figure: Thomas Bayes, 1701-1761. English statistician, philosopher and Presbyterian minister

Bayes' unpublished manuscript was significantly edited by Richard Price, before it was posthumously read at the Royal Society.

Revisiting the Anita/Run Example

Recall the probability tree for the example with Anita going for a run.



$P(R \cap W) = 0.1 \times 0.8$

$P(R \cap W^c) = 0.1 \times 0.2$

$P(R^c \cap W) = 0.9 \times 0.25$

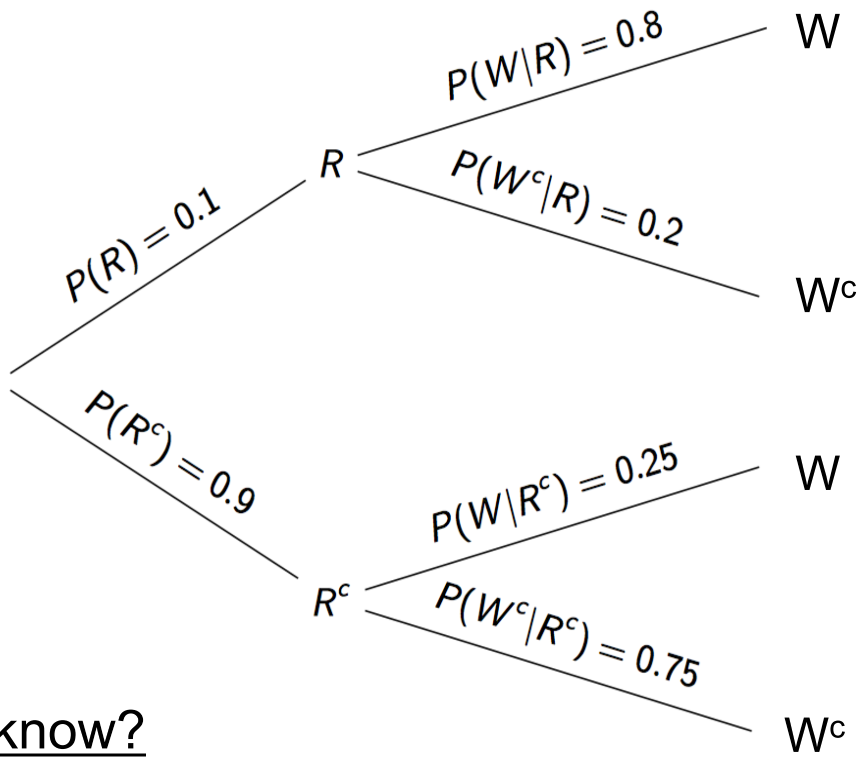
$P(R^c \cap W^c) = 0.9 \times 0.75$

Question:

Suppose you're Anita's friend and you live in a different city. If you know that Anita went for a run, then what is the probability that it rained there?

Revisiting the Anita/Run Example

Question: If you know that Anita went for a run, then what is the probability that it rained? i.e. P(Rain|Run) = P(R|W)



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What do we know?

P(Run|Rain) = P(W|R) = 0.8

P(Run and Rain) = $P(R \cap W) = 0.1 \times 0.8$

Revisiting the Anita/Run Example

<u>Question</u>:

If you know that Anita went for a run, then what is the probability that it rained?

Recall events: R: Rain, W: Went for a run

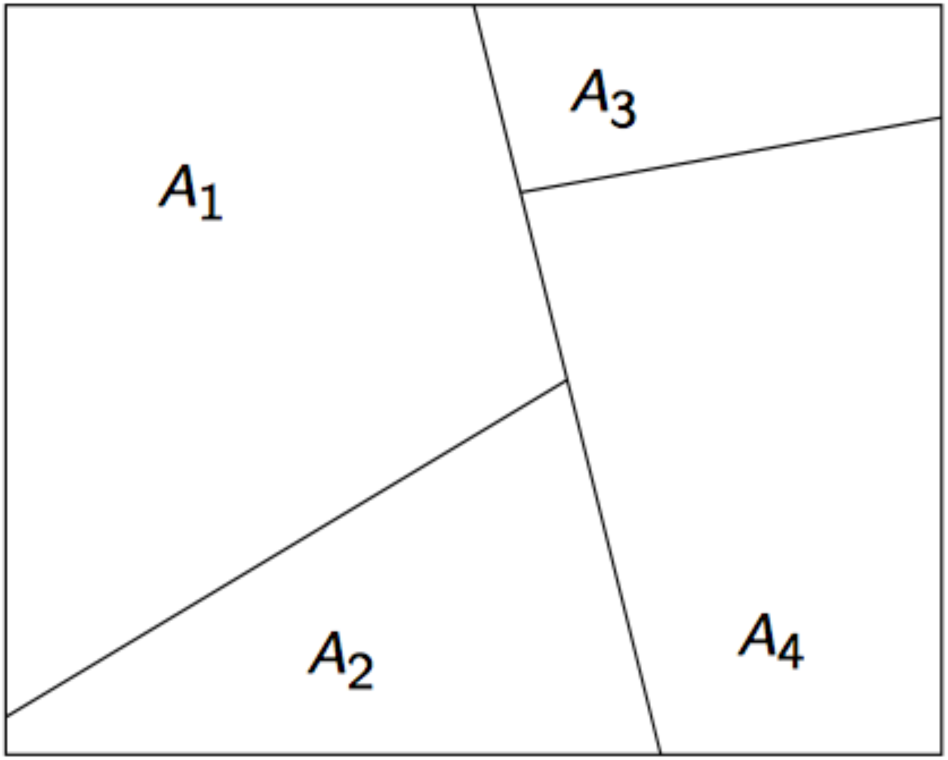
- We want to know $P(R|W)$
- The multiplication rule for two events: $P(R \cap W) = P(R|W)P(W)$
- So $P(R|W) = \dfrac{P(R \cap W)}{P(W)}$
- Numerator: From the tree, and the multiplication rule:

$$P(R \cap W) = P(W|R)P(R) = 0.8 \times 0.1 = 0.08$$

- Denominator: How do we find $P(W)$? Use Total Probability Theorem

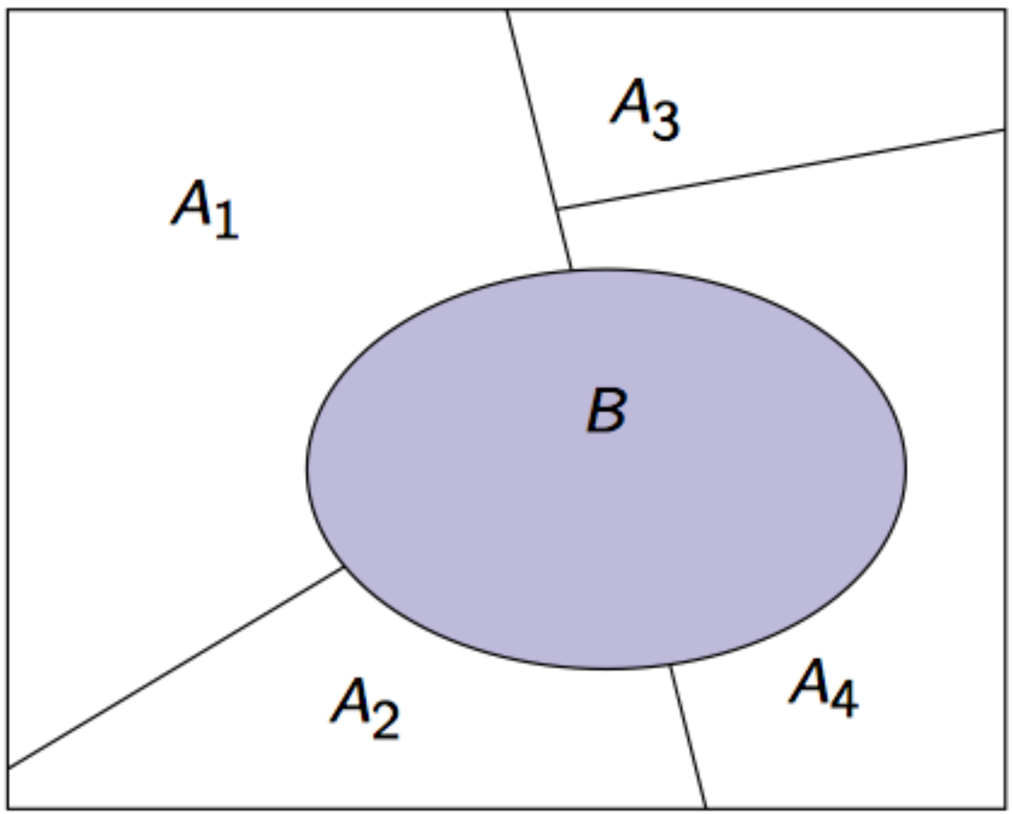
Total Probability Theorem

Let $A_1,\ldots, A_n$ be a partition of Ω, such that $P(A_i) > 0$ for all A_i



Total Probability Theorem

- Let $A_1, \ldots, A_n$ be a partition of Ω, such that $P(A_i) > 0$ for all A_i
- Let B be an event.



Total Probability Theorem

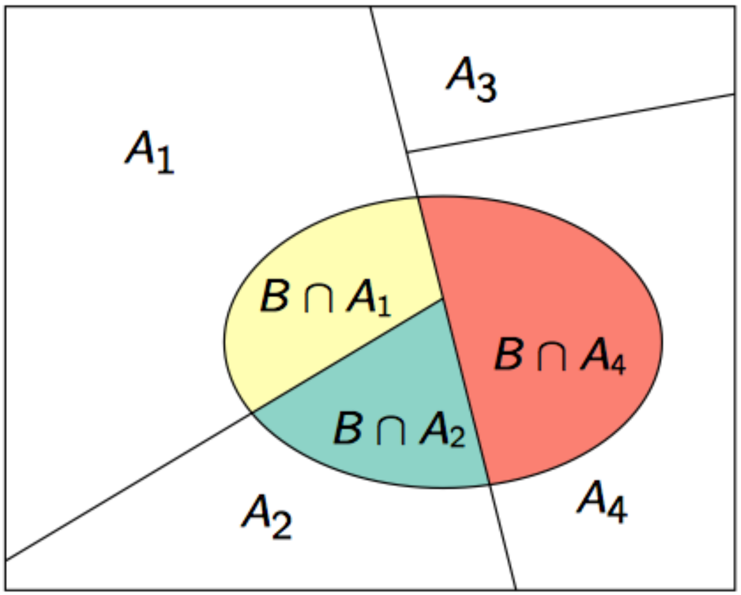
Note that $B = \cup_i (A_i \cap B)$.

Therefore, $P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B) + P(A_4 \cap B)$.

By the multiplication rule, $P(A_i \cap B) = P(A_i)P(B|A_i)$.

So, $P(B) = P(A_1)P(B|A_1) + \cdot\cdot\cdot + P(A_n)P(B|A_n)$.

This is known as the
Total Probability Theorem



Back to the Anita/Run Example

- Finding P(W):
  - First write Ω as an union of two disjoint events (R and R^c).
  - $W = W \cap \Omega = W \cap (R \cup R^c) = (W \cap R) \cup (W \cap R^c)$
  - Now additivity gives

$$P(W) = P(W \cap R) + P(W \cap R^c)$$

$$= P(W|R)P(R) + P(W|R^c)P(R^c)$$

$$= 0.08 + 0.25 \times 0.9 \approx 0.3$$

Revisiting the Anita/Run Example

$$P(R|W) = \frac{P(R \cap W)}{P(W)}$$

$$= \frac{P(W|R)P(R)}{P(W|R)P(R) + P(W|R^c)P(R^c)} = 0.08/0.3 \approx 0.27$$

This last step is known as Bayes Rule.

Bayes Rule

- Simple rule to get conditional probability of A given B, from the conditional formula of B given A.

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

Identifying when to use Bayes Rule

- You have a question asking for P(A|B)
- First check what information is given. Read each sentence and write down the probabilities that you have: Are they conditional? If so, what is the "given" part?
- If you have P(A|B) or $P(A^c|B)$ then you are all set.
- If not, then you have to use Bayes rule.
- To use Bayes Rule, you will need to know P(B|A), $P(B|A^c)$ and P(A)
- Identify and insert these probabilities into the equation for Bayes Rule, and calculate P(A|B)

Bayes Rule: Typical Scenario

- Consider a test for some disease.
- We can directly observe the outcome of the test.
- If the test isn't 100% accurate, we can't directly infer whether we have the disease using the test result.
- We have two possible causes for a positive test result:
  - We have the disease, and the test is correct.
  - We don't have the disease, and the test is a false positive.
- We want to infer which hidden cause underlies our observation.

Bayes Rule: Example 1

Let's add some numbers to the disease testing example:

- The disease affects 2% of the population.
- The false positive rate is 1%.
- The false negative rate is 5%.

If you take the test and the result is positive, we want to know:

Given that you tested positive, what is the chance you have the disease?

Let's define events, and write what we know.

T: Test Positive D: Have disease

$P(D) = 0.02$ $P(T^C|D) = 0.05$ $P(T|D^C) = 0.01$

We want $P(D|T)$

Bayes Rule: Example 1

Given that you tested positive, what is the chance you have the disease?

Bayes Rule gives us:

$$P(D|T) = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)}$$

Bayes Rule helps us go from the conditional probability of an observation given a hidden cause (which we usually know), to the conditional probability of a hidden cause given an observation (which we usually care about!)

Bayes Rule: Example 1

So, let's plug in the numbers. Recall

$$P(D) = 0.02 \qquad P(T^c|D) = 0.05 \qquad P(T|D^c) = 0.01$$

So, $P(T|D) = 1 - 0.05 = 0.95$.

$$\begin{aligned} P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} \\ &= \frac{0.95 \times 0.02}{0.95 \times 0.02 + 0.01 \times 0.98} \\ &= \frac{0.019}{0.0288} = .66 \end{aligned}$$

Bayes Rule: Example 2

Alice is sending a coded message to Bob using “dots” and “dashes,” which are known to occur in the proportion of 3:4 for Morse code.

Because of interference on the transmission line, a dot can be mistakenly received as a dash with probability 1/8 and vice-versa.

If Bob receives a “dot”, what is the probability that Alice sent a “dot”?

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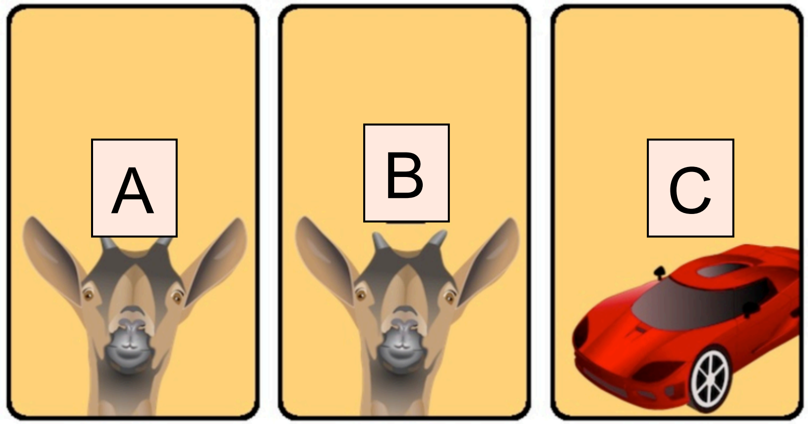
If Bob receives a “dot”, what is the probability that Alice sent a “dot”?

Define Events:

dotS:	dot sent	dotR:	dot received
dashS:	dash sent	dashR:	dash received

The Monty Hall Problem



A game show has three doors. There is a car behind one door and a goat behind each of the other two doors. The host knows behind which door the car is.

You are a contestant on this game show. You pick a door, say A.

To build suspense, the host opens one of the other two doors (say B) revealing a goat. He asks, do you want to stick with your initial choice of door or switch?

What do you do? Does it make a difference?