SDS 321

Lecture 5

Statistical Independence

Statistical Independence

- Let's look at the experiment where two fair coins are tossed together.
- Suppose

  H_1: event that the dime lands a Head $P(H_1)=½$

  H_2: event that nickel lands a Head $P(H_2)=½$

- What is $P(H_1|H_2)$? Read as probability of H_1 "given" H_2

- $$P(H_1|H_2) = \frac{P(H_1 \cap H_2)}{P(H_2)} = \frac{¼}{½} = ½ = P(H_1)$$

- Knowing outcome of H_2 doesn't give me additional info on H_1
- So $P(H_1|H_2) = P(H_1)$, and this means H_1 and H_2 are independent.

- In general,

  If $P(A|B) = P(A)$, we say the events A and B are independent.

Statistical Independence

- The general multiplication rule gives us:
- $P(A \cap B) = P(A|B) \times P(B)$
- If A and B are independent, $P(A \cap B) = P(A) \times P(B)$
- This definition is preferred because it is still defined when $P(B) = 0$
- Independence is a symmetric property, so that if A is independent of B, then B is independent of A.

Therefore:

> If two events A and B are independent,
>
> $$P(A \cap B) = P(A)P(B)$$
>
> Also, $P(A|B) = P(A)$ and $P(B|A) = P(B)$

Statistical Independence

Example:

A gambler is rolls four fair dice. What is the probability that there is at least one 6 in the four rolls?

- Each roll is independent. Let X_i denote the event that there is no six in the i^{th} roll.
- P(at least 1 six in 4 rolls) = 1 – P(no sixes in 4 rolls)

$$= 1 - P(X_1 \cap X_2 \cap X_3 \cap X_4)$$
$$= 1 - P(X_1)P(X_2)P(X_3)P(X_4)$$
$$= 1 - \left(\frac{5}{6}\right)^4 = 0.518$$

Statistical Independence

Theorem. If A and B are independent ($A \perp\!\!\!\perp B$), then so are

- A and B^c
- A^c and B
- A^c and B^c

Statistical Independence

Theorem. If A and B are independent ($A \perp\!\!\!\perp B$), then so are

- A and B^c
  - $$\begin{aligned} P(A \cap B^c) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)(1 - P(B)) = P(A)P(B^c) \end{aligned}$$
- A^c and B
- A^c and B^c

Mutual Independence

Three events A_1, A_2, and A_3 are said to be Mutually Independent if :

A_1, A_2, and A_3 are Pairwise Independent:

1. $P(A_1 \cap A_2) = P(A_1)P(A_2)$
2. $P(A_1 \cap A_3) = P(A_1)P(A_3)$
3. $P(A_2 \cap A_3) = P(A_2)P(A_3)$

And

4. $P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)$

Note:

- The fourth condition does not imply the first three, and the first three do not imply the fourth.
- Mutual Independence implies Pairwise Independence.
- Pairwise Independence does not imply Mutual Independence.

Mutual Independence

$P(A \cap B \cap C) = P(A)P(B)P(C)$ does not imply Pairwise Independence

Example:

Toss two fair dice (red and white).

Event A: Red die outcome is odd:
Event B: Red die outcome is ≤ 3
Event C: Sum of the faces is 9

$P(A) = ½$
$P(B) = ½$
$P(C) = 4/36 = 1/9$

$P(A \cap B \cap C) = 1/36 = P(A)P(B)P(C) = (1/2)(1/2)(1/9) = 1/36$

So the 4^{th} condition is true.

Mutual Independence

$P(A \cap B \cap C) = P(A)P(B)P(C)$ does not imply Pairwise Independence

Let's check if these events are pairwise independent:

Event A: Red die outcome is odd:
Event B: Red die outcome is ≤ 3
Event C: Sum of the faces is 9

$P(A \cap B) = 12/36 = 1/3$
$P(A) \times P(B) = 1/4$. So $P(A \cap B) \neq P(A)P(B)$

$P(A \cap C) = 2/36 = 1/18$
$P(A) \times P(C) = (1/2)(1/9) = 1/18$. So $P(A \cap C) = P(A)P(C)$

$P(B \cap C) = 1/36$
$P(B) \times P(C) = (1/2)(1/9) = 1/18$. So $P(B \cap C) \neq P(B)P(C)$

So A, B, and C are not pairwise independent.

Mutual Independence

Pairwise Independence does not imply $P(A \cap B \cap C) = P(A)P(B)P(C)$

<u>Example</u>:
Toss two fair dice (red and white).

Event A: Sum of the faces is 7
Event B: Red die outcome is 3
Event C: White die outcome is 4

$P(A) = 6/36 = 1/6$
$P(B) = 6/36 = 1/6$
$P(C) = 6/36 = 1/6$

$P(A \cap B) = 1/36 = P(A)P(B)$
$P(A \cap C) = 1/36 = P(A)P(C)$
$P(B \cap C) = 1/36 = P(B)P(C)$

So A, B, and C are pairwise independent.

$P(A \cap B \cap C) = 1/36$

$P(A)P(B)P(C) = 1/216$

Not equal!

So the 4^{th} condition is not met.

Mutual Independence

- **Definition.** Events $A_1, \ldots A_n$ are mutually independent if for any subset S of $\{1, \ldots, n\}$ we have $P(\cap_{i \in S} A_i) = \prod_{i \in S} P(A_i)$.
- Also, any combination of a set of events and the complements of each the remaining events are mutually independent too. i.e. if A, B, C are mutually independent, then so are A^c, B^c, C and A, B^c, C or A^c, B, C^c etc.

- Mutual Independence implies Pairwise Independence.
- Pairwise Independence does not imply Mutual Independence.

Conditional Independence

Definition : Two events A and B are conditionally independent given another event C if $P(A \cap B|C) = P(A|C)P(B|C)$ → equation (1)

An alternative way to state conditional independence is:
$P(A \mid B \cap C) = P(A|C)$

We can derive this second equation from the first one:

$$
\begin{aligned}
P(A \cap B|C) &= P(A \cap B \cap C) / P(C) \\
&= P(A|B \cap C)\, P(B \cap C) / P(C) \\
&= P(A|B \cap C)\, P(B|C)\, P(C) / P(C)
\end{aligned}
$$

So $P(A \cap B|C) = P(A|B \cap C)\, P(B|C)$ → equation (2)

From equations (1) and (2), $P(A|B \cap C) = P(A|C)$, assuming $P(B|C) \neq 0$

Conditional Independence

The meaning of conditional independence is basically, that if
A and B are conditionally independent given C,

If we know that C occurred, then additionally knowing the outcome of B doesn't alter the probability of A.

Toothache example:

A patient goes to the dentist with a toothache. There could be a cavity. The dentist checks the tooth with a probe, and the probe will catch if there is a cavity. Let the events be:

Catch: Probe catches, Cavity: Cavity present, Toothache: Patient has toothache

Catch is conditionally independent of toothache, given cavity.
P(catch | toothache and cavity) = P(catch | cavity)
P(catch | toothache and no cavity) = P(catch | no cavity)

Each of toothache and catch are caused by cavity, but they don't have an effect on each other.

Conditional Independence

Example 1: Experiment involves two independent tosses of a fair coin.

Let H1: 1st toss is a head
H2: second toss is a head
D: the two tosses have different results

Are H1 and H2 independent?
Are H1 and H2 conditionally independent given D?

Conditional Independence

Example 2:

- Consider two urns, each containing 100 balls. The first urn contains all red balls. The second urn contains all blue balls.
- We select an urn at random.
- We select a ball from the urn, note it's color, and put it back. We then select another ball from the urn, note it's color, and put it back.
- Define events:

  A : event that the first urn was chosen

  R_1: event that the first ball was red

  R_2: event that the second ball was red
- Are R_1 and R_2 independent?
- What if we condition on the chosen urn?

Mutual vs. Conditional Independence

In summary, from the previous two examples:

- If A and B are independent, it does not imply that A and B are conditionally in dependent given C.

- If A and B are conditionally independent given C, then A and B don't need to be independent themselves.