SDS 321

Lecture 6

Counting

Calculating probabilities

- Counting is important because we use it to calculate probabilities.
- Suppose we define an event A within a sample space Ω with a finite number of equally likely outcomes, then

$$P(A) = \frac{\#(A)}{\#(\Omega)}$$

- This requires that we accurately count the elements in A and Ω.
- We can also calculate the probability of A if we know that A has a finite number of equally likely outcomes, each with probability *p*, as

$$P(A) = p \text{ x } (\text{number of elements in A})$$

- This again requires that we count the number of elements in A accurately.

The Counting Principle

- One approach is to try and break down the experiment into stages, and looking at the number of outcomes possible at each stage.
- Suppose we have an experiment that can be broken down into k stages, with n_i possible results at the i-th stage.
- Then the total number of possible outcomes of the whole experiment is $n_1 \times n_2 \times \ldots \times n_k$
- Example: Suppose a community consists of 10 women, each of whom has 3 children. If one woman and one of her children are to be chosen as mother and child of the year, how many different choices are possible?
- Answer: We have 2 stages: 10 ways of choosing the mother, and once a mother is chosen, 3 ways of choosing her child. So 10 x 3 = 30 choices.

Some Counting Examples

1. A college planning committee consists of 2 freshmen, 4 sophomores, 5 juniors, and 2 seniors. A subcommittee of 4, consisting of 1 person from each class is to be chosen. How many different subcommittees are possible?

2. How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

3. In the above example, how many license plates are possible if repetition among letters or numbers were not allowed?

Sampling with replacement

There are total n different elements in a population or set. You want to create an (ordered) arrangement of k elements.

- You pick an element, note what it is, and put it back. (sampling with replacement)
- An element can be repeated in your arrangement.
- The first choice can be one of n elements.
- The second choice can again be one of n elements.
- And so on, until the k-th choice can also be one of the n elements.
- Total number of arrangements is ?
- n.n.n….n (k times) = n^k

Sampling with Replacement: Balls and Bins

- k balls and n bins:

- Each ball can be placed in any of the n bins independently.
- So k balls can be placed in n bins in how many ways?

- First ball can be placed in one of the n bins.
- Second ball can be placed in one of the n bins.

One bin can have multiple balls.

- Total $n \times n \times \cdots \times n$ (repeated k times).
- So n^k different ways.

- This is the same as choosing from n bins with replacement, k times.

Sampling without replacement

There are total n different elements in a population or set. You want to create an ordered arrangement of all n elements.

- You pick an element, note what it is, and do not put it back. (sampling without replacement)
- Any element appears at most once in your arrangement.
- The first choice can be one of n elements.
- The second choice can be any one of the remaining (n-1) elements.
- Total number of arrangements is ?
- $n.(n-1).(n-2)\ldots.1 = n!$

Ordered and Unordered Arrangements

- If ordering is distinct, i.e., {a1, a2, a3} ≠ {a1, a3, a2}, then order matters. Otherwise, order doesn't matter.

- In general, when we talk about sampling from a set, there are four possibilities:

- Ordered Sampling with Replacement
- Ordered Sampling without Replacement (Permutation)
- Unordered Sampling without Replacement (Combination)
- Unordered Sampling with Replacement

(1) Ordered Sampling/Arrangement with Replacement

- Suppose we have a set with n elements: A = {1, 2, 3, …, n} and we want to draw k samples from this set.
- Order matters.
- Repetition is allowed.
- Example: A = {1, 2, 3} and k = 2.
- There are 9 arrangements = 3x3 = 3^2
- In general, this is n^k

1. (1,1)
2. (1,2)
3. (1,3)
4. (2,1)
5. (2,2)
6. (2,3)
7. (3,1)
8. (3,2)
9. (3,3)

(2) Ordered Sampling/Arrangement without Replacement (Permutation)

- Suppose we have a set with n elements: A = {1, 2, 3, …, n} and we want to draw all n elements (without replacement) and arrange them.

- Order matters.

- Repetition is not allowed.

Example: A = {1, 2, 3}

1. (1, 2, 3)
2. (1, 3, 2)
3. (2, 1, 3)
4. (2, 3, 1)
5. (3, 1, 2)
6. (3, 2, 1)

- There are 6 arrangements: 3 x 2 x 1

- In general, this is n x (n-1) x (n-2) x … x 2 x 1 = n!

- So the number of n-permutations of an n-element set are n!

- This is also written as $(n)_n$, or n permute n, indicating that it is the number of ways of choosing all n elements from a set of n elements.

(2) Ordered Sampling/Arrangement without Replacement (K-Permutation)

- Now suppose we have a set with n elements: A = {1, 2, 3, …, n} and we want to select k elements from this set.

- Order matters.

- Repetition is not allowed.

Example: A = {1, 2, 3} and k = 2.

1. (1,2)
2. (1,3)
3. (2,1)
4. (2,3)
5. (3,1)
6. (3,2)

- There are 6 arrangements
- In general, this is n x (n-1) x (n-2) x … x (n – k +1)

- So the number of k-permutations of an n-element set are:

$$P_k^n = \frac{n!}{(n-k)!}$$

Permutation: Examples

1. Alex has 10 books that he is going to put on his bookshelf. Of these, 4 are math books, 3 are chemistry books, 2 are history books, and 1 is a language book. He wants to arrange them in such a way that all books of the same subject are placed together on the shelf. How many different arrangements are possible?

2. Find out the number of ways:

   a) 3 boys and 3 girls can sit in a row
   b) 3 boys and 3 girls can sit in a row if the boys and girls are each to sit together?
   c) 3 boys and 3 girls can sit in a row if only the boys must sit together?
   d) 3 boys and 3 girls can sit in a row if no two people of the same sex are allowed to sit together?

3. How many different arrangements can be made using two of the letters of the word TEXAS if no letter is to be used more than once (the two-letter strings don't need to mean anything)?

(3) Unordered Sampling/Arrangement without Replacement (Combination)

- Suppose we have a set with n elements: A = {1, 2, 3, …, n} and we want to draw k elements from this set.
- Order does not matter.
- Repetition is not allowed.

1. {1, 2}
2. {1, 3}
3. {2, 3}

Example: A = {1, 2, 3} and k = 2.

- There are 3 arrangements
- In general, this is $n \times (n-1) \times (n-2) \times \ldots \times (n - k + 1) / k!$
- So the number of k-combinations of an n-element set are:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \text{ for } 0 \le k \le n.$$

Combinations: Example

In how many ways can we pick samples of size 2 without replacement from a set of size 4: {1, 2, 3, 4}? Order doesn't matter. So {1,2} and {2,1} are the same.

The number of 2-permutations is 4! / 2!

There are 4!/2! = 12 possible subsets:

(1,2), (2,1), (1,3), (3,1), (1,4), (4,1), (2,3), (3,2), (2,4), (4,2), (3,4), (4,3)

We note that each pair of numbers appears 2! times. Since order doesn't matter, we need to remove the duplicates.

Therefore, the number of possible combinations is $= \frac{4!}{2!2!}$ = 6

Combinations: Summary

- We start with a collection of ordered subsets of size k chosen without replacement from n objects. This collection is of size $(n)_k$
- However, each unordered subset appears k! times in this collection.
- Hence the total number of unordered subsets of size k is given by

$$C(n, k) = \binom{n}{k} = C_k^n = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

These are also known as Binomial Coefficients.

Example:

A committee of 3 is to be formed from a group of 20 people.
How many different committees are possible?

Combinations: Summary

Example:

A committee of 3 is to be formed from a group of 20 people.
How many different committees are possible?

$$\binom{20}{3} = \frac{20.19.18}{3.2.1} = 1140$$

Binomial Coefficients

1. $$\binom{n}{k} = \binom{n}{n-k}$$

Number of ways of choosing k objects out of n is the same as the number of ways of not choosing (n − k) objects out n.

2. $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This can be easily proved mathematically.

Conceptually, imagine dividing the n objects into one specific object “x”, and a set S of the remaining (n-1) objects.

The subset we choose will either contain “x” or not.

In how many ways can you choose k objects from n such that “x” is never included? We can do this by just picking the k objects from S instead. This is C(n-1, k).

In how many ways can you choose k objects from n such that “x” is always included? Choose “x”, then choose the remaining (k-1) objects from the set of (n-1) objects. This is C(n-1, k-1).

Binomial Coefficients

3. $$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

This is the total number of subsets of a set with n elements.
Or the total number of possible binary strings.

This can be proved using induction.

Partitions

Suppose we have an n-element set, which can be partitioned into r disjoint subsets, with the *i*-th subset containing n_i elements.

This is similar to having n objects, of which n_1 are alike, n_2 are alike, … n_r are alike.

The number of permutations of the n-element set to form this configuration is given by:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

These are also called Multinomial Coefficients.

Note that when we only have two subsets, this becomes a Binomial Coefficient!

Partitions & Multinomial Coefficients

- $\binom{n}{k}$:= #ways to divide n elements into two disjoint groups, where the first group has size k and the second size $n-k$.
- $\binom{n}{n_1, n_2, n_3}$:= #ways to divide n elements into 3 disjoint groups of sizes n_1, n_2 and $n_3 = n - n_1 - n_2$ respectively.
- First we choose n_1 out of n. Next from the remaining $n - n_1$ we choose n_2. Rest are assigned to the third group.
- $\binom{n}{n_1, n_2, n_3} = \binom{n}{n_1} \times \binom{n-n_1}{n_2} = \frac{n!}{n_1!n_2!n_3!}$
- Generalizing to r groups of sizes $n_1, \ldots, n_r$ with $n_1 + \cdots + n_r = n$ we have:

$$\binom{n}{n_1, \ldots n_r} := \frac{n!}{n_1! \ldots n_r!}$$

Partitions

Example 1:

How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags, if all flags of the same color are identical?

Answer: $$\frac{9!}{4!3!2!}$$

Partitions

Example 2:

How many different letter arrangements can be formed using the letters PEPPER ?

Answer:

There are 6! permutations of $P_1E_1P_2P_3E_2R$, when the 3 P's and 2 E's are distinguished from each other.

But if we permute the 3 P's and the 2 E's amongst themselves, we still get the same arrangement. i.e., it's not necessary to distinguish between them.

This is 3! x 2! arrangements.

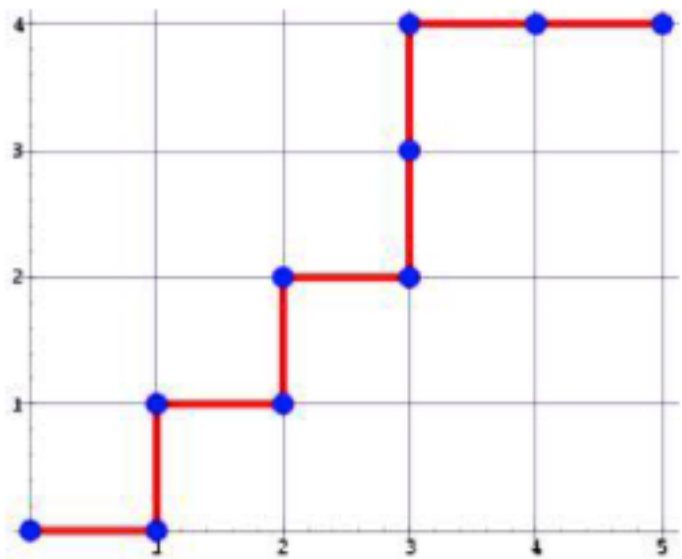
So the answer is 6! / 3!2! = 60 possible arrangements

Example 1

You are walking on a grid. You can either go right or up by one step. You start from (0,0).

(1) How many paths are there to (5, 10)?

(2) How many of the above paths go via (4, 4)?



Part of the grid

Example 2

Question: How many binary sequences of length n are there with k 1s?

Example 3

How many configurations of length 10 strings are there with three 0’s, four 1’s and three 2’s?