SDS 321

Lecture 7

More Counting

Partitions

Suppose we have an n-element set, which can be partitioned into r disjoint subsets, with the *i*-th subset containing n_i elements.

This is similar to having n objects, of which n_1 are alike, n_2 are alike, … n_r are alike.

The number of permutations of the n-element set to form this configuration is given by:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

These are also called Multinomial Coefficients.

Note that when we only have two subsets, this becomes a Binomial Coefficient!

Partitions

Example:

How many different letter arrangements can be formed using the letters PEPPER ?

Answer:

There are 6! permutations of $P_1E_1P_2P_3E_2R$, when the 3 P's and 2 E's are distinguished from each other.

But if we permute the 3 P's and the 2 E's amongst themselves, we still get the same arrangement. i.e., it's not necessary to distinguish between them.

This is 3! x 2! arrangements.

So the answer is 6! / 3!2! = 60 possible arrangements

Partitions are Permutations with Indistinguishable Objects

How many different letter arrangements can be formed using the letters PEPPER ?

In this above example, we basically figured out the number of different permutations of n objects where there are

n_1 indistinguishable objects of type 1,
n_2 indistinguishable objects of type 2, . . . , and
n_k indistinguishable objects of type k:

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

Occupancy Problems: Boxes and Objects

- The techniques we have been seeing are all approaches or strategies that help solve counting problems faster, if we are able to frame them correctly.
- So another approach for solving counting problems is to formulate them as occupancy problems: balls in bins, or objects in boxes, or stars and bars problems.
- Many counting problems can be solved by enumerating the ways objects can be placed into boxes (where the order these objects are placed into the boxes does not matter).
- The objects can be
  - *distinguishable* → different from each other, or
  - *indistinguishable* → identical.
- Similarly, boxes can be
  - *distinguishable* → different, or
  - *indistinguishable* → identical.

We will discuss: Distinguishable Objects and Distinguishable Boxes
Indistinguishable Objects and Distinguishable Boxes

Occupancy Problems

(1) Distinguishable Objects and Distinguishable Boxes

Example:

How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?

- The distinguishable objects are the 52 cards, and the five distinguishable boxes are the hands of the four players and the rest of the deck.
  - We will use the product rule to solve this problem.
  - The first player can be dealt 5 cards in C(52, 5) ways.
  - The second player can be dealt 5 cards in C(47, 5) ways.
  - The third player can be dealt 5 cards in C(42, 5) ways.
  - Finally, the fourth player can be dealt 5 cards in C(37, 5) ways.
- Hence, the total number of ways to deal four players 5 cards each is

$$C(52,5)C(47,5)C(42,5)C(37,5) = \frac{52!}{47!\,5!} \cdot \frac{47!}{42!\,5!} \cdot \frac{42!}{37!\,5!} \cdot \frac{37!}{32!\,5!}$$

$$= \frac{52!}{5!\,5!\,5!\,5!\,32!}.$$

Occupancy Problems

(1) Distinguishable Objects and Distinguishable Boxes (contd.)

Therefore, the number of ways of distributing n distinguishable objects into r distinguishable boxes so that n_i objects are placed into box i, for i = 1,2,...,r, equals

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \ldots n_r!}$$

Note that the solution to this example equals the number of permutations of 52 objects, with 5 indistinguishable objects of each of four different types, and 32 indistinguishable objects of a fifth type!

Occupancy Problems

(2) Indistinguishable Objects and Distinguishable Boxes

Example:

In how many ways can you divide 10 fruits among 4 children so that everyone gets at least one?

- Here, we are not interested in which fruit went to which child. We only care about how many of the fruits went to a child.
- So we think of the fruits as indistinguishable objects and the children as distinguishable bins.
- Suppose x_i is the number of fruits going to child i. Then we are looking for the number of solutions for

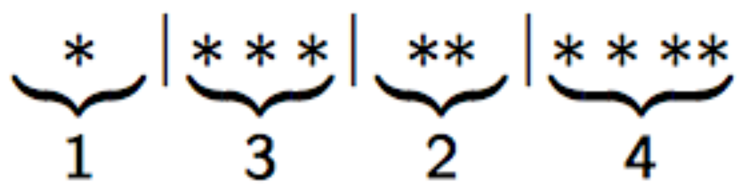
$$x_1 + x_2 + x_3 + x_4 = 10, \quad \text{where } x_i > 0$$

($x_i > 0$ because every child gets at least one fruit).

Occupancy Problems

(2) Indistinguishable Objects and Distinguishable Boxes (contd.)

- Let's look at this a little differently.
- Suppose the 1st child got 1 fruit, the 2nd got 3 fruits, the 3rd got 2 fruits and the 4th got 4 fruits.
- How can we represent this solution: $x_1=1$, $x_2=3$, $x_3=2$, and $x_4=4$?
- Write 10 stars for 10 fruits. Place bars in between the stars to represent bins/children.
- In order to divide 10 stars into 4 groups, we will need 3 bars.



- There are 9 spaces between 10 stars. We want to place 3 bars in these 9 spaces. There are total C(9,3) ways of doing this.

Occupancy Problems

(2) Indistinguishable Objects and Distinguishable Boxes (contd.)

In general,

Suppose we want to distribute n indistinguishable objects into k boxes so that every box gets at least one object,

- If x_i is the number of objects going into box i, we we are looking for the number of solutions to

$$x_1 + x_2 + \cdots + x_k = n, \text{ where } x_i > 0.$$

- If we express this as stars and bars, you want to place (k − 1) bars in (n − 1) spaces between the stars.
- So the answer is $\binom{n-1}{k-1}$

Occupancy Problems

(2) Indistinguishable Objects and Distinguishable Boxes (contd.)

Example:

Now suppose we want to divide 10 fruits among 4 children. A child may or may not get a fruit. In how many ways can we do this?

- Expressing as stars and bars, I have 10 stars and 3 bars.
- But I can have any number of of the 3 bars in each space between two stars.
- So essentially, I am looking at how many arrangements are there, of 10 stars and 3 bars in a row.
- The answer is $\binom{13}{3}$ or $\binom{13}{10}$

Occupancy Problems

(2) Indistinguishable Objects and Distinguishable Boxes (contd.)

In general,

Suppose we want to distribute n indistinguishable objects into k boxes so that every box may or may not get an object.

The number of ways of doing this is $\binom{n+k-1}{k-1}$

Note that this is also the number of ways can we write n as the sum of k non-negative integers.

$$x_1 + x_2 + \cdots + x_k = n, \text{ where } x_i \geq 0.$$

Occupancy Problems

(2) Indistinguishable Objects and Distinguishable Boxes (contd.)

Remember we have not yet addressed:

(# 4) Unordered Sampling/Arrangement with Replacement

Turns out, this case can be formulated as one with indistinguishable objects in distinguishable boxes!

Think about it and ask me if you have questions. We will not cover this case in class.

How are occupancy problems different from combinations?

When do you use a combination as opposed to the linear equation?

Example:
In how many ways can 4 passengers get off a bus in 2 stops (assume passengers are indistinguishable, so we are only counting the number of configurations in which bodies leave the bus)?

Possibilities are:

Stop 1	Stop 2
4	0
3	1
2	2
1	3
0	4

$x_1 + x_2 = 4$ with $x_i \geq 0$

k=2, n=4 gives solution C(4+2-1, 4) = 5

Now if we want to see in how many ways can we have, for example, 3 people getting off at stop 1, and one getting off at stop 2, with the passengers being distinct, that would be a combination C(4,3) or C(4,1) = 4 ways.

Probability and Counting

Example 1:

Our population consists of ten digits {0, 1, . . . , 9}.
I pick a 5 digit number at random. Such a number can start with zero, and may have repetitions. What is the probability that the five digits are all different?

Probability and Counting

Example 2:

A bus with 5 (distinguishable) passengers makes 10 stops. All configurations of discharging the passengers are equally likely. What is the probability p that no two passengers get down at the same stop?

Probability and Counting

Example 3:

The birthdays of $r \leq 365$ people form a sample of size r from the population of all birthdays (365 days in the year). We assume that a person is equally likely to be born on any of the 365 days and no one was born on Feb 29th. What is the probability that no two people will have the same birthday?

What is the probability if $r = 366$?