Lecture 11

CS 384R, CES 392

Geometric (Bio-) Modeling and Visualization

http://www.cs.utexas.edu/~bajaj/cs384R2012/

Cross-Sectional Contour Reconstruction (Surface, Volume) and Mesh Improvement

Computational Visualization Center (CVC) http://cvcweb.ices.utexas.edu
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Surface Tiling Steps (revisited)

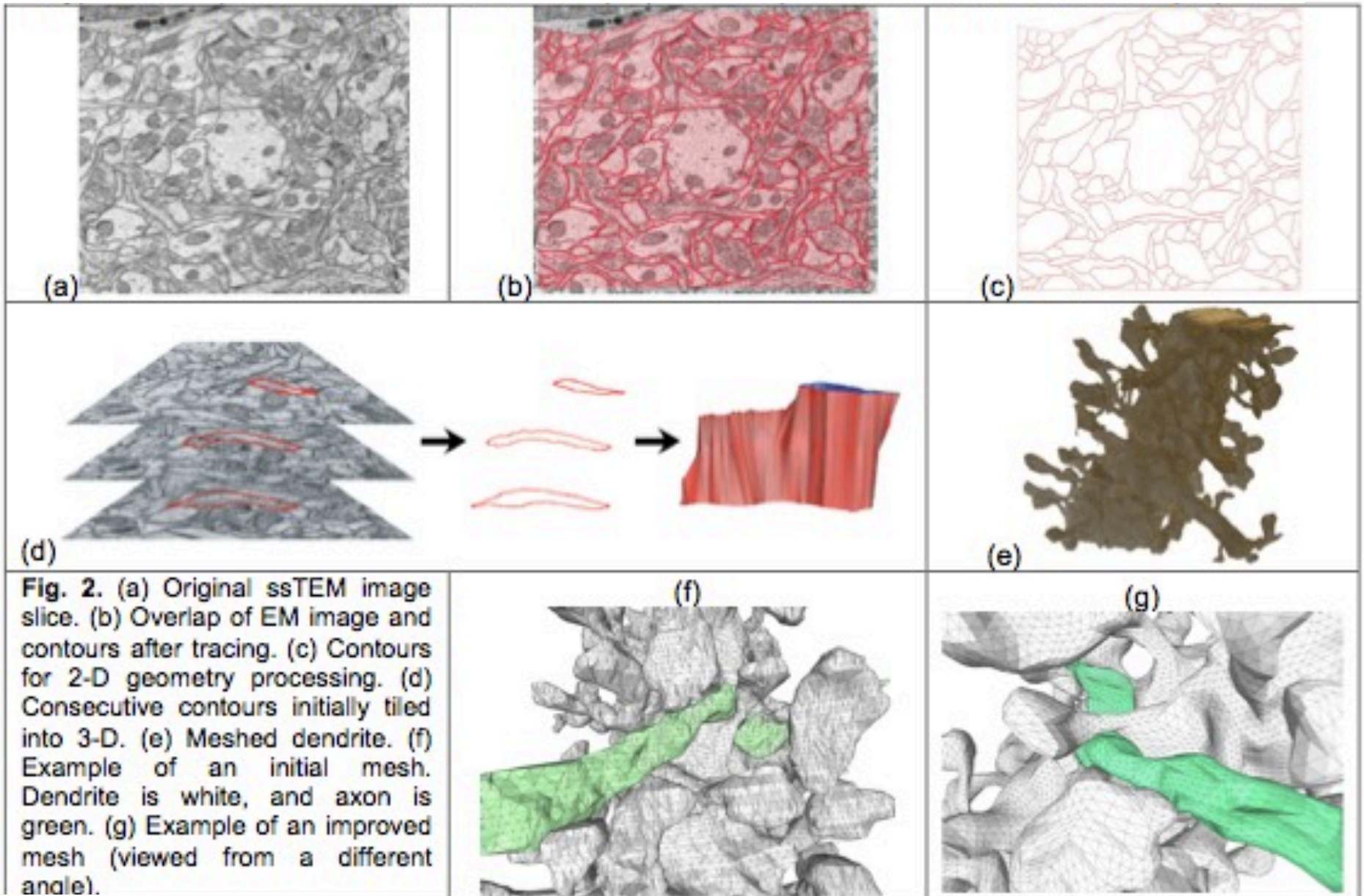
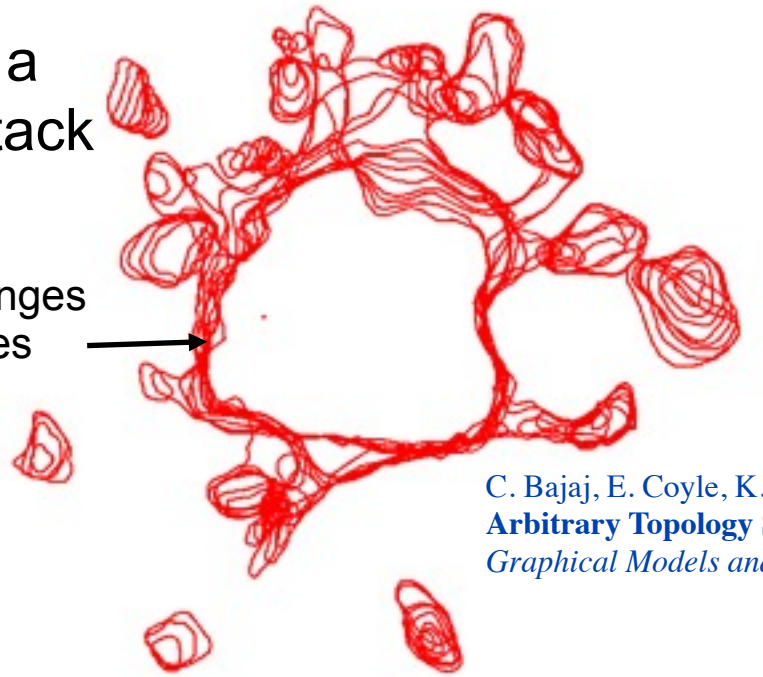
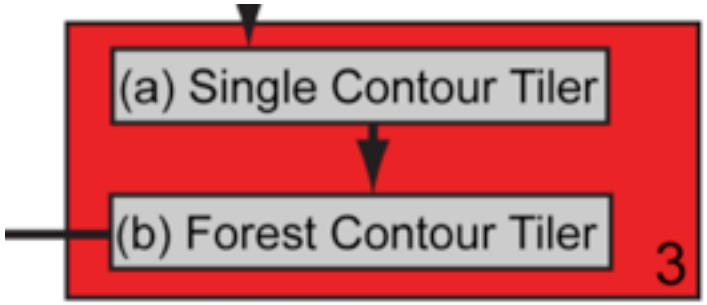


Fig. 2. (a) Original ssTEM image slice. (b) Overlap of EM image and contours after tracing. (c) Contours for 2-D geometry processing. (d) Consecutive contours initially tiled into 3-D. (e) Meshed dendrite. (f) Example of an initial mesh. Dendrite is white, and axon is green. (g) Example of an improved mesh (viewed from a different angle).

Reconstructing a Single Contour Stack

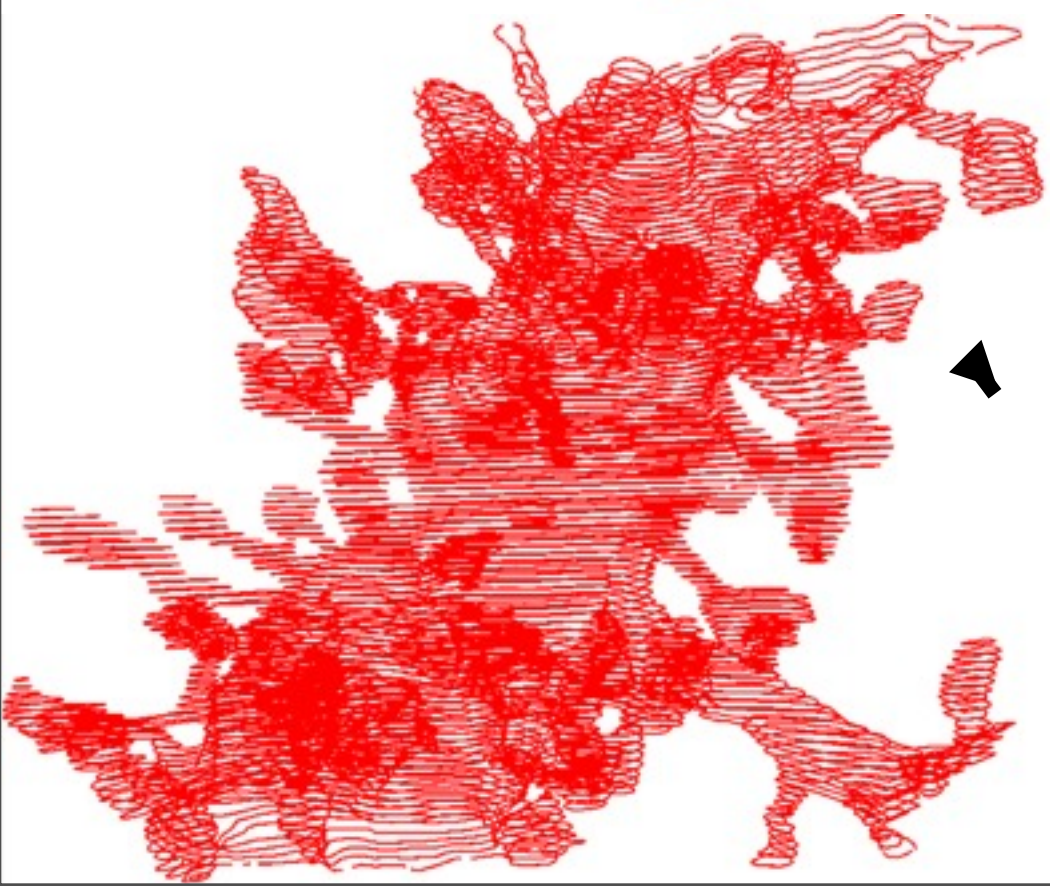
Complex Topological Changes
Between Adjacent Slices

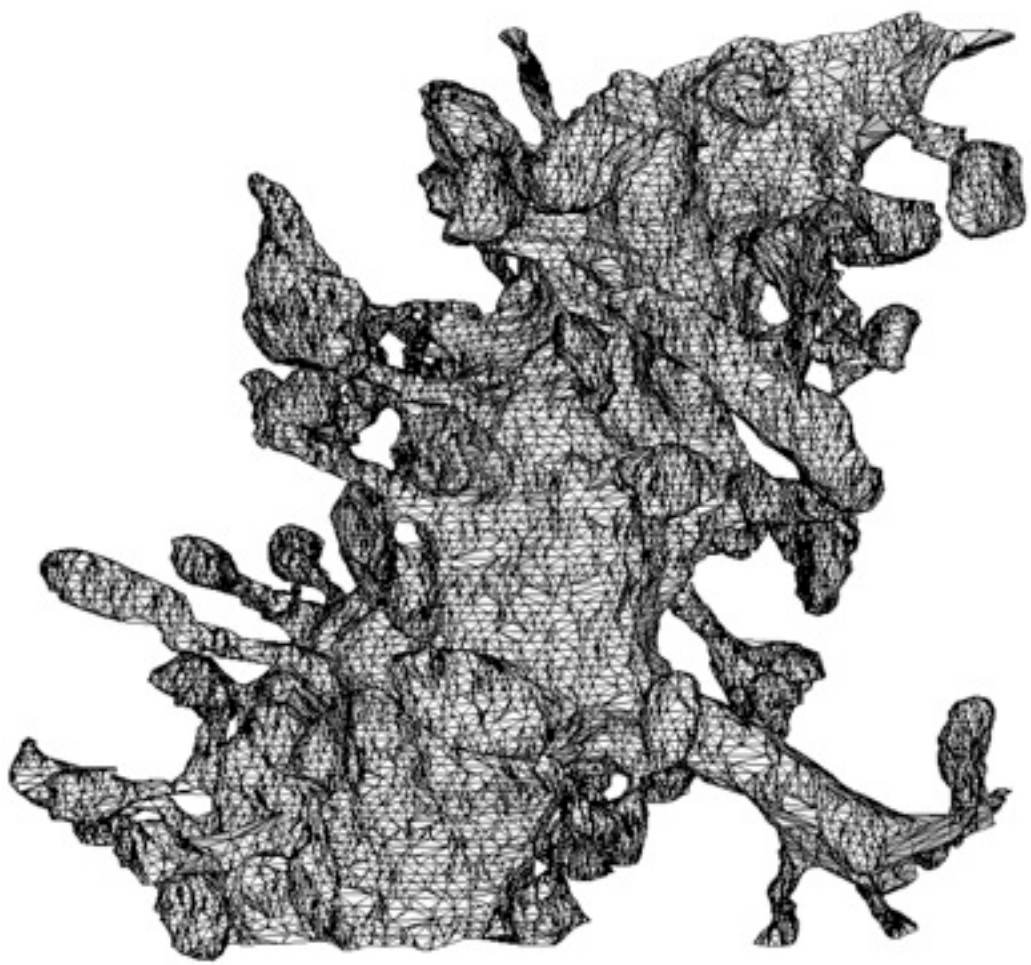




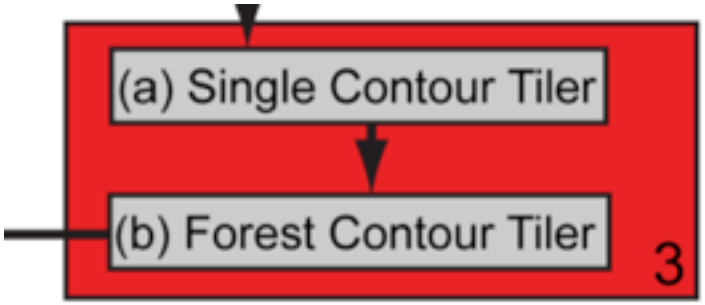
2-D to 3-D Reconstruction

C. Bajaj, E. Coyle, K. Lin
Arbitrary Topology Shape Reconstruction from Planar Cross Sections
Graphical Models and Image Processing, 58:6, (1996), 524-543.



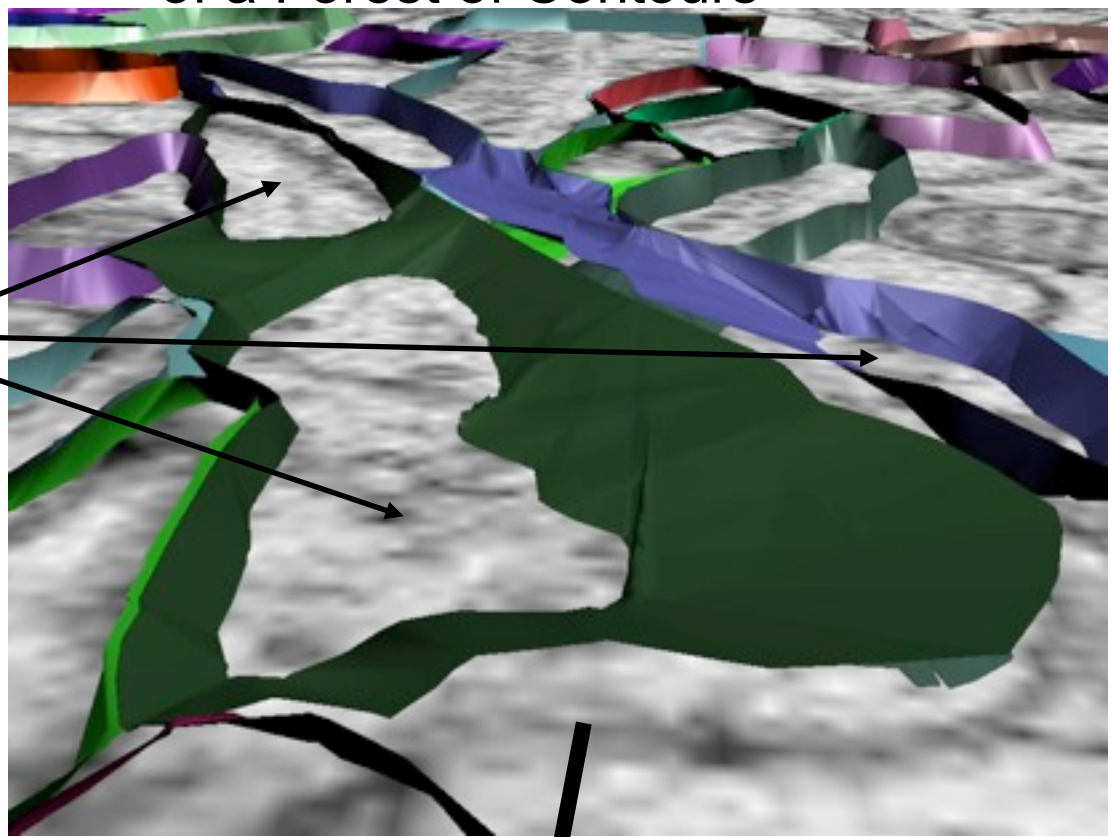


Topological/Geometric Consistent Reconstruction of a Forest of Contours



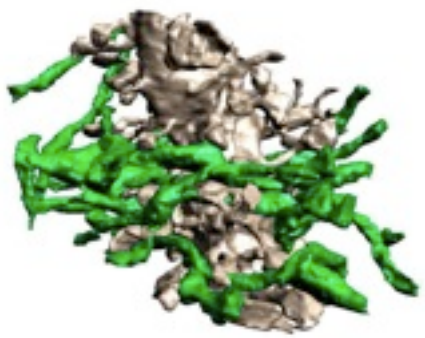
2-D to 3-D Reconstruction

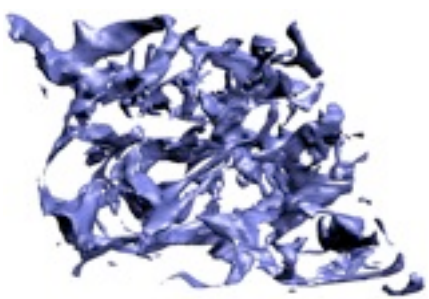
Many Contours Must Be Reconstructed Simultaneously and Consistently



J. Edwards and C. Bajaj
Topologically correct reconstruction of tortuous contour forests
Computer-Aided Design, 43(10): 1296-1306, 2011

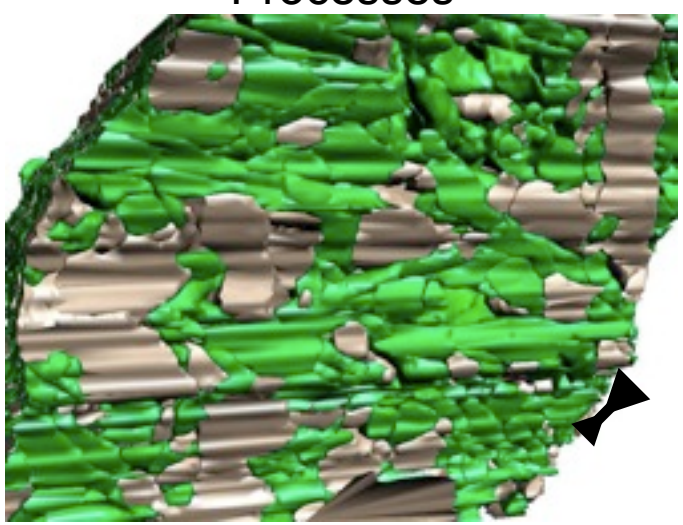
Partial Set of Reconstructed Neuronal Processes



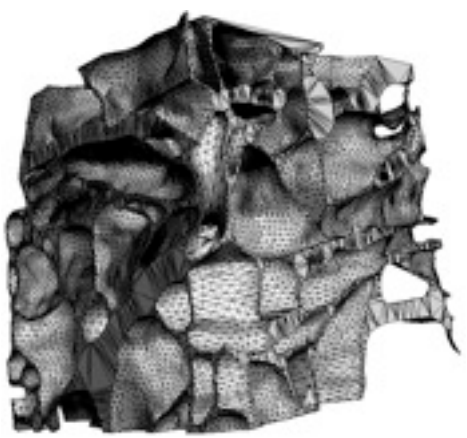


Dendrites (golden)
Axons (green)
Glial (purple)

Full Set of Reconstructed Neuronal Processes



Portion of Extracellular Space

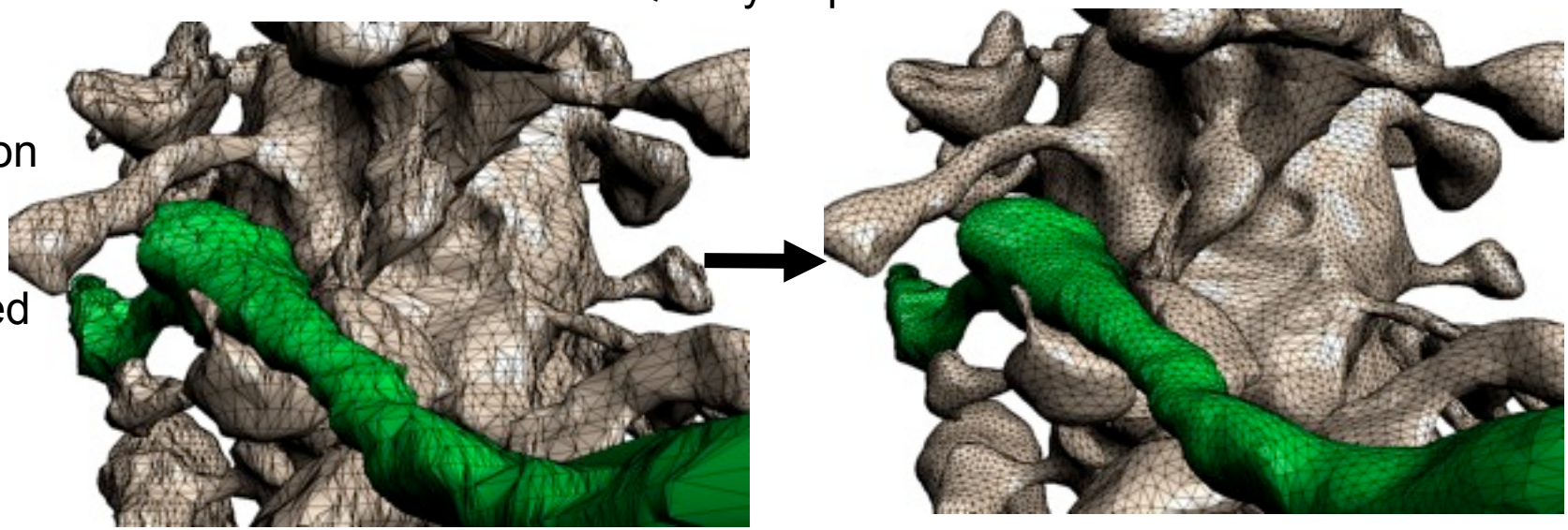


2012

3-D Geometry Processing

Mesh Quality Improvement

Reconstruction often produces poorly-shaped triangles



C. Bajaj, E. Coyle, K. Lin

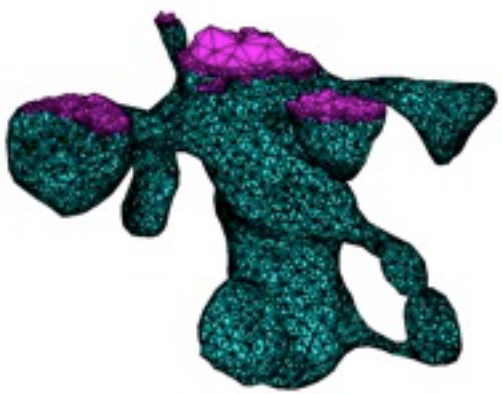
Tetrahedral Meshes from Planar Cross Sections

CMAME, 1999

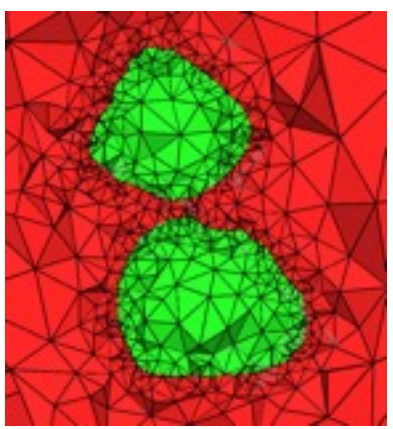
Y. Zhang, G. Xu, C. Bajaj,
Quality Meshing with Geometric Flow
Comm. in Numerical Methods in Engineering, 2008.

Q. Zhang, G. Xu, C. Bajaj
Quality Multi-domain meshing for Volumetric Data
Biomed Egg & Informatics 2010

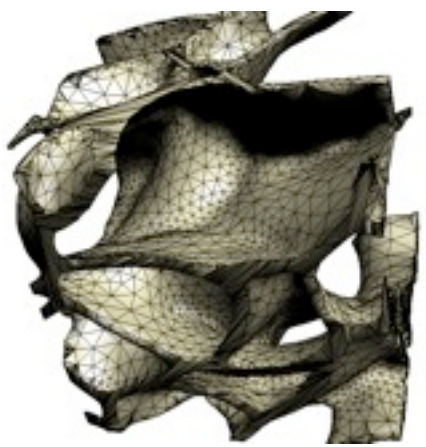
Tetrahedralization



Intra-Dendritic

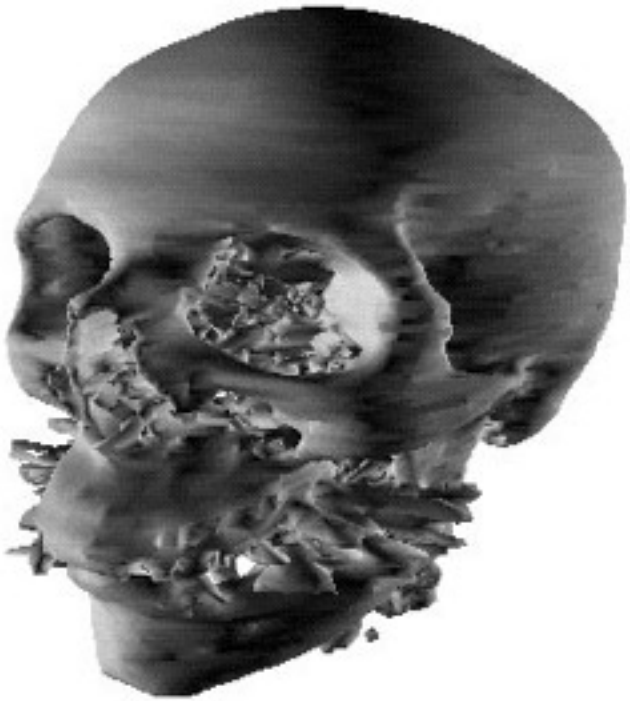


Intra & Extra-cellular

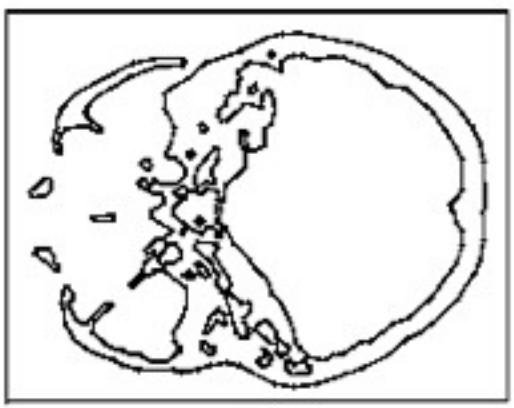


Extra-cellular

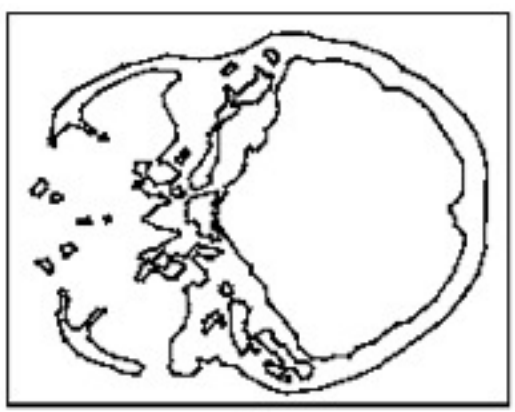
Tiling/Meshing 1



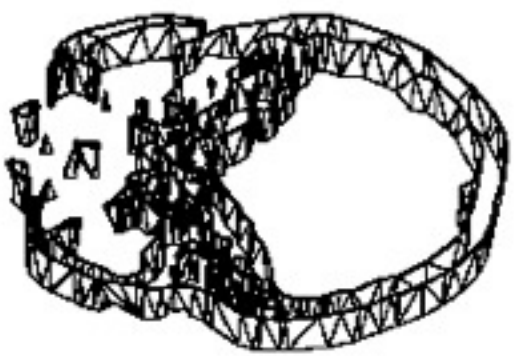
(a)



(b)



(c)

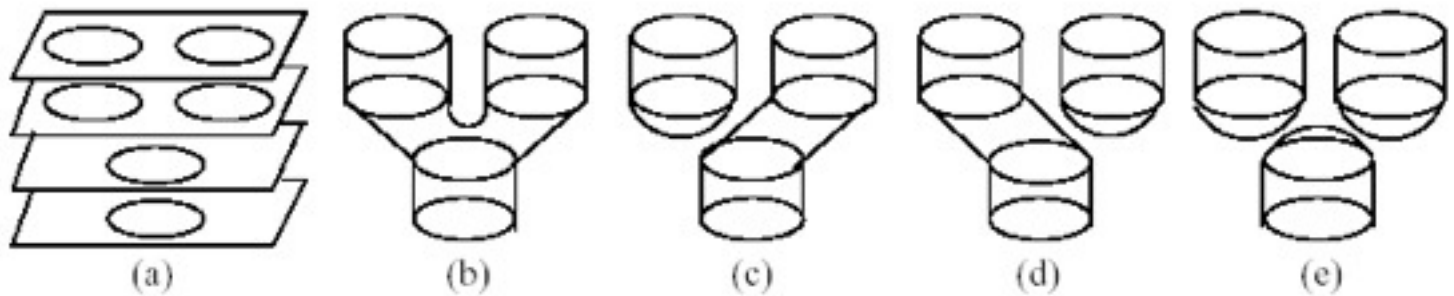


(d)

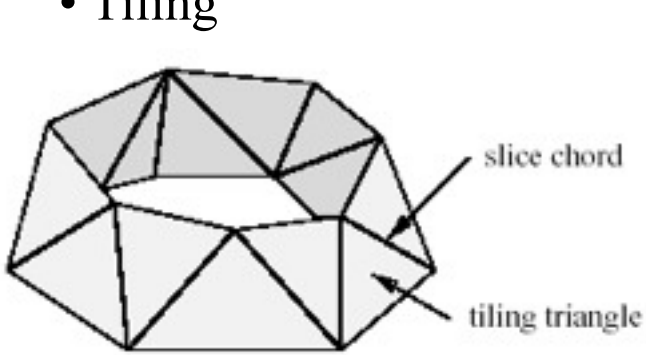
- To generate a boundary element triangular mesh from a set of cross-section polygonal slice data.
- Subproblems
  - The correspondence problem
  - The tiling problem
  - The branching problem

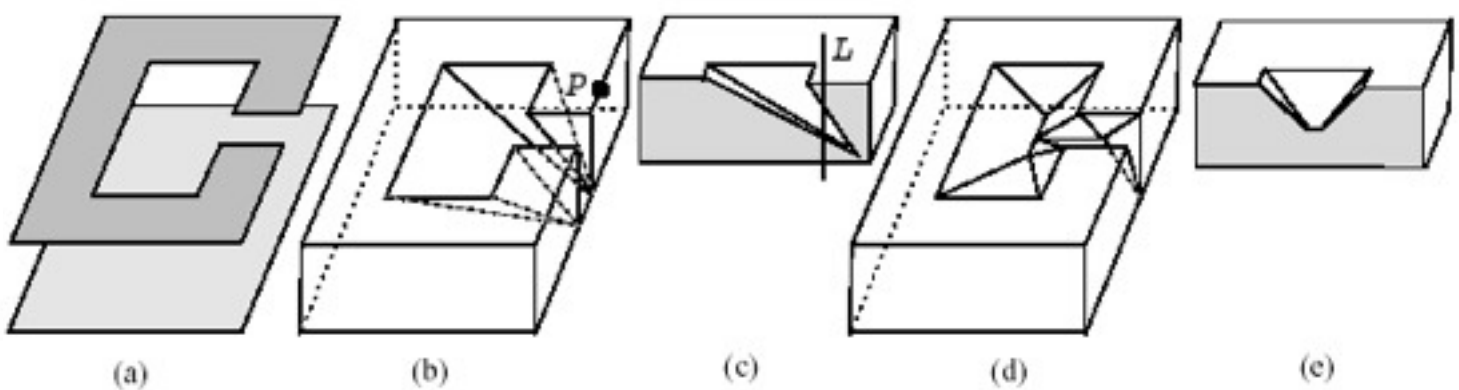
Sub-problems

- Correspondence

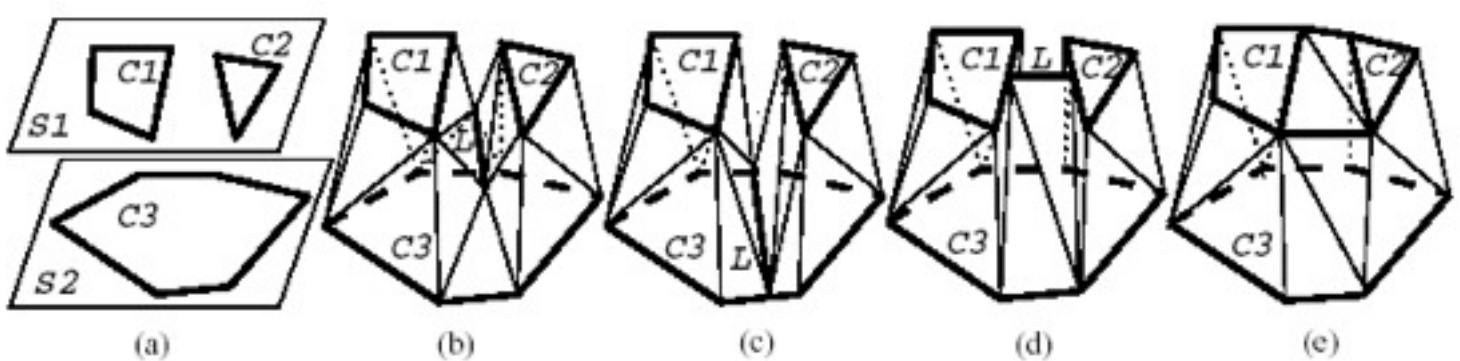


- Tiling





- Branching



Criteria for Tiled (Ribbon) Mesh

Criterion 1 *The reconstructed surface and solid regions form piecewise closed surfaces of polyhedra.*

Criterion 2 *Any vertical line (a line perpendicular to the slice) between two slices intersects the reconstructed surface at zero points, one point, or along line segment (Fig. 6(a)).*

Criterion 3 *Re-sampling of the reconstructed surface on the slice should produce the original contours.*

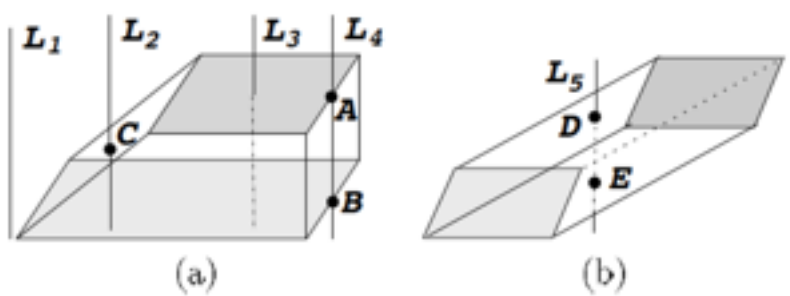
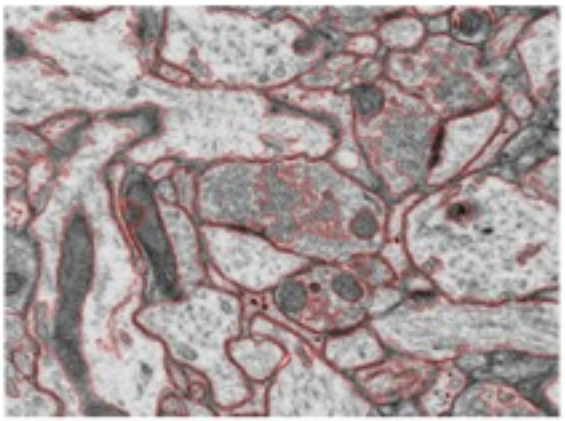
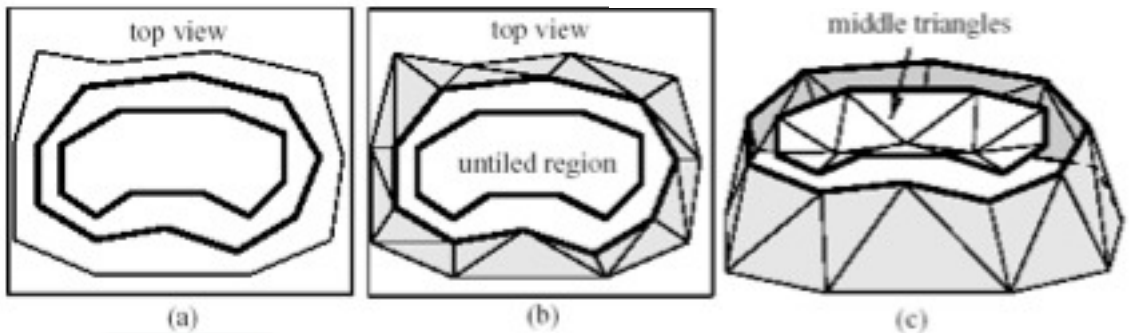


Figure 6: Criterion 2: The shaded regions are the solid regions of two slices, and the reconstructed surface is the surface covering the space between two solid regions. (a) A vertical line between two slices intersects the reconstructed surface at zero points (e.g. L_1 and L_3), one point (e.g. L_2), or one line segment (e.g. L_4). (b) The reconstructed surface shown is not allowed because the vertical line L_5 intersects the reconstructed surface at two points, D and E.

Sketch of Tiling Algorithm





Step 1: Segment closed contours from image slices.

Step 2: Create any required augmented contours.

Step 3: Find correspondences between contours.

Step 4: Form the tiling region of each vertex.

Step 5: Construct the tiling.

Step 6: Collect the boundaries of untiled regions.

Step 7: Form triangles to cover untiled regions based on their edge Voronoi diagram (EVD).

C. Bajaj, E. Coyle, K. Lin
Arbitrary Topology Shape Reconstruction from Planar Cross Sections
Graphical Models and Image Processing, 58:6, (1996), 524-543.

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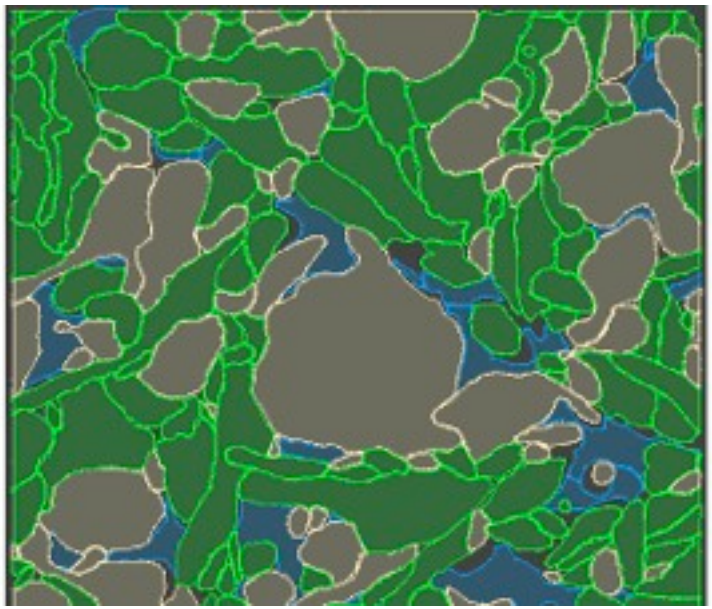
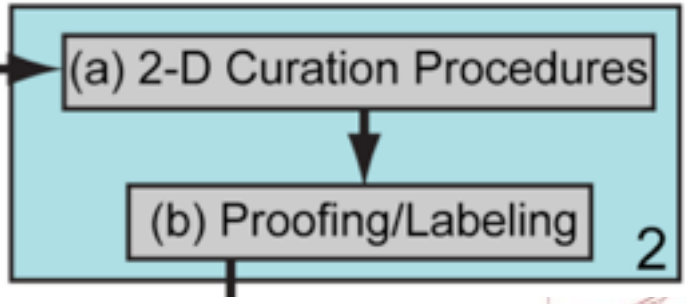
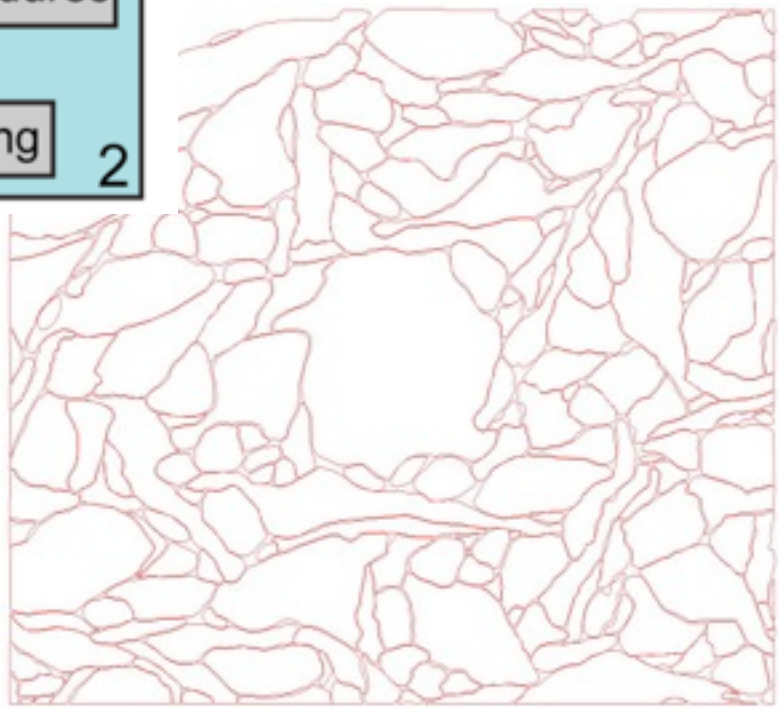
Steps 1&2

2-D Geometry Processing



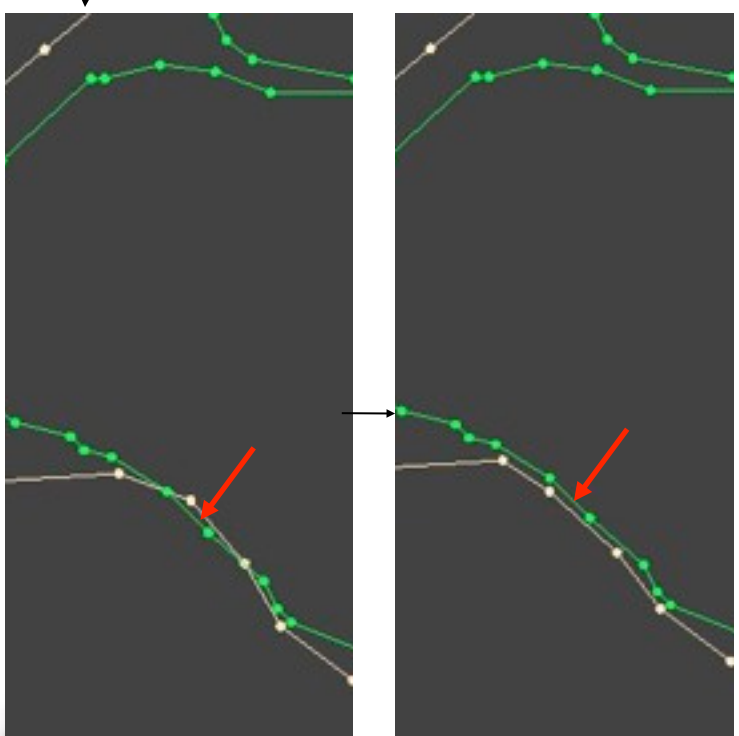


Spline-Fitted Contours

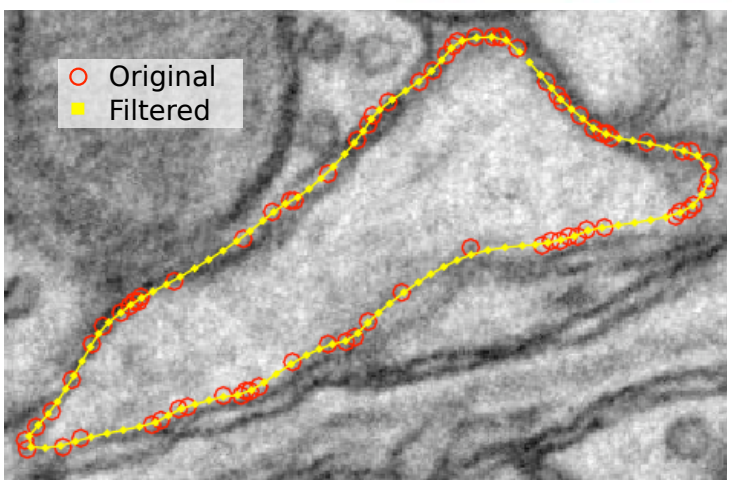


Traced Contours

2D Curation



Contour Resampling



C. Bajaj, G. Xu, Q. Zhang

A Fast Variational Method for the Construction of Adaptive Resolution C^2 Smooth Surfaces

CMAME, (2009)

C. Bajaj, A. Gillette, S. Goswami

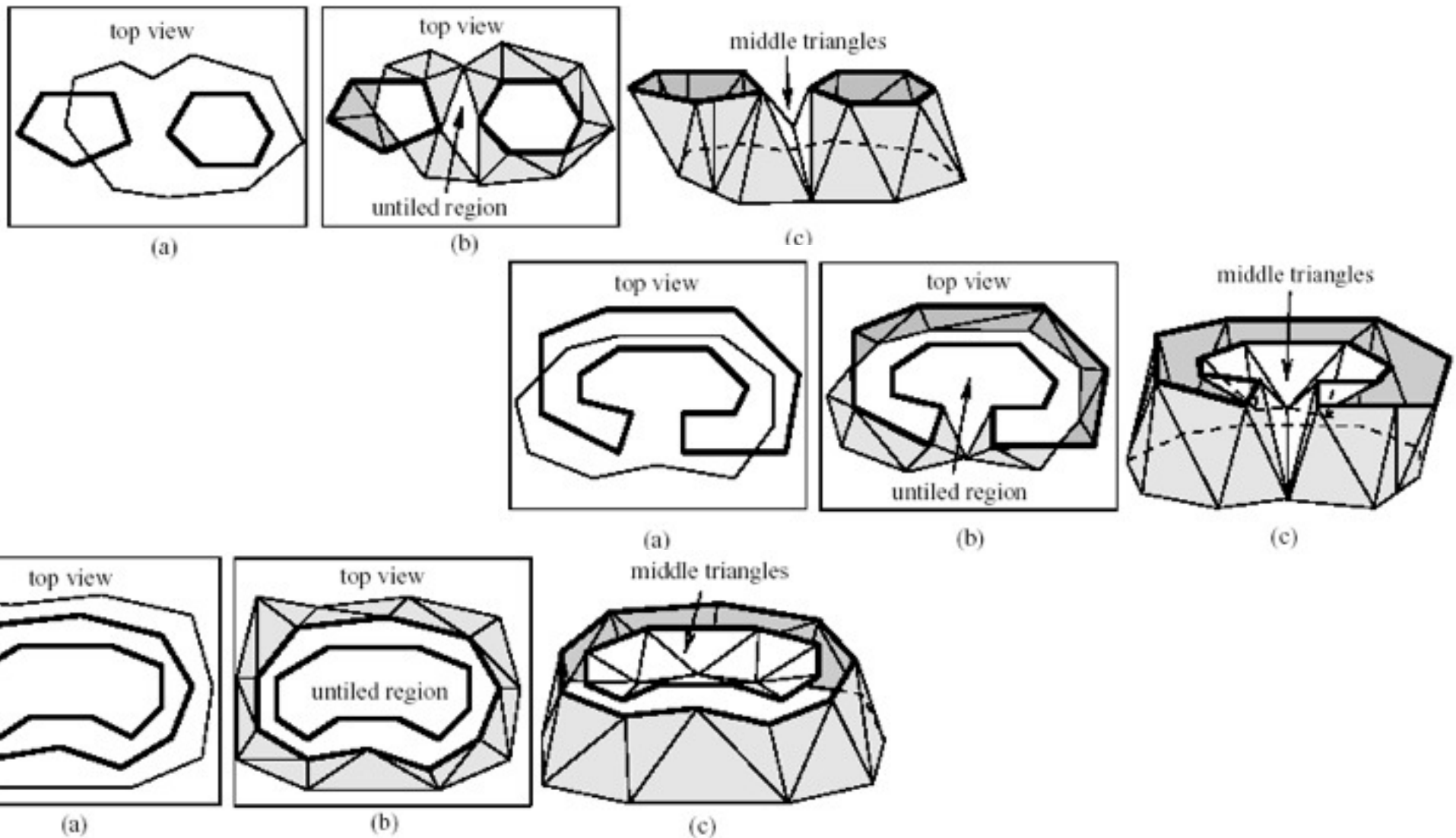
Topology Based Selection and Curation of Level Sets

Topology-in-Visualization, ed. by A. Wiebel, H. Hege, K. Polthier, G. Scheuermann, 2009

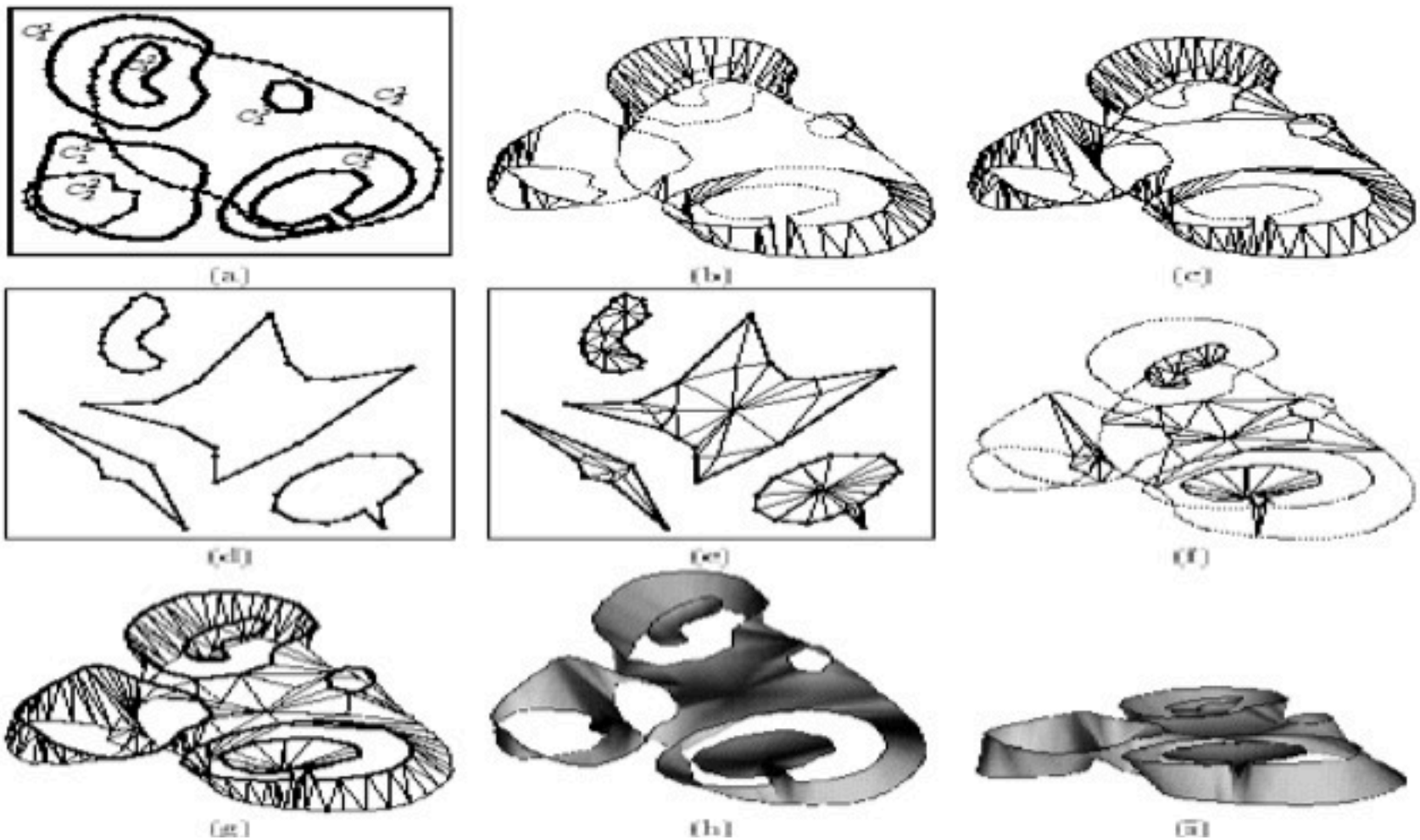
2012

Tiling Steps 3 - 6

- A multi-pass tiling approach followed by the postprocessing of untiled regions



UnTiled Regions for Real Data



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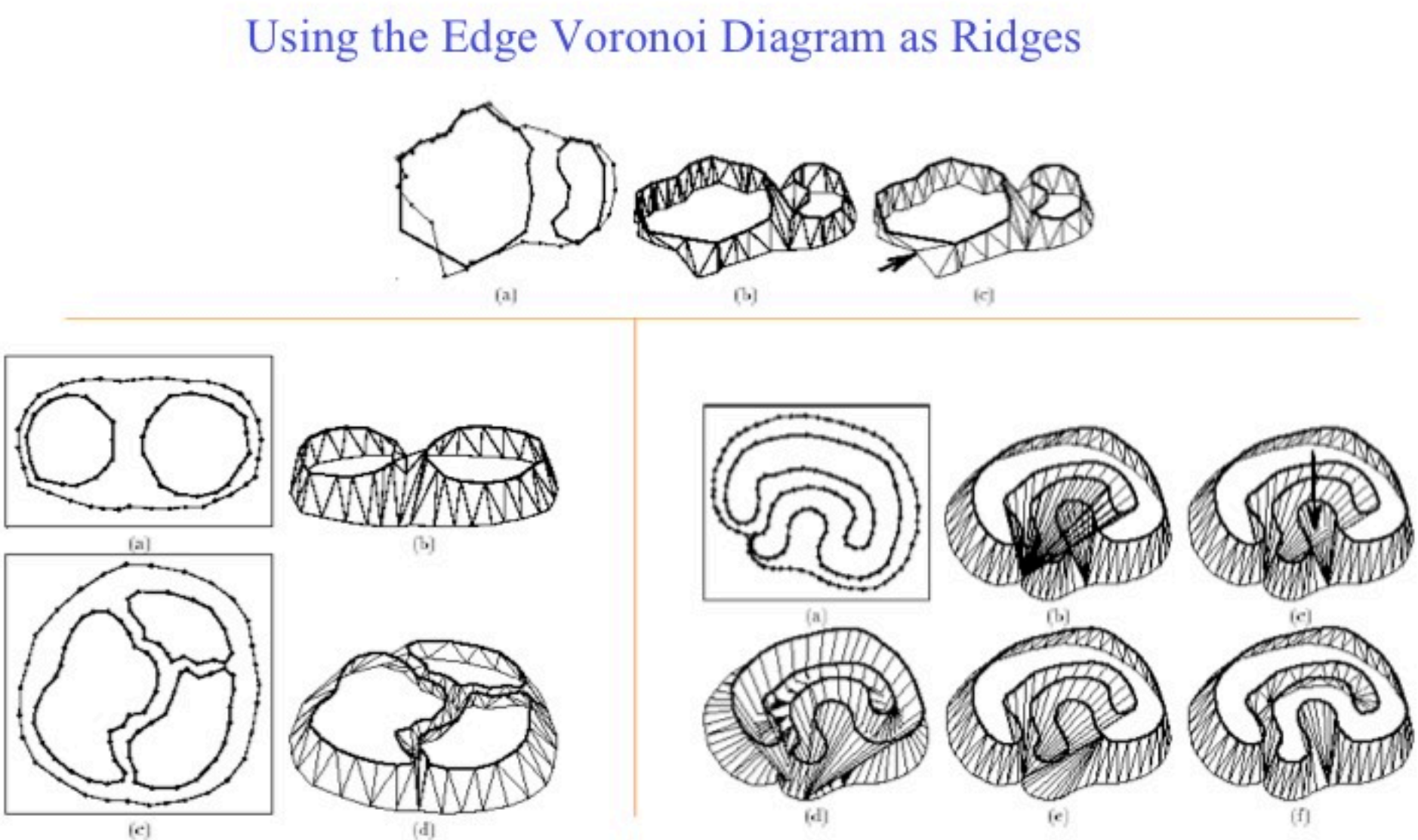
Processing Un-Tiled Regions with Branching



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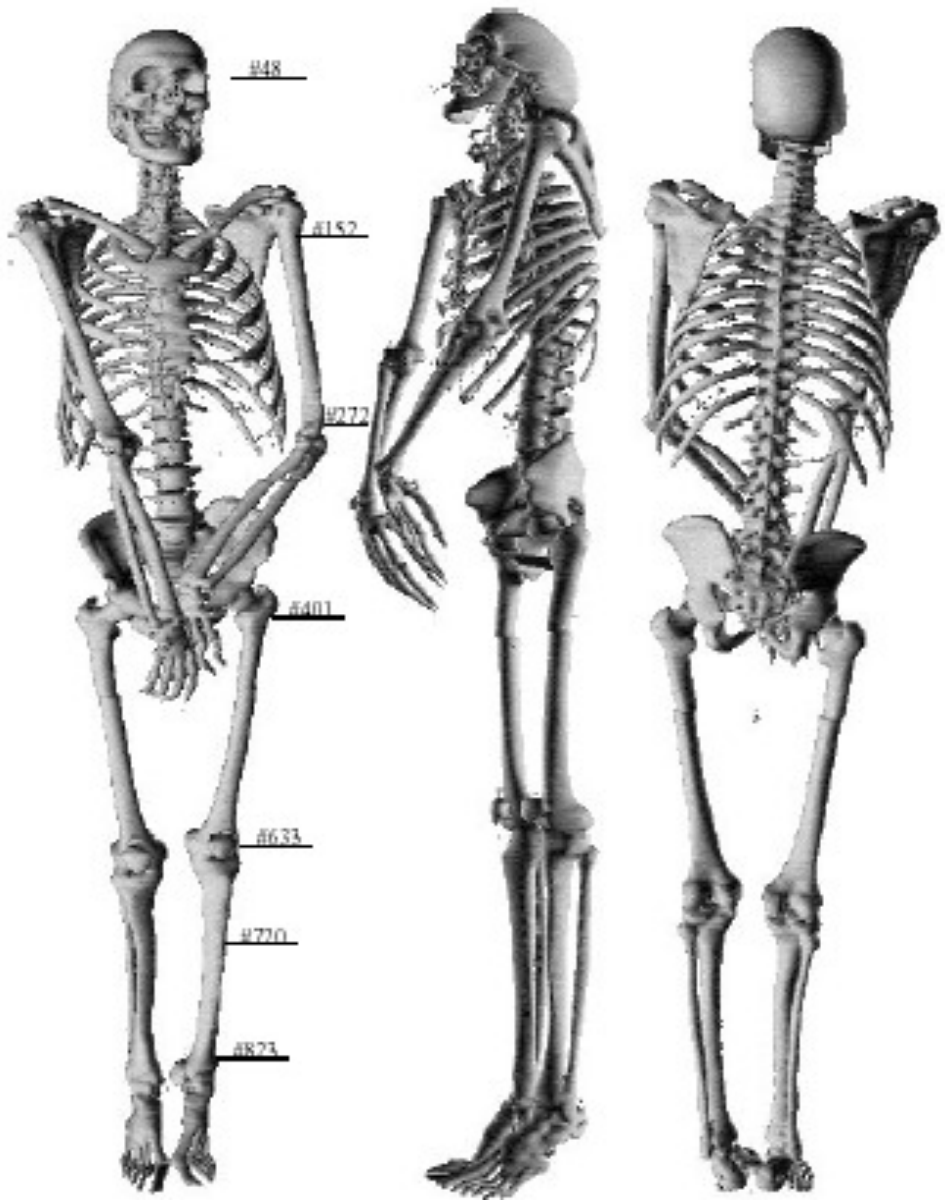
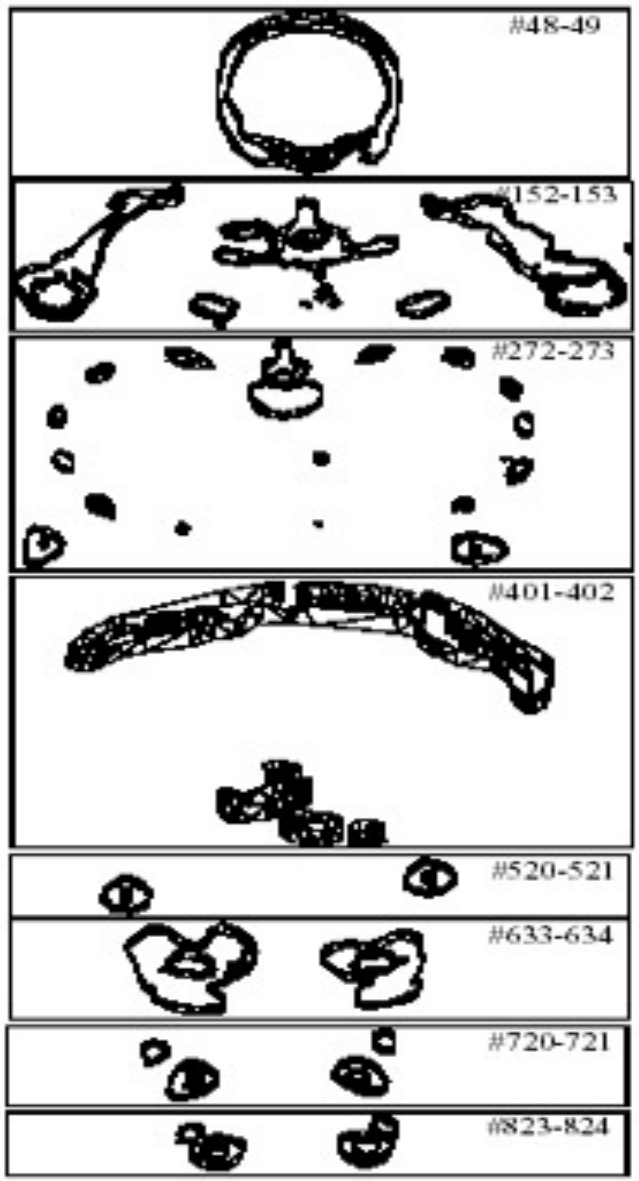
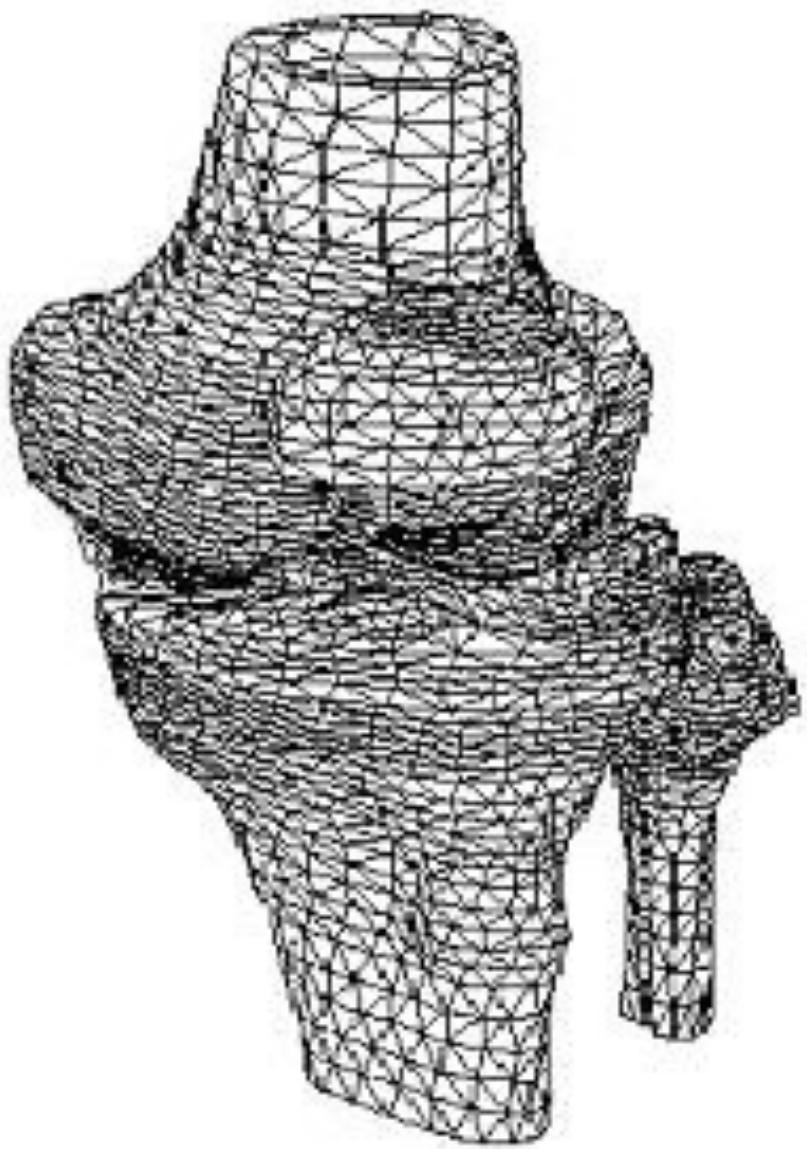
Boundary Element Triangular Mesh





Meshing II



- To generate a 3D finite element tetrahedral mesh of the simplicial polyhedron obtained via the BEM construction of cross-section polygonal slice data.

- Subproblems
  - The shelling of tetrahedra to reduce polyhedron to prismatoids
  - The tetrahedralization of prismatoids

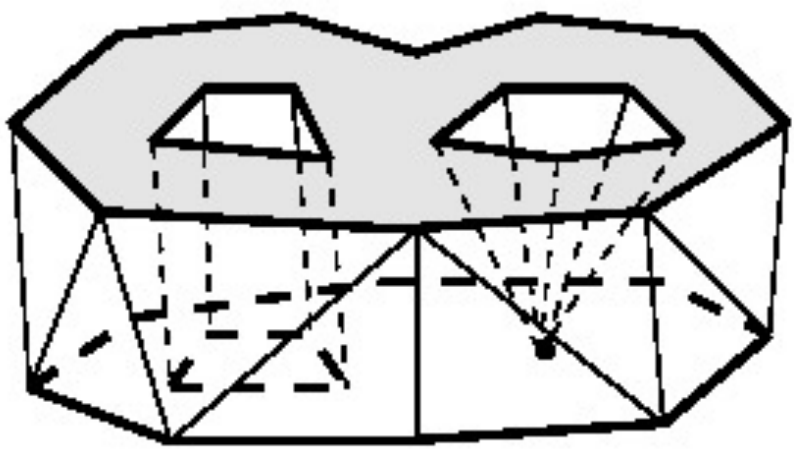
C. Bajaj, E. Coyle, K. Lin

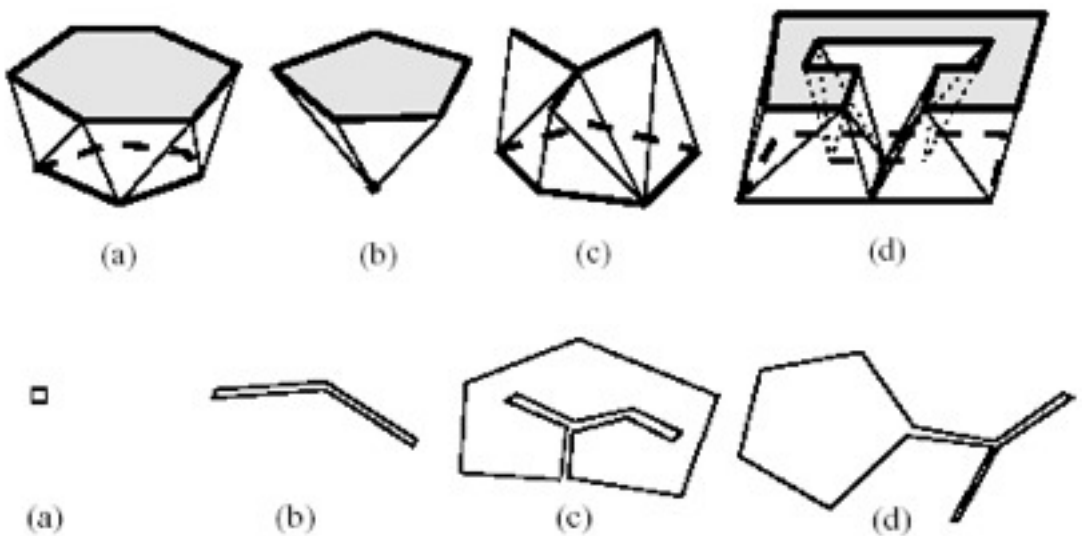
Tetrahedral Meshes from Planar Cross Sections

CMAME, 1999

What is prismatoid?

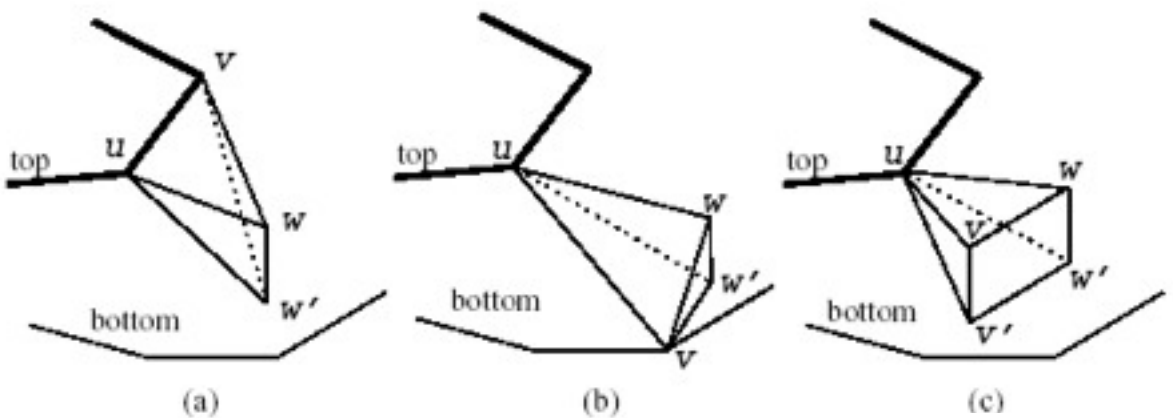
A prismatoid is a polyhedron having for bases two polygons in parallel planes, and for lateral faces triangles or trapezoids with one side lying in one base, and the opposite vertex or side lying in the other base, of the polyhedron.

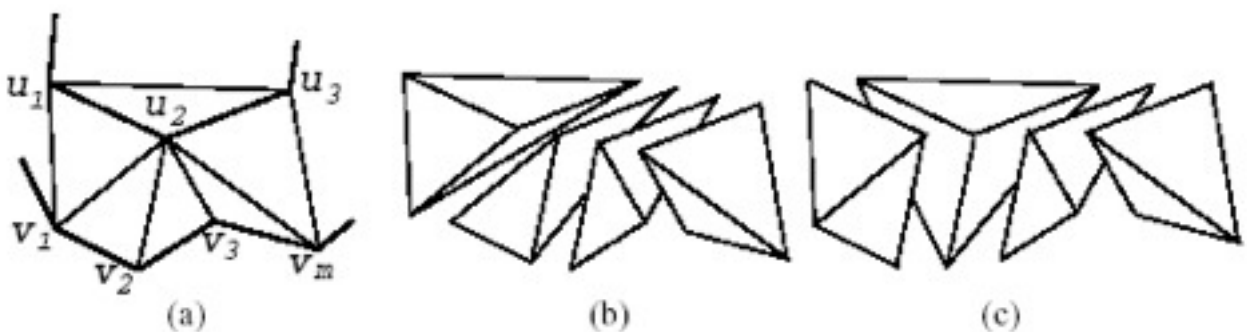




The Shelling Step

- Shell tetrahedra from the polyhedron, so the remaining part is a prismatoid or can be divided into prismatoids.





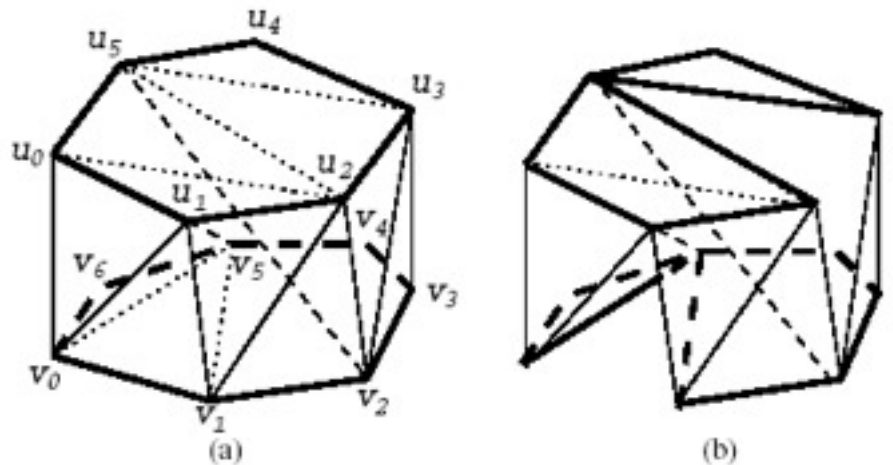
Prismatoid → Tetrahedra

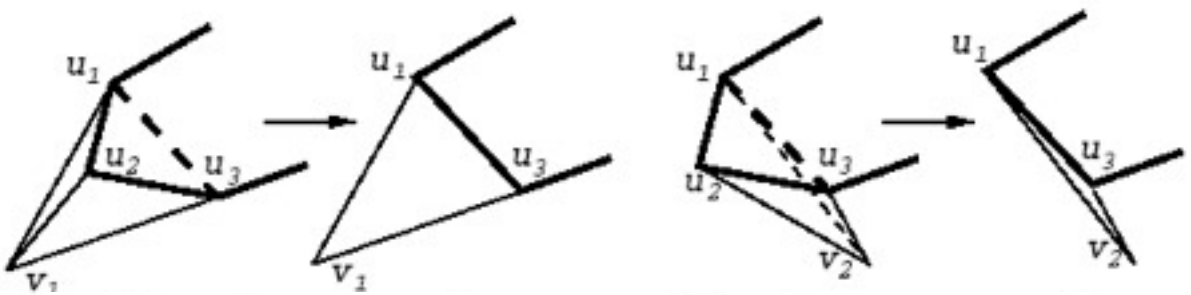
- To tetrahedralize a non-nested prismatoid without Steiner points.
  1. For each boundary triangle on both slices, calculate its metric.
  2. Pick up the boundary triangle with the best metric and form one set of tetrahedra.
  3. Update the advancing front and go to Step 1.
  4. If the remaining part is non-tetrahedralizable, backtrack and select another choice in Step 1.

Metric, Weight Factor, Grouping

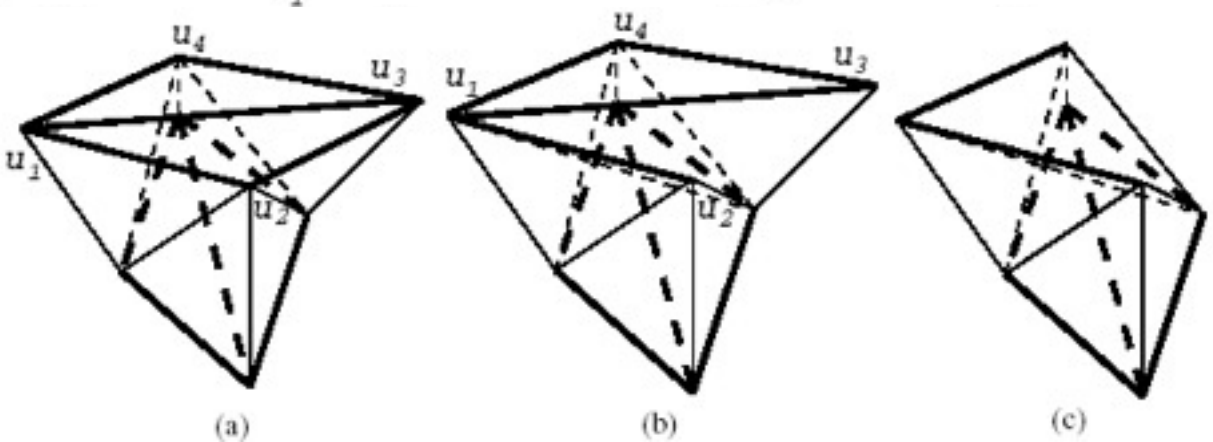
- Metric = volume/(edge)^3
- Weight factor

$$w = \begin{cases} 2(1-\frac{d}{h}) & \text{if } d \le 0.5h \\ 1 & \text{if } 0.5h < d < h \\ \frac{h}{d} & \text{if } d \ge h \end{cases}$$



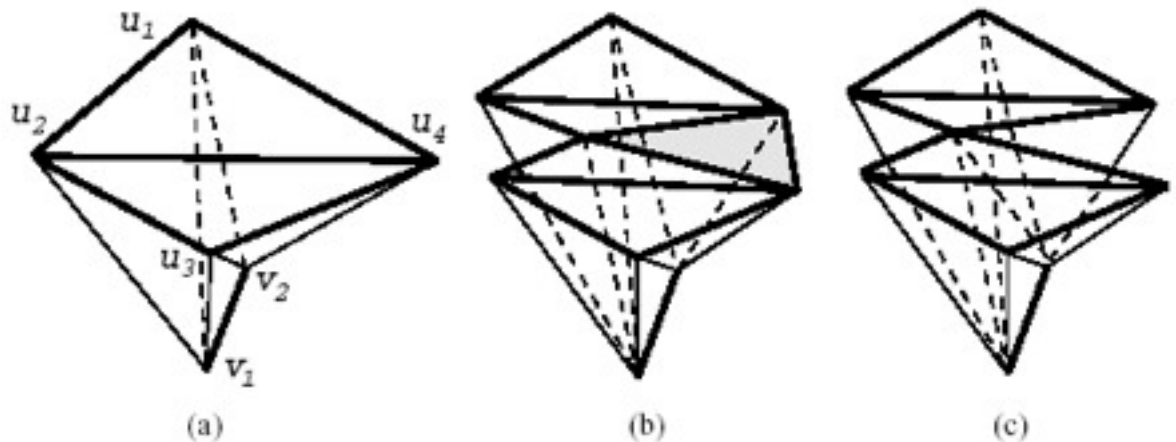


- Grouping can avoid irregular remaining part



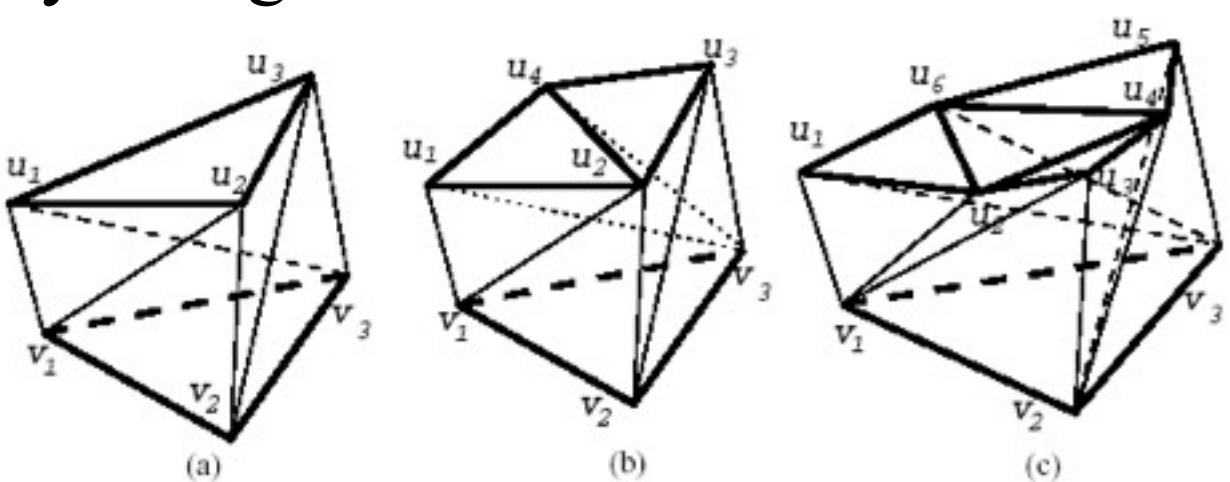
Classification of Untetrahedralizable Prismatoids

1. Has two boundary triangles on the top face and one line segment on the bottom face.



2. Has one bottom triangle which is treated as three boundary triangles.

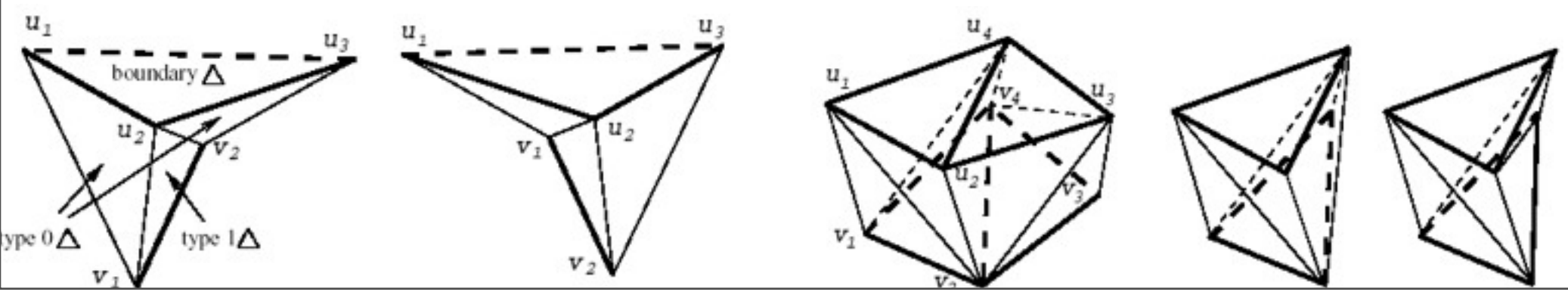
Schonhardt



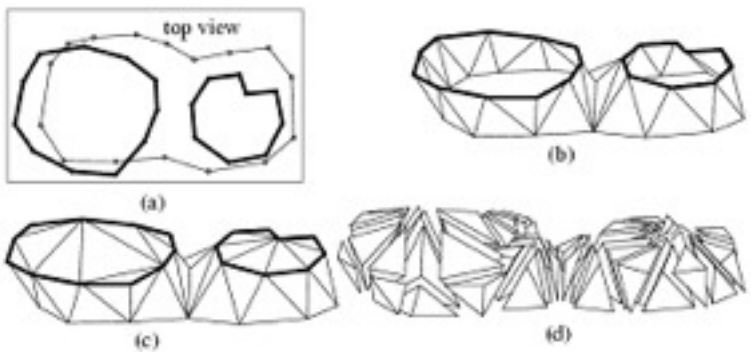
Protection Rules

Lemma 1: Suppose a top boundary triangle $\Delta u_1u_2u_3$ is under the constraint that no more than one type 1 triangle is between the two type 0 triangles containing the contour segments u_1u_2 and $u_2\ u_3$. Furthermore, let the bottom vertices of the two type 0 triangles be v_1 and v_2. Our grouping operation cannot apply to $\Delta u_1u_2u_3$ to form a set of tetrahedra, if and only if all the following conditions are satisfied.

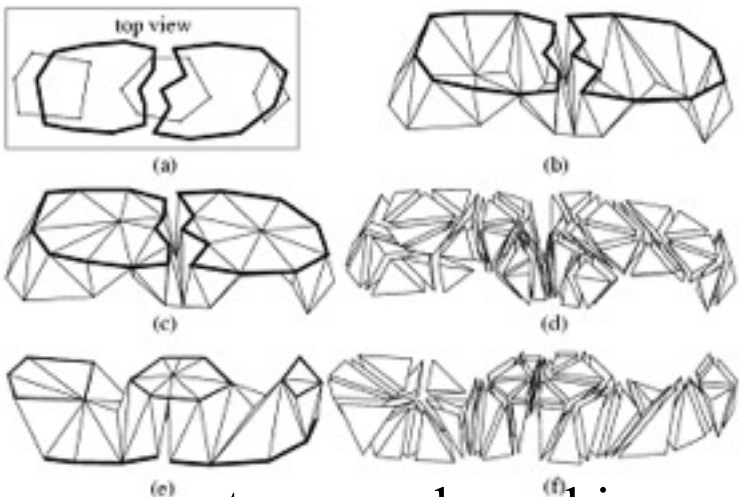
1. v_1v_2 is exactly one contour segment.
2. One of the slice chords u_2v_1 and u_2v_2 is reflex and the other is convex.
3. Both u_1v_2 and u_3v_1 are not inside the prismatoid.



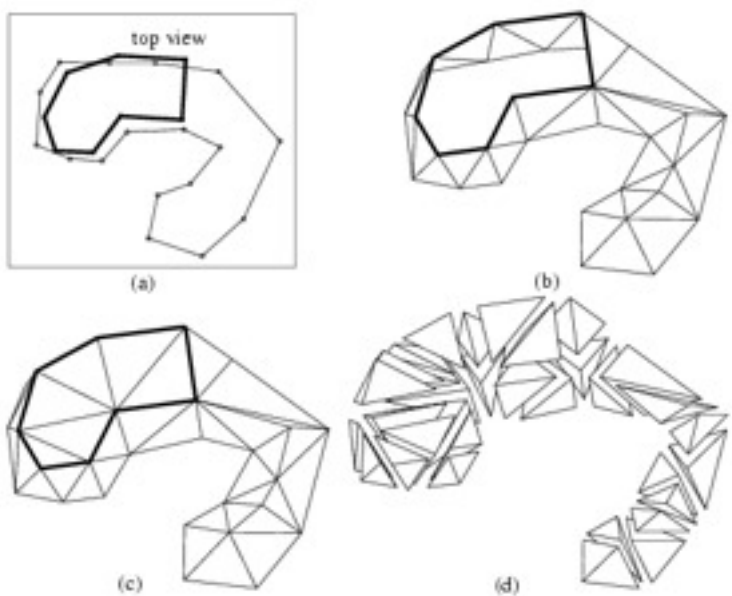
Multiple Tetrahedralizable Cases



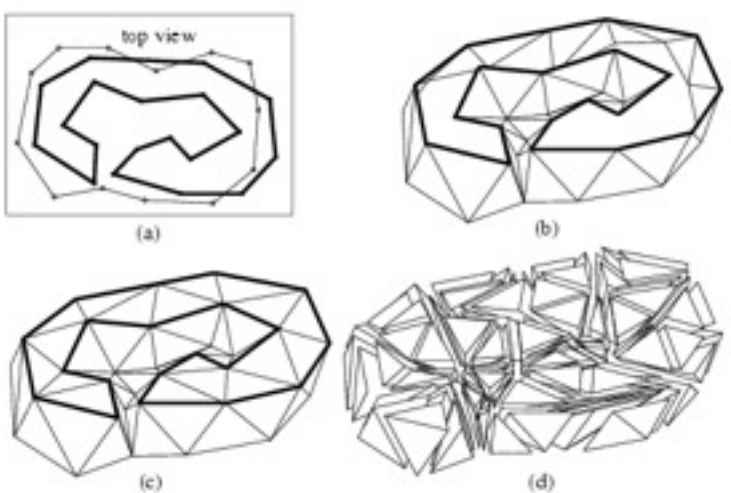
One-to-many branching



many-to-many branching

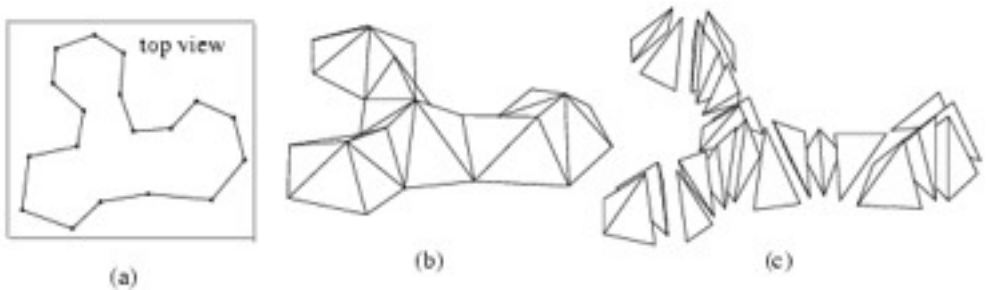


Dissimilar region (the right bottom portion of the bottom contour)

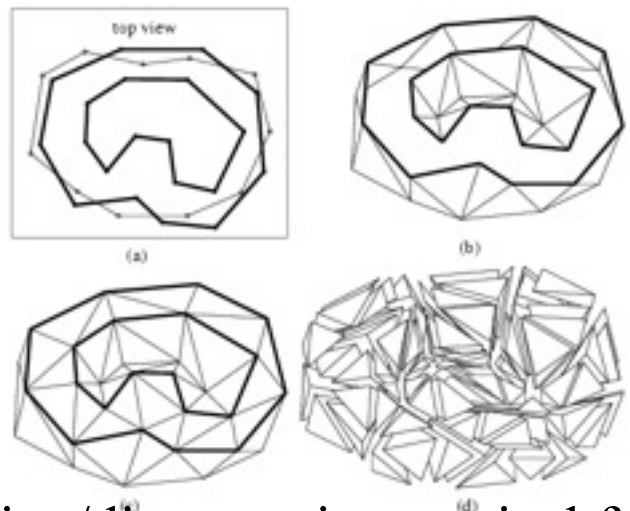


Dissimilar region (the inner portion of the top contour)

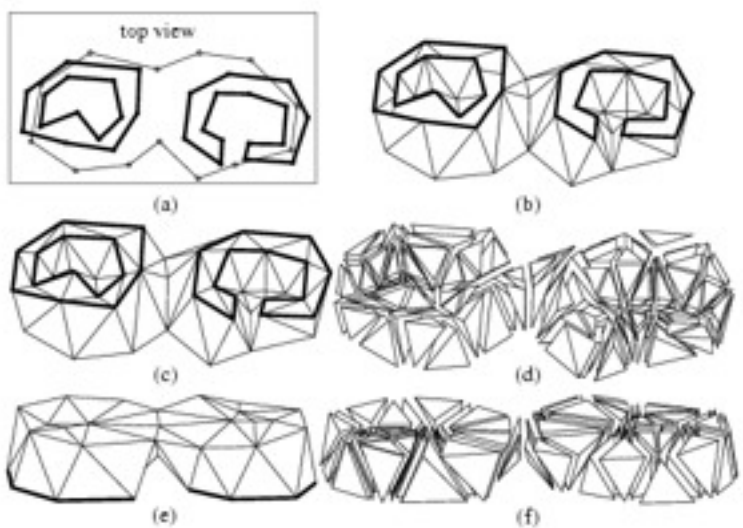
Multiple Tetrahedralizable Cases



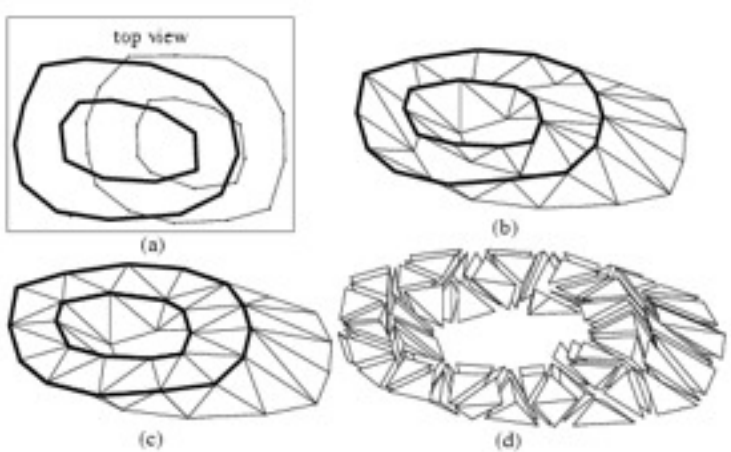
Appearing/disappearing vertical feature of a solid interior



Appearing/disappearing vertical feature (the top inner contour) of a void interior

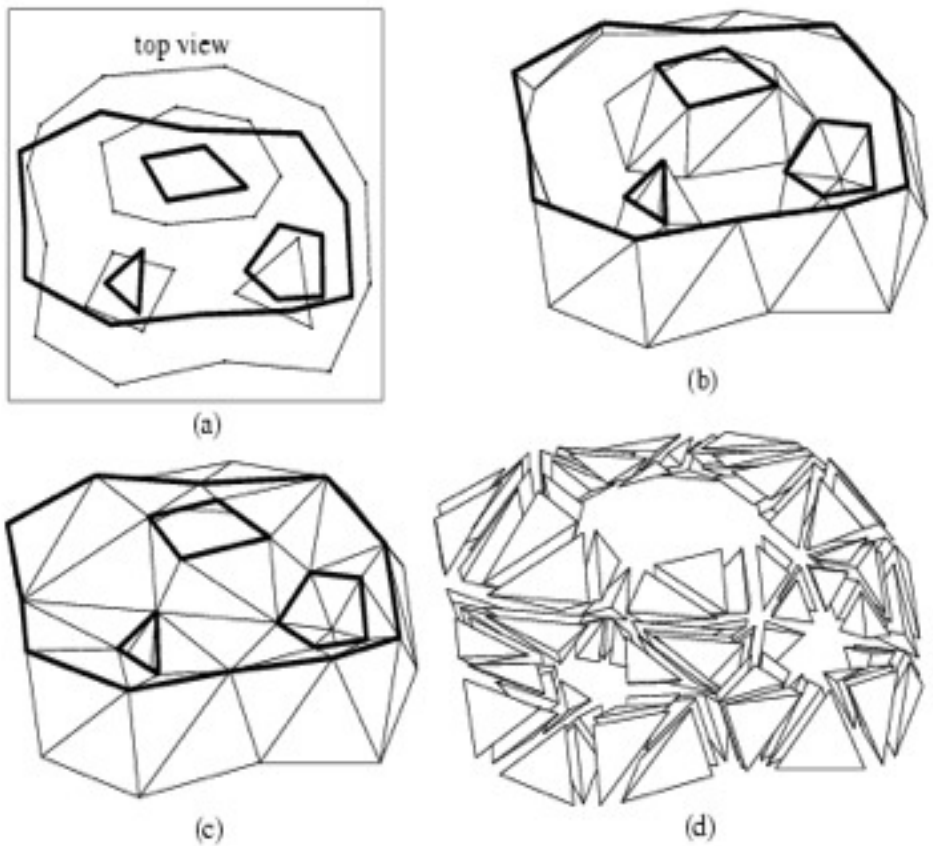


A branching, a dissimilar portion (the inner portion of the top right contour), and an appearing/disappearing vertical feature (the inner contour at the left of the top slice)

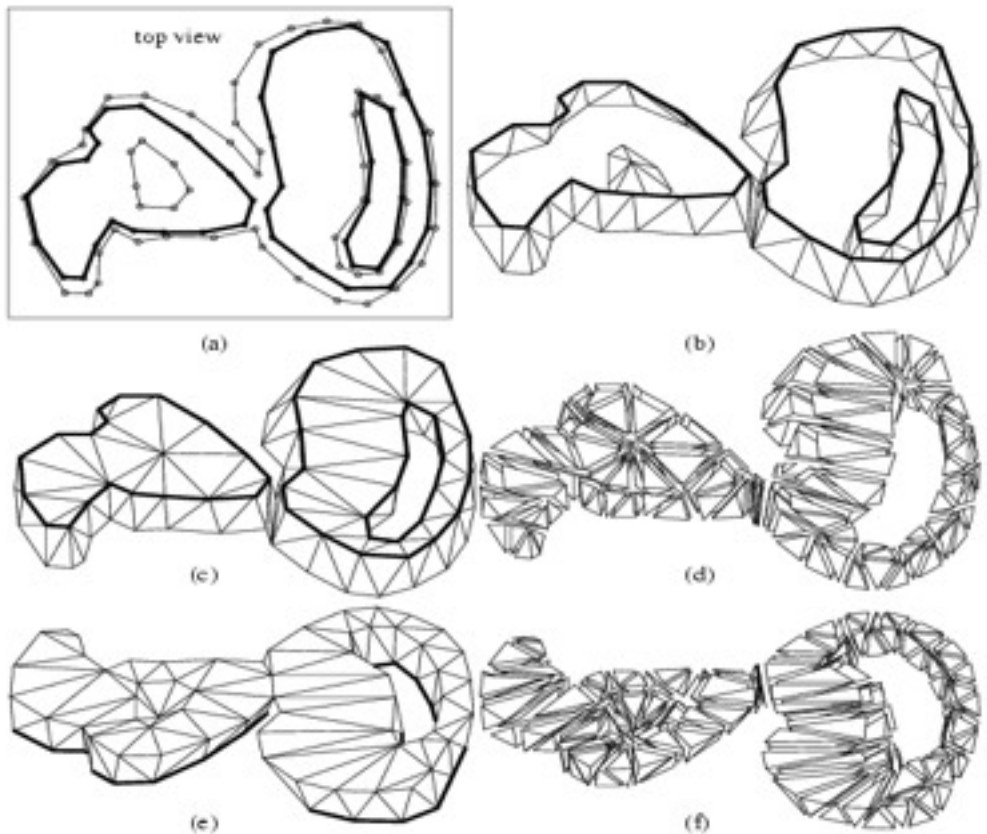


Nested prismatoids

Multiple Tetrahedralizable Cases

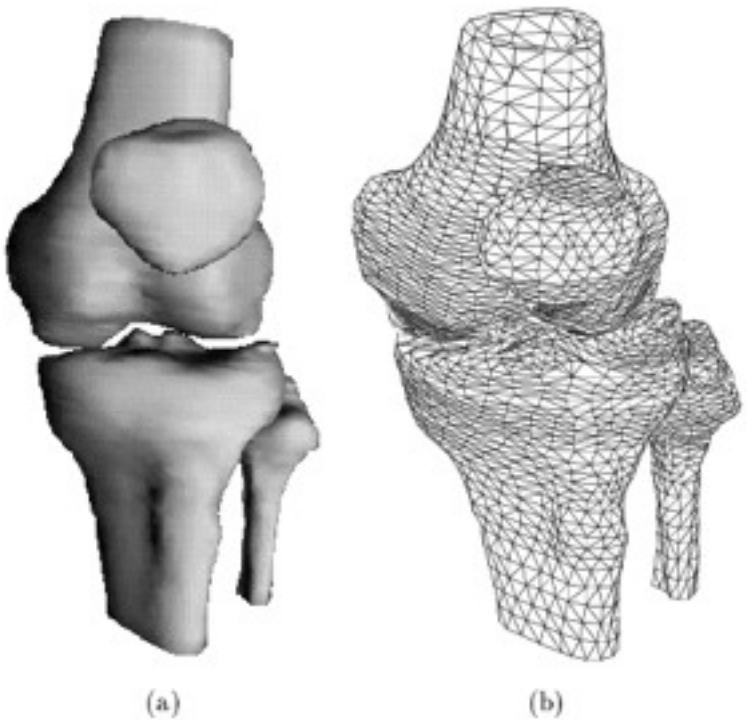


Multiply-nested prismatoid



Solid region between two slices
of a human tibia

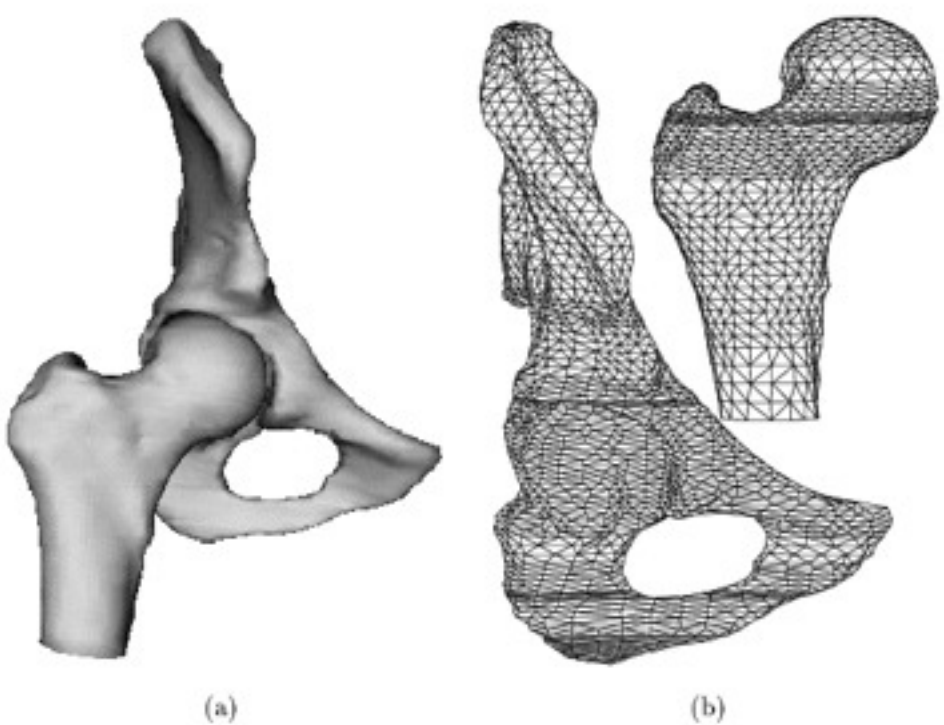
Examples



Knee joint (the lower femur, the pper tibia and fibula and the patella)

(a) Gouraud shaded

(b) The tetrahedralization



Hip joint (the upper femur and the pelvic joint)

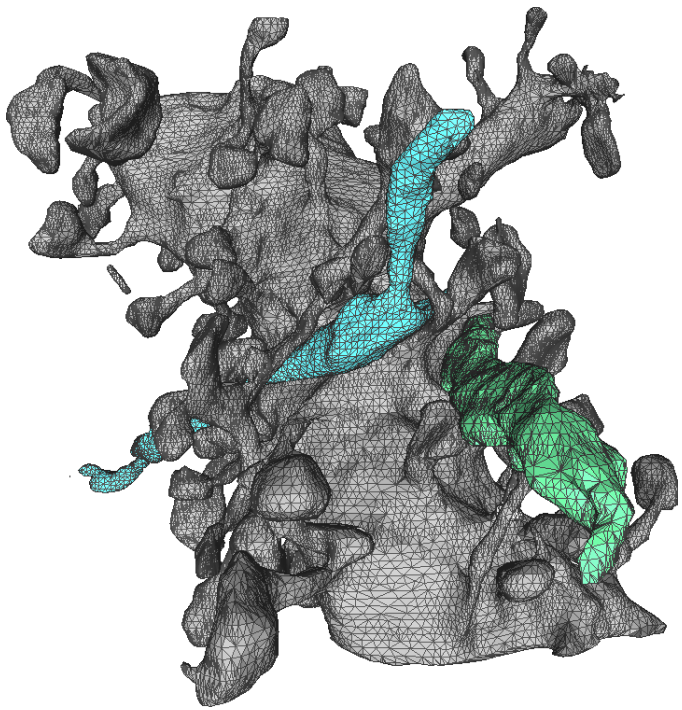
(a) Gouraud shaded

(b) The tetrahedralization

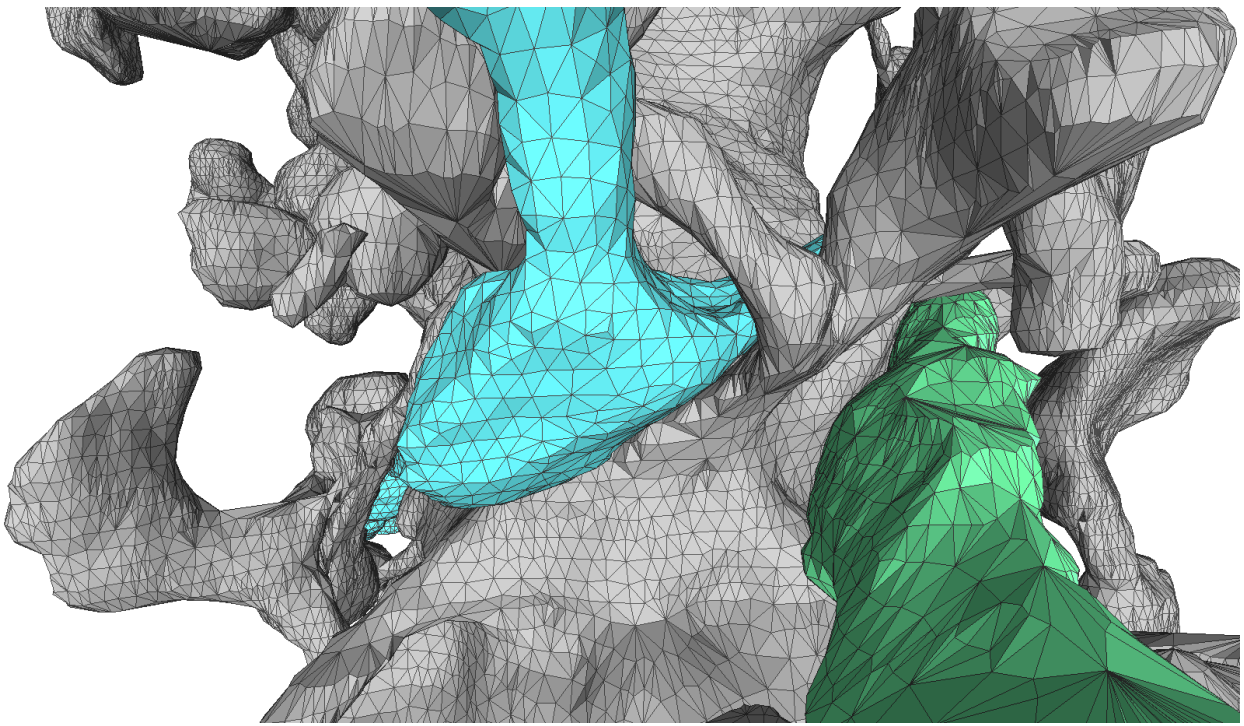
Mini-summary

- The characterization, avoidance of non-tetrahedralizable polyhedra is one of the main challenges
- The mix of numerical precision and topological decision making needs precise rules so errors don’t propagate.

Decimation & Mesh Improvement



(a)



(b)

Decimation

Mesh Quality Improvement

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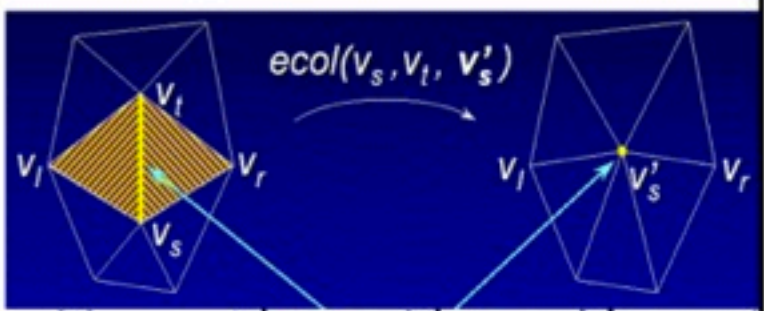
Decimation Techniques

- **Edge Contraction**
  [Hoppe96] [Popovic,Hoppe97] [Staadt,Gross98]
  [Trotts,Hamann,Joy,Wiley98][Gueziec96]



- **Vertex Removal**
  [Bajaj,Schikore98][De Berg,Dobrindt98]
  [Lee,Dobkin,Sweldens,Cowsar,Schroder98]

- **Triangle Contraction**
  [Hamann97]

- **Hole Re-triangulation**
  [Defloriani89] [Cohen,Varshney,Manocha,Turk96]

- **General Re-triangulation** [Turk92]

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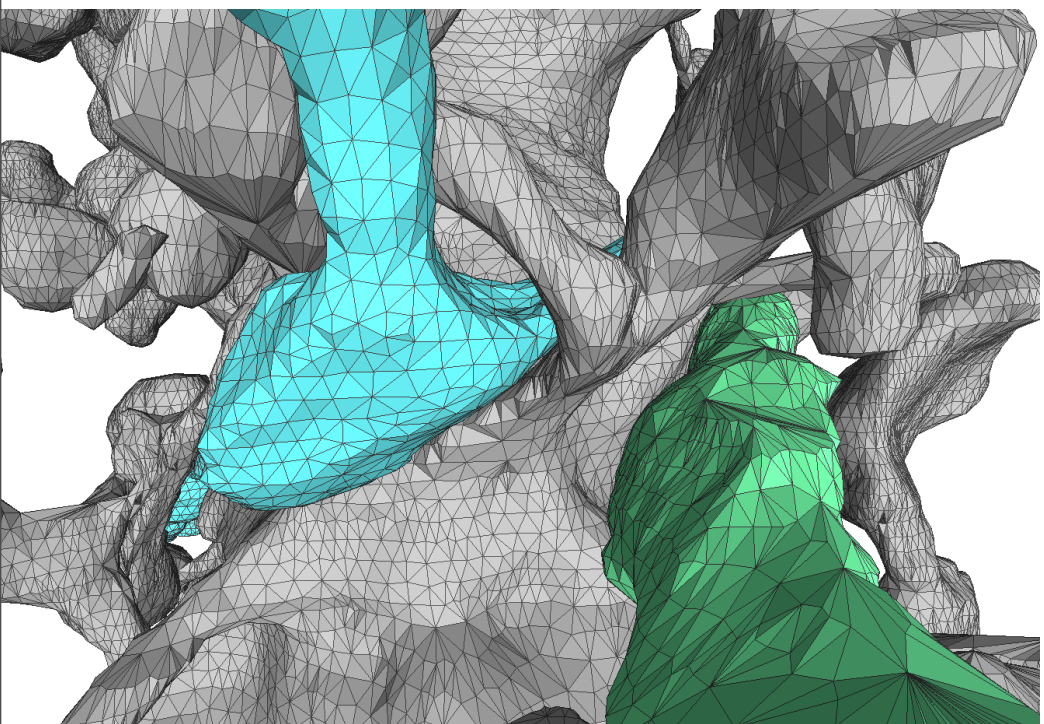
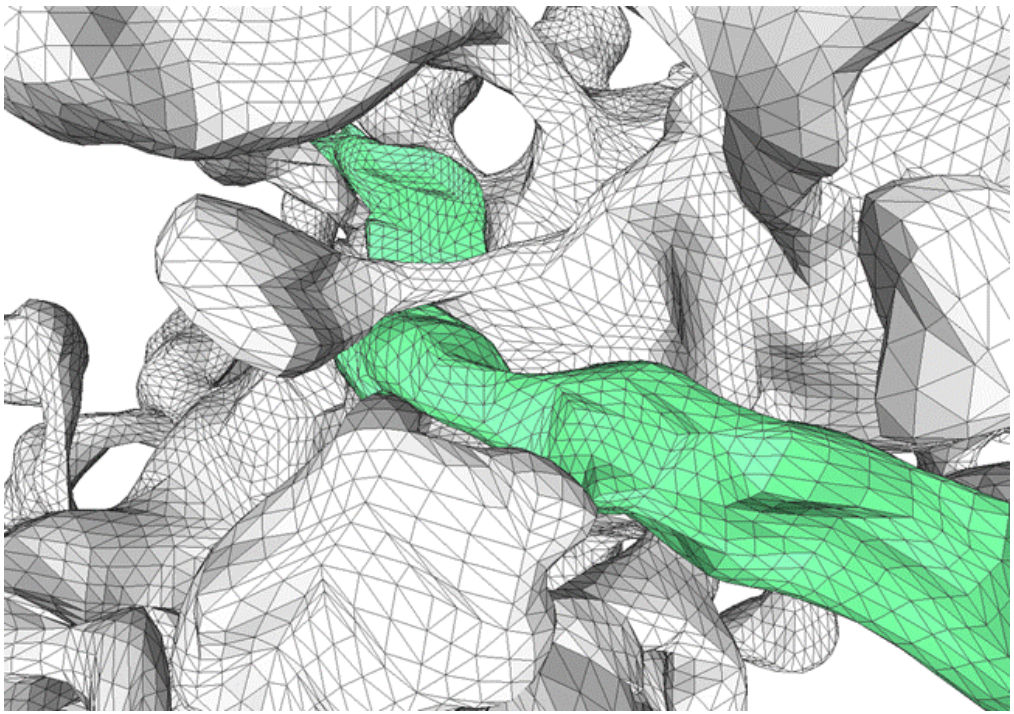
Mesh Quality Improvement



(a)



(b)

1. Tri/Tetra Quality Improvement Metrics

- Criteria:
  - Triangle: the aspect ratio = inscribed sphere radius / circumsphere radius
  - Tetrahedron:
    - Edge-ratio
    - Joe-Liu parameter
    - Minimum volume bound

- **Joe-Liu parameter:** Let $|s|$ denote the volume (which may be negative) of a given d-simplex 's', $v_i (i = 0, \ldots, d)$ denote the vertices of s, and e_{ij} denote the edges of s that connect v_i to v_j. We compute the Joe-Liu parameter of a d-simplex s as:

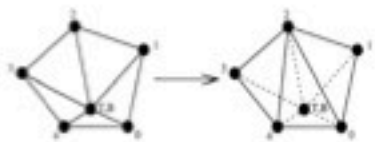
$$F(s,d) = \frac{f(s,d)}{g(s,d)} = \frac{2^{2(1-\frac{1}{d})} \times 3^{\frac{d-1}{2}} \times |s|^{\frac{2}{d}}}{\sum_{0 \le i < j \le 3} |e_{ij}|^2} \qquad (6)$$

For tetrahedral meshes, d is set to be 3. $F(s,3)$ is normalized to yield maximal value of 1 for the unit tetrahedron (with volume 1/6).

2. Tri/Tetra Quality Improvement Schemes

- Local refinement/coarsening – inserting/deleting veritces
  - refinement: edge bisection (Rivara 1997, Bajaj 1996), point insertion and templates.
  - coarsening: edge contraction
- Local remeshing - face/edge swapping (Freitag 1997)
- Mesh smoothing - relocating vertices

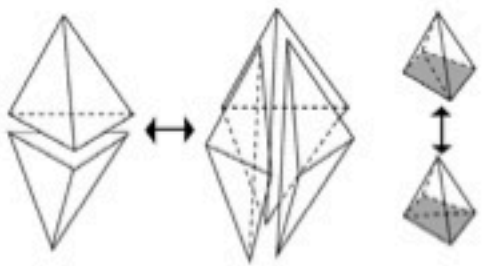
Edge swapping



– Averaging methods
  - Laplacian smoothing (Field 1988): replace the node with the average of its neighbors.
  - Constrained/weighted Laplacian smoothing (Canann 1998, Bajaj 2002)

– Optimization-based methods (Canann 1998, Freitag 1995, 1997)
  - It measures the quality of the surrounding elements to a node, and attempt to optimize by computing the local gradient of the element quality w.r.t. the node location. The node moves along the increasing gradient direction until an optimum is reached.
  - Combined Laplacian/optimization-based approach

– Physically-based methods
  - Lohner (1986) simulates the force between neighboring nodes as a system of spring.
  - Shimada (1997) and Bossen (1996) view the nodes as the center of bubbles that are repositioned to attain equilibrium. Ansiostropic meshes can be achieved.



Face swapping

Quality Improvement with Surface Diffusion Flow

- Nonlinear geometric PDEs have been used to efficiently solve surface modeling problems: surface blending, N-sided filling and free-form surface fitting [1][2]. These nonlinear equations are discretized based on discrete differential geometry operators.

$$\frac{\partial x}{\partial t} = V_n(k_1, k_2, x)N(x), \quad M(0) = M_0.$$

Where *X(t)* – a surface point on a closed surface *M(t)*

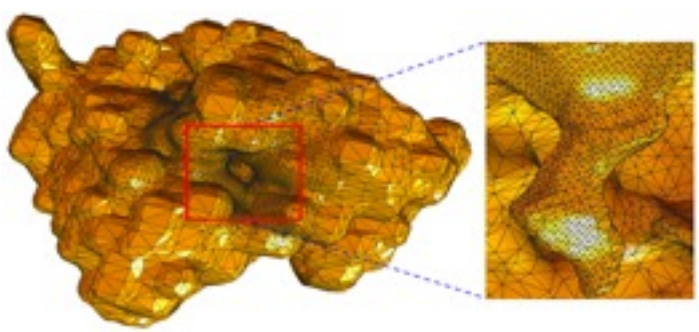
Vn(k_1, K_2, x) – the normal velocity of *M(t)*

N(x) – the unit normal of the surface at *x(t)*

- Mean Curvature Flow: *Vn = -H = -(k1 + k2)/2*
- Average Mean Curvature Flow: *Vn = h(t) – H(t)*

$$\text{where } h(t) = \int_{M(t)} H d\sigma / \int_{M(t)} d\sigma$$

- Surface Diffusion Flow: $V_n = \Delta H$
- High Order Flow: $\frac{\partial x}{\partial t} = (-1)^{k+1}\Delta^k H N(x)$



- Discretized Laplace-Beltrami operator over triangles [1][3]

$$\Delta f(p_i) = \frac{1}{A(p_i)} \sum_{j \in N_1(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} [f(p_j) - f(p_i)],$$

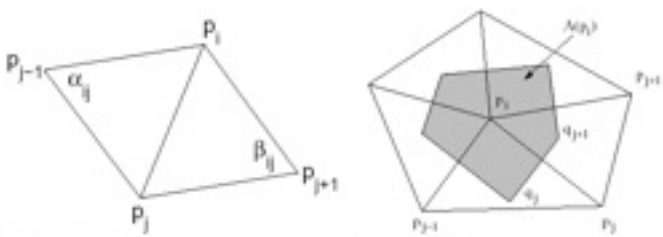
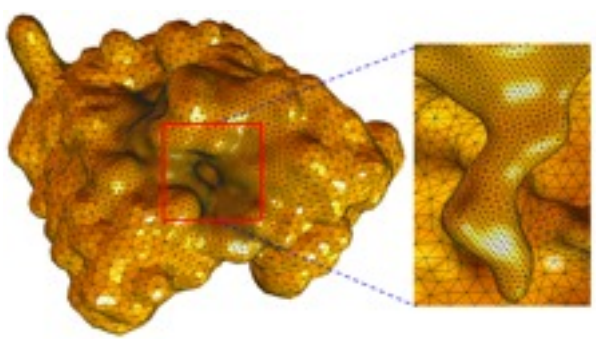


Fig 3.1: Left: The definition of the angles α_{ij} and β_{ij}. Right: The definition of the area $A(p_i)$.

$$H(p_i)N(p_i) = \frac{1}{2A(p_i)} \sum_{j \in N_1(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} (p_i - p_j)$$

$$\Delta^k f(p_i) = \Delta(\Delta^{k-1} f)(p_i) = \frac{1}{A(p_i)} \sum_{j \in N_1(i)} \frac{\cot \alpha_{ij} + \cot \beta_{ij}}{2} [\Delta^{k-1} f(p_j) - \Delta^{k-1} f(p_i)]$$



Xu G., Pan Q., Bajaj C. **Discrete Surface Modeling using Geometric Flows.** *Computer Aided Geometric Design,* **2006.**

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Surface Diffusion Flow Properties

1. Vertex normal movement:
   - The surface diffusion flow preserves volume
   - The surface diffusion flow preserves a sphere accurately if the initial mesh is close to a sphere.
2. Vertex tangent movement:
   - The tangent movement doesn't change the surface shape
   - Area-weighted averaging method, suitable for adaptive meshes.

Quality Metrics for Hex (Quads)

- Three error metrics [Kober et. al. 2000] [Knupp 2000] [Oddy et. al. 1988]

Edge vector $e_i = x_i - x,\ i = 1, \ldots, m$

Jacobian matrix $J = [e_1\, e_2\, \ldots\, e_m]$

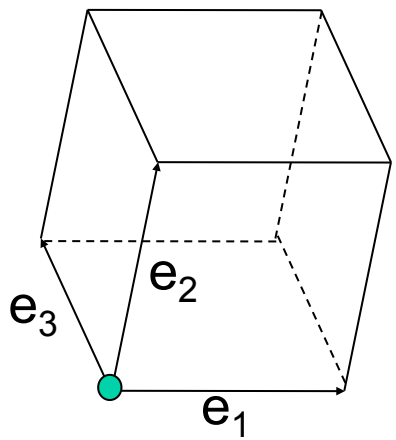
The condition number $\kappa(J) = |J||J^{-1}|$

Where $|J| = (tr(J^T J)^{1/2})$ $\quad |J^{-1}| = \frac{|J|}{det(J)}$

$$Jacobian(x) = det(J) = \sqrt{det(J^T J)} \quad (1)$$

$$\kappa(x) = \frac{1}{m}|J^{-1}||J| \quad (2)$$

$$Oddy(x) = \frac{(|J^T J|^2 - \frac{1}{m}|J|^4)}{det(J)^{\frac{4}{m}}} \quad (3)$$



- Set the condition number as the objective function, and use the conjugate gradient method to find an optimized position for a node with the worst condition number.


Quadrilateral Surface Mesh Improvement

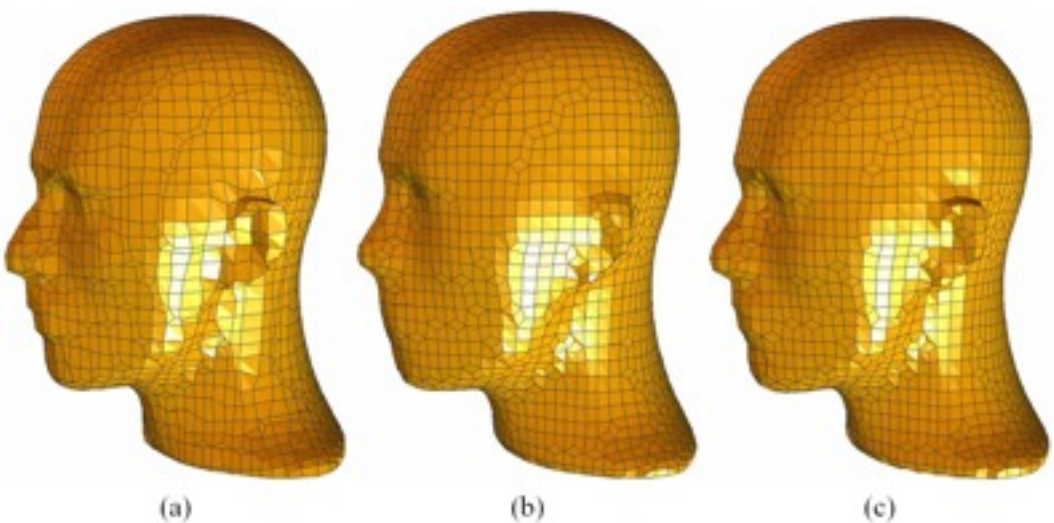
- Surface diffusion geometric flow

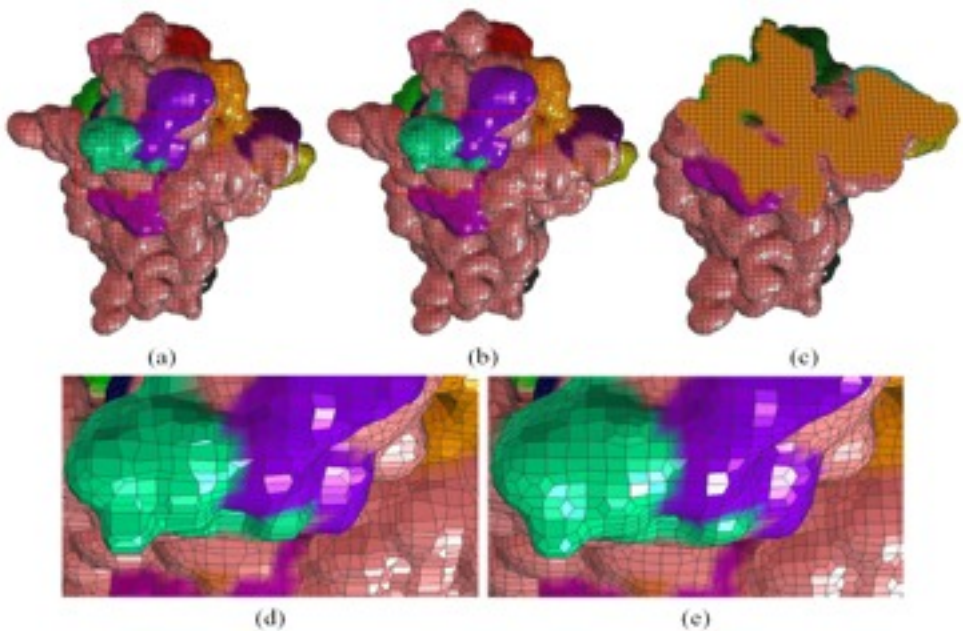
$$\frac{\partial x}{\partial t} = \Delta H(x)\mathbf{n}(x)$$

where Δ - the Laplace-Beltrami (LB) operator

H - the mean curvature

$\mathbf{n}(x)$ - the unit normal vector at the node x.





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Quad Area Calculation

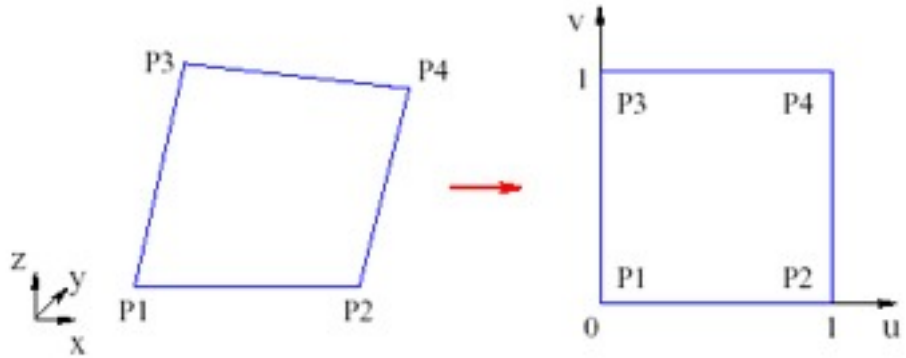


Fig. 2. A quad $[p_1p_2p_4p_3]$ is mapped into a bilinear parametric surface.

$$S(u,v) = (1-u)(1-v)p_1 + u(1-v)p_2 + (1-u)vp_3 + uvp_4.$$

The tangents:

$$S_u(u,v) = (1-v)(p_2-p_1) + v(p_4-p_3),$$
$$S_v(u,v) = (1-u)(p_3-p_1) + u(p_4-p_2).$$

The area:

$$A = \int_0^1 \int_0^1 \sqrt{\| S_u \times S_v \|^2}\, dudv = \int_0^1 \int_0^1 \sqrt{\| S_u \|^2 \| S_v \|^2 - (S_u, S_v)^2}\, dudv.$$

Four-point Gaussian quadrature rule:

$$\int_0^1 \int_0^1 f(u,v)dudv \approx \frac{f(q_1)+f(q_2)+f(q_3)+f(q_4)}{4}$$

$$q^- = \frac{1}{2} - \frac{\sqrt{3}}{6}, \qquad q^+ = \frac{1}{2} + \frac{\sqrt{3}}{6},$$

$$q_1 = (q^-, q^-), \qquad q_2 = (q^+, q^-),$$
$$q_3 = (q^-, q^+), \qquad q_4 = (q^+, q^+).$$

Discretized Laplace-Beltrami Operator (I)

$$\lim_{diam(R)\to 0} \frac{2\nabla A}{A} = \mathbf{H}(p)$$

where A is the area of a region R over the surface around the surface point p, $diam(R)$ denotes the diameter of the region R, and $\mathbf{H}(p)$ is the mean curvature normal.

$$\begin{aligned}
\nabla A &= \int_0^1 \int_0^1 \nabla \sqrt{\| S_u \|^2 \| S_v \|^2 - (S_u, S_v)^2}\, du dv \\
&= \int_0^1 \int_0^1 \frac{S_u(S_v, (v-1)S_v - (u-1)S_u))}{\sqrt{\| S_u \|^2 \| S_v \|^2 - (S_u, S_v)^2}}\, du dv \\
&+ \int_0^1 \int_0^1 \frac{S_v(S_u, (u-1)S_u - (v-1)S_v)}{\sqrt{\| S_u \|^2 \| S_v \|^2 - (S_u, S_v)^2}}\, du dv \\
&= \alpha_{21}(p_2 - p_1) + \alpha_{43}(p_4 - p_3) \\
&+ \alpha_{31}(p_3 - p_1) + \alpha_{42}(p_4 - p_2),
\end{aligned}$$

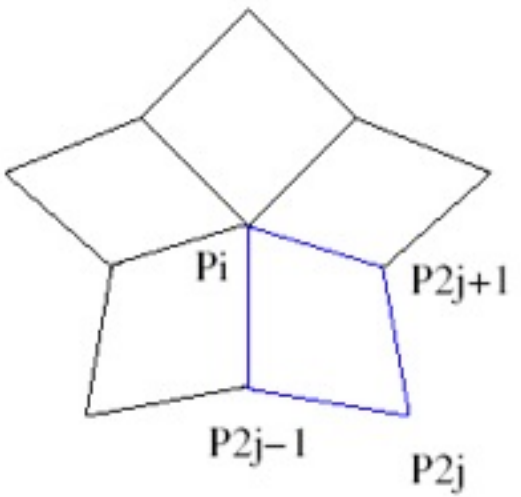
$$\alpha_1 = -\alpha_{21} - \alpha_{31}, \quad \alpha_2 = -\alpha_{21} + \alpha_{42},$$
$$\alpha_3 = \alpha_{31} - \alpha_{43}, \quad \alpha_4 = \alpha_{43} + \alpha_{42}.$$

$$\begin{aligned}
\nabla A &= \alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3 + \alpha_4 p_4 \\
&= \alpha_2(p_2 - p_1) + \alpha_3(p_3 - p_1) + \alpha_4(p_4 - p_1).
\end{aligned}$$

Discretized Laplace-Beltrami Operator (II)

$$\mathbf{H}(p_i) \approx \frac{2}{A(p_i)} \sum_{j=1}^{n} [\alpha_j^i (p_{2j-1} - p_i) + \beta_j^i (p_{2j+1} - p_i) + \gamma_{j+1}^i (p_{2j} - p_i)]$$

$$= \sum_{k=1}^{2n} w_k^i (p_k - p_i)$$



$$w_{2j}^i = \frac{2\gamma_j^i}{A(p_i)}, \quad w_{2j-1}^i = \frac{2(\alpha_j^i + \beta_{j-1}^i)}{A(p_i)}, \quad w_{2j+1}^i = \frac{2(\alpha_{j+1}^i + \beta_j^i)}{A(p_i)}.$$

Using the relation $\Delta x = 2H(p_i)$

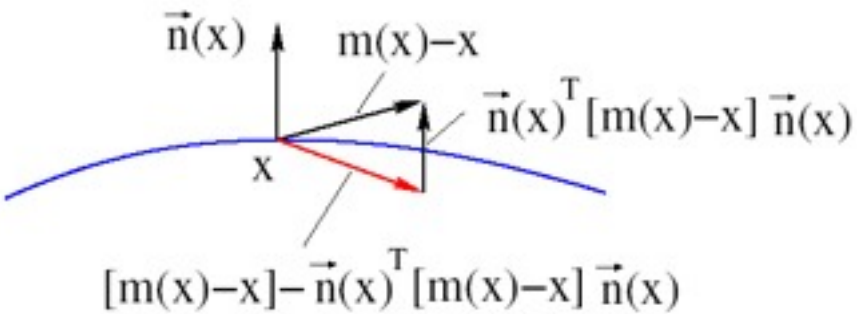
$$\Delta f(p_i) \approx 2 \sum_{k=1}^{2n} w_k^i (f(p_k) - f(p_i)).$$

$$\Delta H(p_i)\mathbf{n}(p_i) \approx 2 \sum_{k=1}^{2n} w_k^i (H(p_k) - H(p_i))\mathbf{n}(p_i)$$

$$= 2 \sum_{k=1}^{2n} w_k^i \left[\mathbf{n}(p_i)\mathbf{n}(p_k)^T \mathbf{H}(p_k) - \mathbf{H}(p_i)\right]$$

Tangent Movement

$$\frac{\partial x}{\partial t} = \Delta H(x)\mathbf{n}(x) + v(x)\mathbf{T}(x)$$



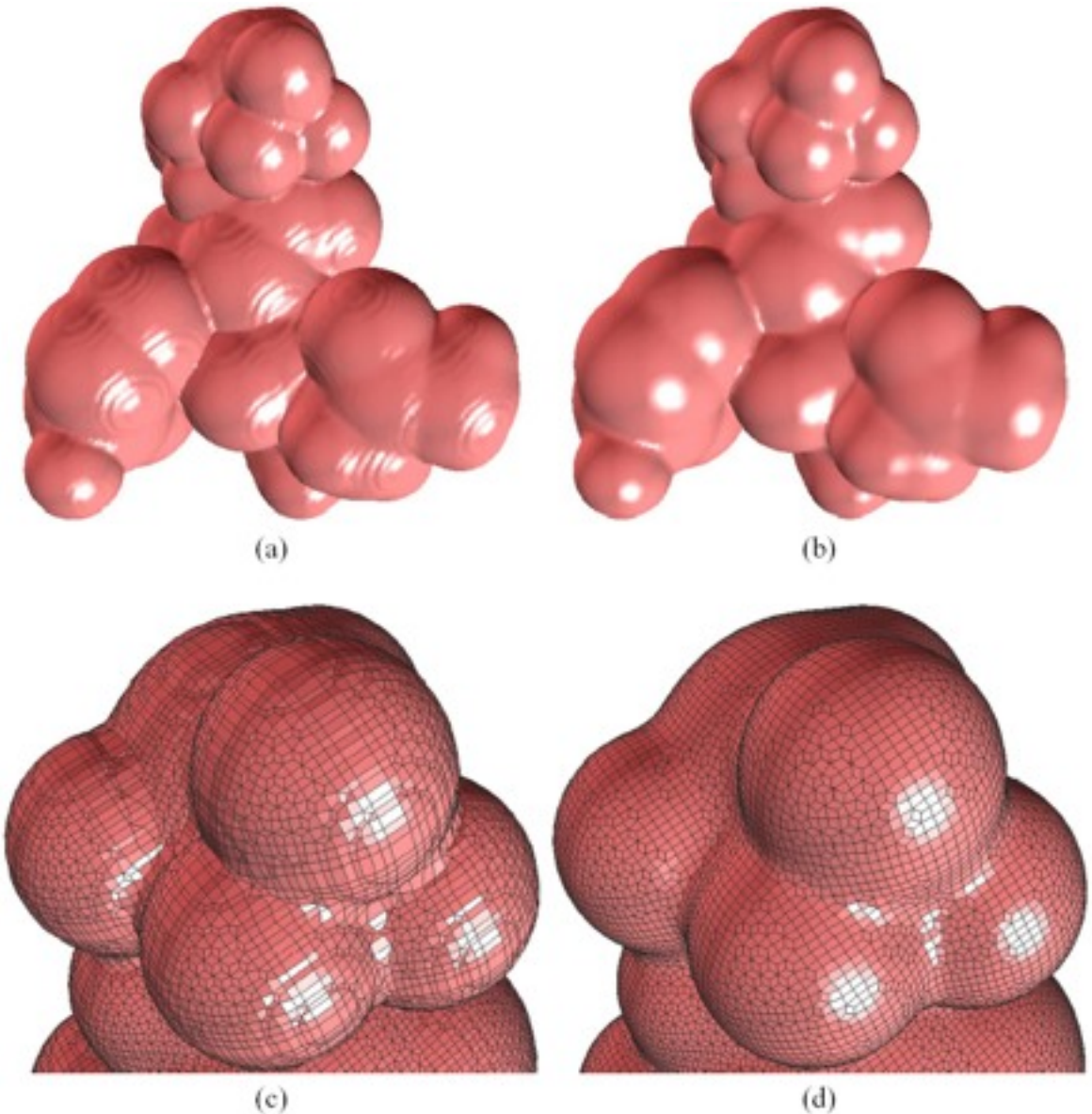
Mass center:

$$\int_S \| y - p \|^2 \, d\sigma = min$$

$$m(p_i) = \sum_{j=1}^{n} \left(\frac{p_i + p_{2j-1} + p_{2j} + p_{2j+1}}{4} A_j \right) / A^i_{total}$$

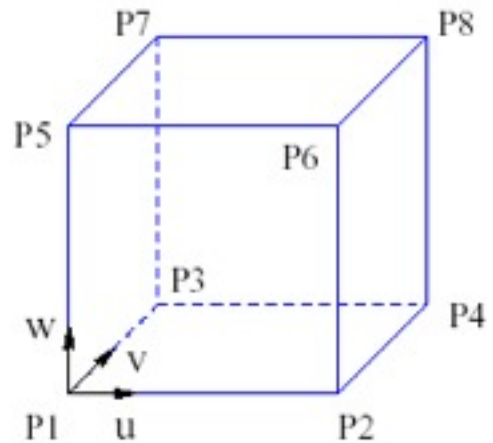
Temporal discretization:

- semi-implicit Euler scheme
- conjugate gradient iterative method



Hexahedral Volumetric Mesh

- Surface Vertex Movement
- Interior Vertex Movement



$$
\begin{aligned}
V(u,v,w) &= (1-u)(1-v)(1-w)p_1 \\
&+ u(1-v)(1-w)p_2 + (1-u)v(1-w)p_3 \\
&+ uv(1-w)p_4 + (1-u)(1-v)wp_5 \\
&+ u(1-v)wp_6 + (1-u)vwp_7 \\
&+ uvwp_8.
\end{aligned}
$$

$$
\begin{aligned}
V_u(u,v,w) &= (1-v)(1-w)(p_2-p_1) + v(1-w)(p_4-p_3) \\
&+ (1-v)w(p_6-p_5) + vw(p_8-p_7), \\
V_v(u,v,w) &= (1-u)(1-w)(p_3-p_1) + u(1-w)(p_4-p_2) \\
&+ (1-u)w(p_7-p_5) + uw(p_8-p_6), \\
V_w(u,v,w) &= (1-u)(1-v)(p_5-p_1) + u(1-v)(p_6-p_2) \\
&+ (1-u)v(p_7-p_3) + uv(p_8-p_4).
\end{aligned}
$$

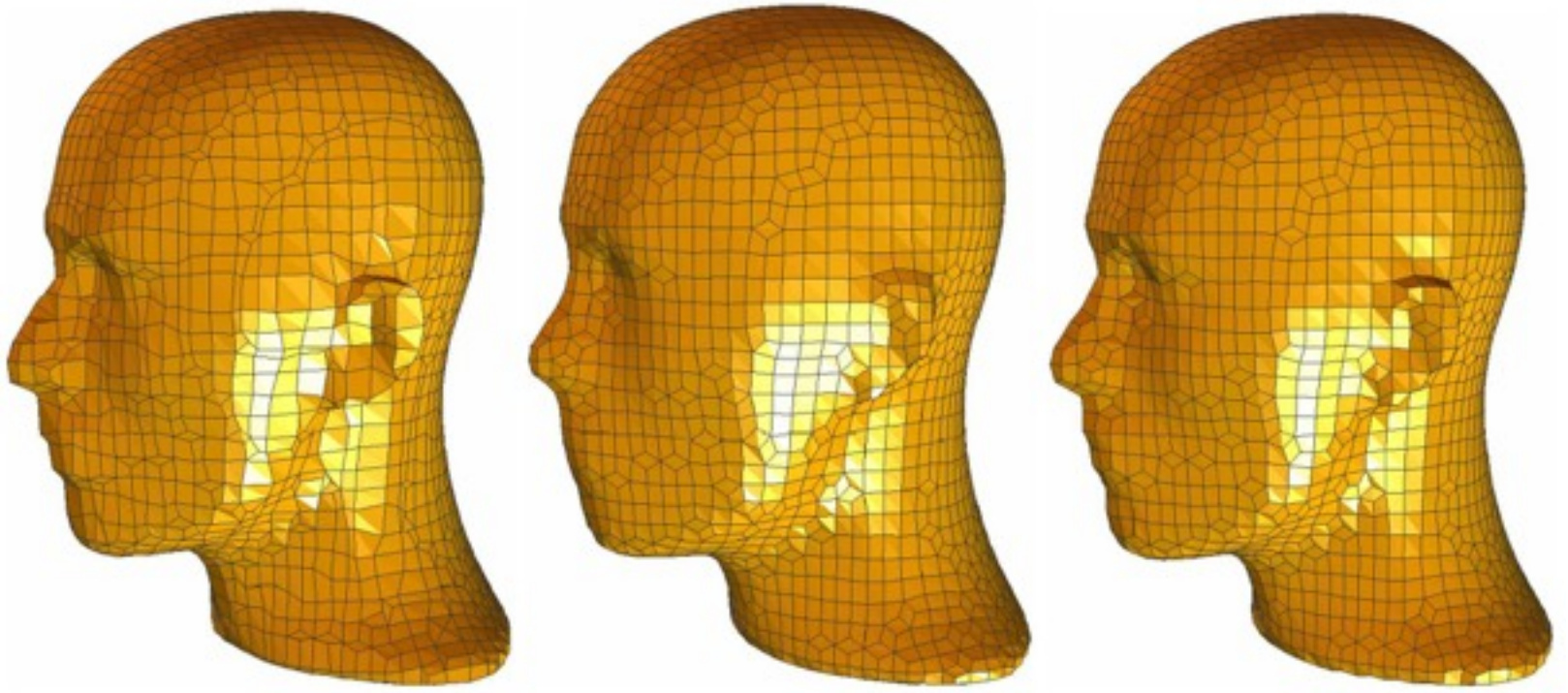
$$
V = \int_0^1 \int_0^1 \int_0^1 \sqrt{\bar{V}}\, dudvdw \qquad \bar{V} = \| (V_u \times V_v) \cdot V_w \|^2
$$

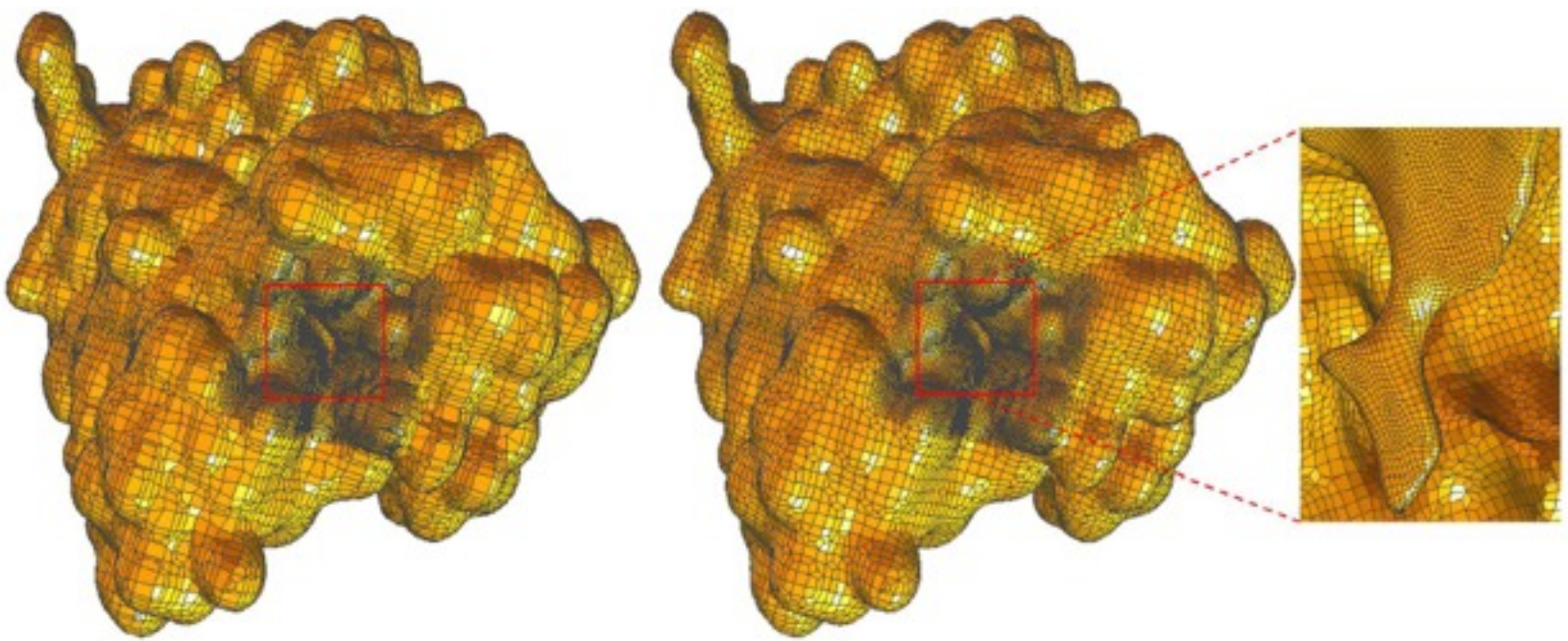
$$
\int_0^1 \int_0^1 \int_0^1 f(u,v,w)dudvdw \approx \frac{\sum_{j=1}^{8} f(q_j)}{8},
$$

$$
q^- = \frac{1}{2} - \frac{\sqrt{3}}{6}, \qquad q^+ = \frac{1}{2} + \frac{\sqrt{3}}{6},
$$

$$
\begin{aligned}
q_1 &= (q^-, q^-, q^-), & q_2 &= (q^+, q^-, q^-), \\
q_3 &= (q^-, q^+, q^-), & q_4 &= (q^+, q^+, q^-), \\
q_5 &= (q^-, q^-, q^+), & q_6 &= (q^+, q^-, q^+), \\
q_7 &= (q^-, q^+, q^+), & q_8 &= (q^+, q^+, q^+).
\end{aligned}
$$

$$
m(p_i) = \sum_{j \in N(i)} \left(\frac{1}{8} \sum_{j=1}^{8} p_j V_j\right) / V^i_{total},
$$


(a)
(b)
(c)


(a)
(b)

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