Multi-dimensional Hermite Interpolation and Approximation for Modelling and Visualization

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Abstract

In this paper we use some well known theorems of algebraic geometry in reducing polynomial Hermite interpolation and approximation in any dimension to the solution of linear systems. We present a mix of symbolic and numerical algorithms for low degree curve fits through points in the plane, surface fits through points and curves in space, and in general, hypersuface fits through points, curves, surfaces, and sub-varieties in n dimensional space. These interpolatory and (or) approximatory fits may also be made to match derivative information along all the sub varieties. Such multi-dimensional hypersurface interpolation and approximation provides mathematical models for scattered data sampled in three or higher dimensions and can be used to compute volumes, gradients, or more uniform samples for easy and realistic visualization.

1. Introduction

Interpolation and approximation provide efficient ways to fit analytic functions to sampled data. Various scientific applications [12] require data visualization as well as mathematical models to be constructed from their data samples in quite high dimensions. Motivated by computational efficiency, this paper deals with constructing mathematical models using polynomials as opposed to arbitrary analytic forms. One distinguishes between multivariate polynomial functions $\mathcal{F} : x_n = f_1(x_1, \ldots, x_{n-1})$, multivariate rational functions $\mathcal{R} : x_n = \frac{f_1(x_1,\ldots,x_{n-1})}{f_2(x_1,\ldots,x_{n-1})}$ and polynomial algebraic functions or implicitly defined hypersurfaces $\mathcal{H} : f_1(x_1, \ldots, x_n) = 0$, where all f_i are multivariate polynomials with coefficients in $\mathbb{R}$. While prior work on interpolation has dealt with multivariate polynomial functions $\mathcal{F}$ and rational functions $\mathcal{R}$, see for e.g. [1, 7–9, 11], little work has been reported on interpolation with implicitly defined hypersurfaces $\mathcal{H}$. This paper extends the results of [5, 6] of two and three dimensions to arbitrary dimensions. See also [4] which summarizes prior work on implicit surface interpolation in three dimensions and provides several additional references.

One primary motivation for considering implicit hypersurfaces is the extra degrees of freedom that are available, while manipulating polynomials of the same degree. In $\mathbb{R}^n$ if we consider only manipulating polynomials of degree d then functions $\mathcal{F}$ and $\mathcal{R}$ have

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respectively $\binom{d+n-1}{d}$ and $2\binom{d+n-1}{d}-1$ degrees of freedom In comparison, the hypersurface $\mathcal{H}$ has the considerably larger $\binom{d+n}{d}-1$ degrees of freedom.

The rest of the paper is as follows. Section 2 establishes the mathematical notation and facts that are required later in the paper. Section 3 presents several motivating examples along with computational and implementation details of the multivariate interpolation and approximation algorithms. In sections 4 and 5, we present details of the multivariate Hermite interpolation and approximation algorithms, which generalizes the usual curve fits through points in the plane and surface fits through both points and curves in space to general hypersurface fits through points, curves, surfaces, and any sub-varieties upto dimension $n-2$ in n dimensional space together with the matching of specified derivative information along the sub-varieties. Section 6 presents the use of weighted least-squares approximation for hypersurface selection from the families of Hermite interpolants. Finally, section 6 provides computational and implementation details of the multivariate interpolation and approximation algorithms and also presents several examples.

2. Preliminaries

In this section we review some basic definitions and theorems from algebraic geometry that we shall be using in subsequent sections. These and additional facts can be found for example in [13, 14].

The set of real and complex solutions (or *zero* set $Z(S)$) of a collection S of polynomial equations

$$\begin{aligned} \mathcal{H}_1 : f_1(x_1, ..., x_n) &= 0 \\ &\cdots \\ \mathcal{H}_m : f_m(x_1, ..., x_n) &= 0 \end{aligned} \tag{1}$$

with coefficients in $\mathbb{R}$ is referred to as an *algebraic set.* The algebraic set defined by a single equation ($m = 1$) is also known as a hypersurface. A algebraic set that cannot be represented as the union of two other distinct algebraic sets, neither containing the other, is said to be *irreducible.* An irreducible algebraic set is also known as an algebraic variety V.

A hypersurface in $\mathbb{R}^n$, an n dimensional space, is of *dimension* $n-1$. The *dimension* of an algebraic variety V is k if its points can be put in $(1,1)$ rational correspondence with the points of an irreducible hypersurface in $k+1$ dimensional space. An algebraic set $Z(S)$ on the other hand may have irreducible components or *sub-varieties* of different dimension. An algebraic set is called *unmixed* if all of its sub-varieties are of the same dimension, and *mixed* otherwise. The *dimension* of the algebraic set $Z(S)$ is considered the maximum dimension of any of its sub-varieties. An algebraic variety of dimension 1 is also called an *algebraic space curve* and of dimension 2 is also called an *algebraic surface.* The following two lemmas summarize the resulting dimension of intersections of varieties and sub-spaces of different dimensions.

Lemma 2.1 *In $\mathbb{R}^n$, an n dimensional space, a variety V_1 of dimension k intersects a general sub-space $\mathbb{R}^{n-k+h}$, with $k > h$, in a variety V_2 of dimension h.*

Lemma 2.2 *In $\mathbb{R}^n$, a variety V_1 of dimension k intersects a a variety V_2 of dimension h, with $h \geq n-k$, in an algebraic set $Z(S)$ of dimension at least $h+k-n$.*

In the above lemma, the resulting intersection is termed *proper* if all subvarieties of $Z(S)$ are of the same minimum dimension $h+k-n$. Otherwise the intersection is termed *excess* or *improper*.

The *degree* of an algebraic hypersurface is the maximum number of intersections between the hypersurface and a line, counting both real and complex intersections and at infinity. This degree is also the same as the degree of the defining polynomial. A degree 1 hypersurface is also called a *hyperplane*. The *degree* of an algebraic space curve is the maximum number of intersections between the curve and a hyperplane, counting both real and complex intersections and at infinity The *degree* of a variety V of dimension h in $\mathbb{R}^n$ is the maximum number of intersections between V and a sub-space $\mathbb{R}^{n-h}$, counting both real and complex intersections and at infinity. The *degree* of an *unmixed* algebraic set is the sum of the degrees of all its sub-varieties.

The following theorem, perhaps the oldest in algebraic geometry, summarize the resulting degree of intersections of varieties of different degrees.

Theorem 2.1 (Bezout) *A variety of degree d which* properly *intersects a variety of degree e does so either in an algebraic set of degree at most $d * e$ or infinitely often.*

The *normal* or *gradient* of a hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ is the vector $\nabla f = (f_{x_1}, f_{x_2}, \dots f_{x_n})$. A point $\mathbf{p} = (a_0, a_1, \dots a_n)$ on a hypersurface is a *regular* point if the gradient at $\mathbf{p}$ is not null; otherwise the point is *singular*. A singular point $\mathbf{q}$ is of multiplicity e for a hypersurface $\mathcal{H}$ of degree d if any line through $\mathbf{q}$ meets $\mathcal{H}$ in at most $d-e$ additional points. Similarly a singular point $\mathbf{q}$ is of multiplicity e for a variety V in $\mathbb{R}^n$ of dimension k and degree d if any sub-space $\mathbb{R}^{n-k}$ through $\mathbf{q}$ meets V in at most $d-e$ additional points. It is important to note that even if two varieties intersect in a *proper* manner, their intersection in general may consist of sub-varieties of various multiplicites. The total degree of the intersection, however is bounded by the above Bezout's theorem.

3. Computational Details

In this section, we discuss some computational aspects of Hermite interpolation and approximation, and give several examples of design, modelling and computer graphics visualization. The basic method followed is:

1. properties of a desired hypersurface interpolant are described in terms of a combination of points, curves, surfaces, etc., with possibly associated "normal" directions,

2. these properties are translated into a homogeneous linear system of equations $\mathbf{M_I x} = \mathbf{0}$

3. nontrivial solutions of the above system are computed and a desirable solution hypersurface is selected by weighted least-squares approximation from additional points or simple hypersurfaces, *minimize* $= \| \mathbf{M_A x} - \mathbf{b} \|^2$.

We therefore solve the following, simultaneous interpolation and weighted least-squares approximation problem below.

$$\begin{array}{ll} \textit{minimize} & \| \mathbf{M_A x} - \mathbf{b} \|^2 \\ \textit{subject to} & \mathbf{M_I x} = \mathbf{0}, \end{array}$$

where $\mathbf{M_I} \in \mathbb{R}^{n_i \times q}$ is the Hermite interpolation matrix, and $\mathbf{M_A} \in \mathbb{R}^{n_a \times q}$ and $b \in \mathbb{R}^{n_a}$ are matrix and vector, respectively, for contour level approximation, and $\mathbf{x} \in \mathbb{R}^q$ is a vector containing coefficients of a degree d algebraic hypersurface $f(x_1, \ldots, x_n) = 0$. Hence, $q = \binom{d+n}{n})$.

3.1. Computing Nontrivial Interpolation Solutions

We first solve the linear system $\mathbf{M_I x} = \mathbf{0}$, in a computationally stable manner, by computing the singular value decomposition (SVD) of $\mathbf{M_I}$ [10]. Hence, $\mathbf{M_I}$ is decomposed as $\mathbf{M_I} = U \Sigma V^T$ where $U \in \mathbb{R}^{n_i \times p}$ and $V \in \mathbb{R}^{q \times q}$ are orthonormal matrices, and $\Sigma = diag(\sigma_1, \sigma_2, \cdots, \sigma_r) \in \mathbb{R}^{n_i \times q}$ is a diagonal matrix with diagonal elements $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r \geq 0$ $(r = min\{n_i, q\})$. It can be proved that the rank s of $\mathbf{M_I}$ is the number of the positive diagonal elements of Σ, and that the last $q - s$ columns of V span the null space of $\mathbf{M_I}$. Hence, the nontrivial solutions of the homogeneous linear system are expressed as :

$\{\mathbf{x} (\neq \mathbf{0}) \in \mathbb{R}^q \,|\, \mathbf{x} = \sum_{i=1}^{q-s} w_i \cdot \mathbf{v}_{s+i},$ *where* $w_i \in \mathbb{R}$, *and* $\mathbf{v}_j$ *is the jth column of* $V\}$.
Furthermore, $\mathbf{x} = V_{q-s}\mathbf{w}$ compactly expresses all the degree d hypersurfaces which satisfy the Hermite interpolation constraints.

3.2. Computing Least Squares Approximation

To solve the simultaneous interpolation and weighted least-squares approximation problem, we reduce $\| \mathbf{M_A x} - \mathbf{b} \| = \| \mathbf{M_A} V_{q-s}\mathbf{w} - \mathbf{b} \|$, by subsituting the basis of solution hypersurfaces $\mathbf{x} = V_{q-s}\mathbf{w}$, which satisfy the interpolation constraints. Then, an orthogonal matrix $Q \in \mathbb{R}^{n_a \times n_a}$ is computed such that

$$Q^T \mathbf{M_A} V_{q-s} = R = \begin{pmatrix} R_1 \\ \mathbf{0} \end{pmatrix}$$

where $R_1 \in \mathbb{R}^{(q-s)\times(q-s)}$ is upper triangular. (This factorization is called a *Q-R factorization* [10]). Now, let

$$Q^T b = \begin{pmatrix} c \\ d \end{pmatrix}$$

where c is the first $q - s$ elements. Then, $\| \mathbf{M_A} V_{q-s}\mathbf{w} - \mathbf{b} \|^2 = \| Q^T \mathbf{M_A} V_{q-s}\mathbf{w} - Q^T\mathbf{b} \|^2$ $= \| R_1\mathbf{w} - c \|^2 + \| d \|^2$. The solution $\mathbf{w}$ can be computed by solving $R_1\mathbf{w} = c$, from which the final fitting surface is obtained as $\mathbf{x} = V_{q-s}\mathbf{w}$.

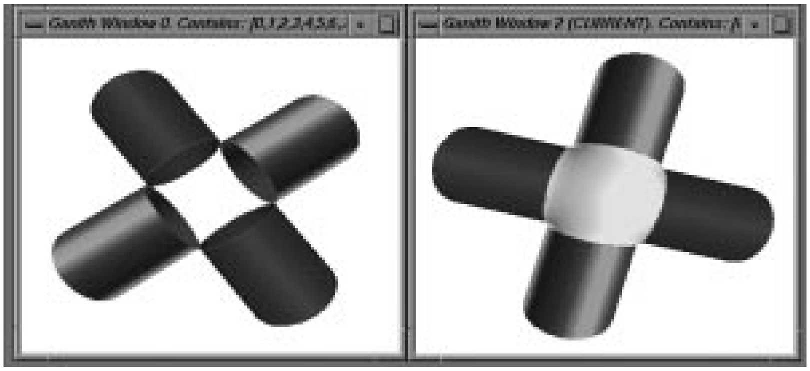

Figure 1. A C^1 Join of Cylinders with a Quartic Surface

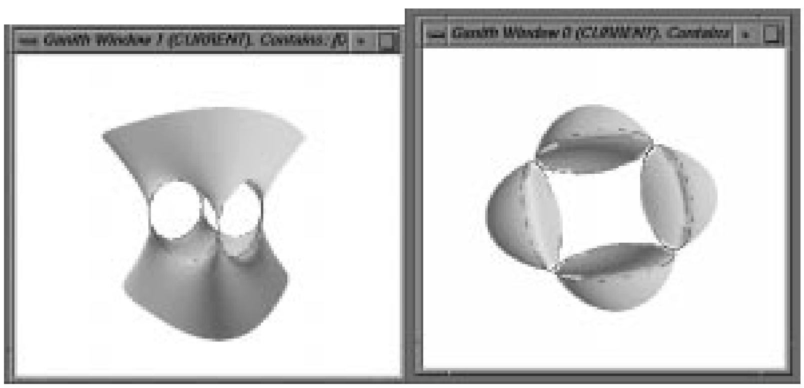

Figure 2. (left) A Good C^1 Join Quartic Surface (right) A Degenerate C^1 Join Quartic Surface

3.3. Examples

These examples were generated using the special case of Hermite interpolation in three and four dimensional space.

Example 3.1 *A Quartic Surface for Joining Four Orthogonal Cylinders*

Here Hermite interpolation yields a quartic surface which smoothly joins the orthogonal cylinders. The input is defined by $CYL_1 : x^2+y^2-1=0$ for $z \geq 1$, $CYL_2 : x^2+y^2-1=0$ for $z \leq -1$, $CYL_3 : y^2+z^2-1=0$ for $x \geq 1$, and $CYL_4 : y^2+z^2-1=0$ for $x \leq -1$.

Algorithms 1 and 2 produce 64 linear equations from the input, and so $\mathbf{M_I} \in \mathbb{R}^{64\times 35}$. The Σ in the SVD[2] of $\mathbf{M_I}$ tells us that the rank of $\mathbf{M_I}$ is 33, and the null space of $\mathbf{M_I}$ is $\mathbf{x} = r_1 \cdot \mathbf{v}_34 + r_2 \cdot \mathbf{v}_{35}$. Hence, the Hermite interpolating surface is $f(x,y,z) = r_1 * (-0.239737466 * x^2 - 0.632096672 * y^2 - 0.239737466 * z^2 + 0.239737466 - 0.038155435 * x^4 + 0.316048336 * x^2 * y^2 + 0.316048336 * x^2 * z^2 + 0.354203771 * y^4 + 0.316048336 * y^2 *$

[2]The subroutine dsvdc of Linpack was used to compute the SVD of a matrix.

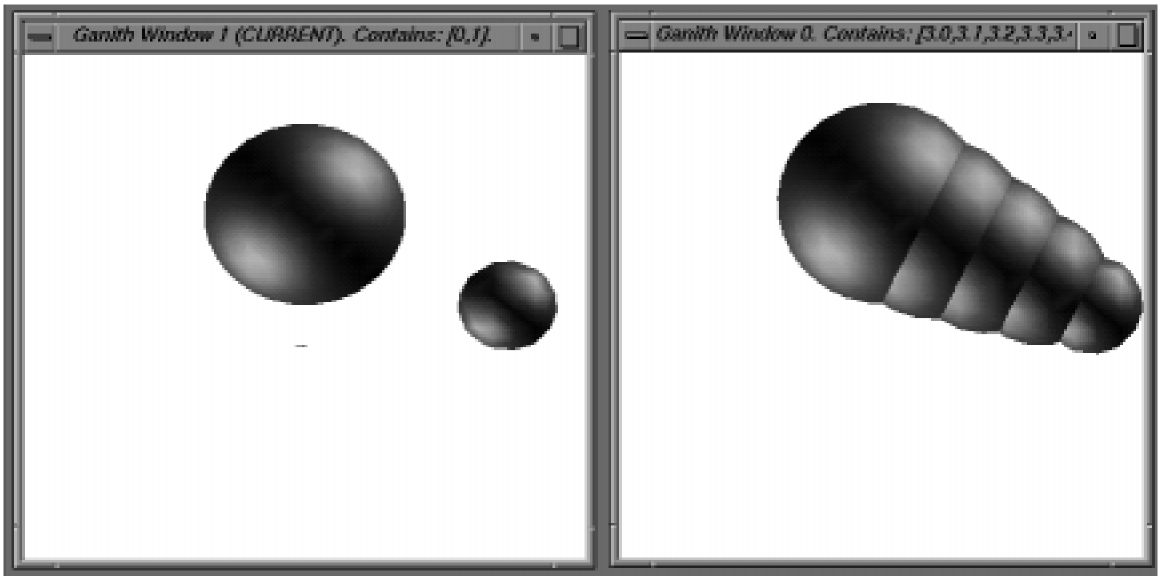

Figure 3. A Quartic HyperSurface Interpolant of Two Spheres

$z^2 - 0.038155435 * z^4) + r_2 * (-0.470209927 * x^2 + 0.170367911 * y^2 - 0.470209927 * z^2 + 0.470209927 + 0.277696942 * x^4 - 0.085183956 * x^2 * y^2 - 0.085183956 * x^2 * z^2 - 0.362880897 * y^4 - 0.085183956 * y^2 * z^2 + 0.277696942 * z^4) = 0$ which has one remaining independent degree of freedom. Suitable solutions are obtained by computing values for the parameters, r_1 and r_2, which yield a Hermite interpolating surface which is also least-squares approximate from some given, simple algebraic surface.

The selected interpolating surface is shown in the left part of Figure 2 and its use is shown in Figure 1. The least-squares approximating surface used was an ellipsoid $x^2 + 0.2 * y^2 + z^2 - 1 = 0$, which yielded values $r_1 = -0.458394$ and $r_2 = -0.746194$. It should be noted that not every instance of the Hermite interpolant is always appropriate for the geometric design application on hand. The degenerate surface shown in the right part of Figure 2, is not useful (for several reasons) for the joining task shown in Figure 1, however is in the same Hermite interpolating family with $r_1 = -0.899483$ and $r_2 = -0.436957$ and was produced for the least-squares approximating surface $x - z = 0$.

Example 3.2 *A Quartic Hypersurface Interpolant of Two Spheres*

The input data consists of the unit two dimensional sphere given by $[(x-4)^2 + y^2 + z^2 - 1 = 0,\ w = 0]$ centered at $(4, 0, 0, 0)$ and the unit two dimensional sphere given by $[x^2 + (y-2)^2 + z^2 - 4,\ w = 2]$ centered at $(0, 2, 0, 2)$. The rank of the $15x32$ interpolation matrix is 13 and the two parameter, hypersurface interpolant family in four dimensional space is given by $f(x, y, z, w) = r_1 * (0.834299517 + -0.444959743 * (x^1) + 0.055619968 * (x^2) + 0.055619968 * (y^2) + 0.055619968 * (z^2) + 0.222479871 * (x^1 * w^1) + -0.111239936 * (y^1 * w^1) + -0.166859903 * (w^2) + -0.083429952 * (w^1)) + r_2 * (0.447213595 * (w^2) + -0.894427191 * (w^1))$ Figure 3, shows isosurfaces of the interpolating family with $r_1 = 1$ and $r_2 = -1$ and for values of $w = 0, 0.5, 1, 1.5, 2$.

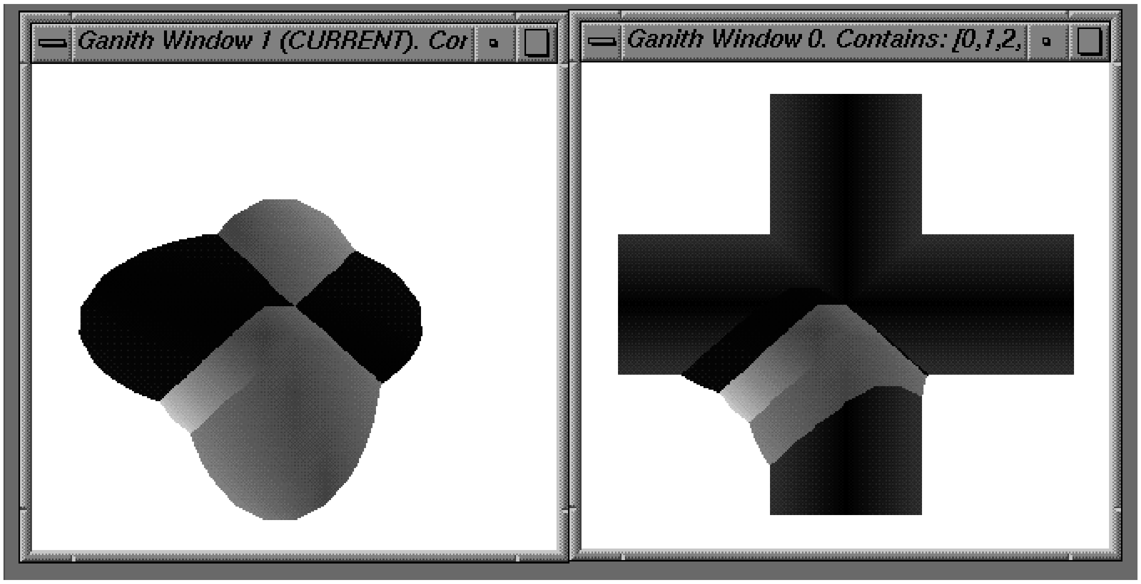

Figure 4. A Quartic HyperSurface Interpolant of Two Orthogonal Cylindrical Surfaces

Example 3.3 *A Quartic Hypersurface Interpolant of Two Orthogonal Cylindrical Surfaces*

The input data consists of the two dimensional cylinders given by $[x^2 + z^2 - 1 = 0, w = 0]$ and $[y^2 + z^2 - 1 = 0, w = 2]$. The rank of the 21X58 interpolation matrix is 18 and the three parameter, hypersurface interpolant family in four dimensional space is given by $f(x, y, z, w) = r_1 * (0.534522210 * (1) + -0.534522210 * (x^2) + -0.534522210 * (z^2) + 0.000904961 * (x^1 * w^1) + 0.267261105 * (x^2 * w^1) + -0.267261105 * (y^2 * w^1) + 0.000018271 * (y^1 * w^1) + -0.000452481 * (x^1 * w^2) + -0.000009135 * (y^1 * w^2)) + r_2 * (-0.000537584 * (1) + 0.000537584 * (x^2) + 0.000537584 * (z^2) + 0.886712345 * (x^1 * w^1) + -0.000268792 * (x^2 * w^1) + 0.000268792 * (y^2 * w^1) + 0.117219484 * (y^1 * w^1) + -0.443356173 * (x^1 * w^2) + -0.058609742 * (y^1 * w^2)) + r_3 * (-0.000060052 * (1) + 0.000060052 * (x^2) + 0.000060052 * (z^2) + 0.117219442 * (x^1 * w^1) + -0.000030026 * (x^2 * w^1) + 0.000030026 * (y^2 * w^1) + -0.886712802 * (y^1 * w^1) + -0.058609721 * (x^1 * w^2) + 0.443356401 * (y^1 * w^2))$

Figure 3, shows isosurfaces of the interpolating family with $r_1 = 2$, $r_2 = -1$ and $r_3 = 1$ and for values of $w = 0, 0.5, 1, 1.5, 2$.

Example 3.4 *Locally supported triangular C^1 interpolants for smoothing polyhedra*

The input is an arbitrary genus polyhedron. First a single normal is chosen at each vertex endpoint, a necessary condition for obtaining a globally C^1 smooth polyhedra. Next a wireframe of cubics are constructed where each cubic replaces an edge and C^1 interpolates the corresponding vertices of the edge. Furthermore, normals are constructed for each curvilinear cubic edge of the wireframe and varying cubically along the edge. See the left part of Figure 5.

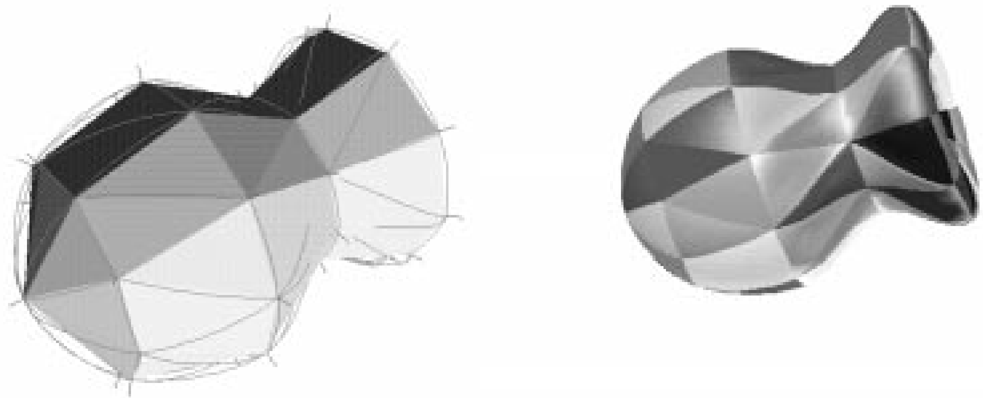

Figure 5. (left) An input polyhedron with a C^1 cubic curve wireframe (right) A smooth object with locally supported triangular C^1 interpolants

The Hermite interpolation algorithm then constructs triangular C^1 interpolants - a 5 parameter family of degree 7 surfaces, one family per triangular facet of the wireframe. Instances of degree 7 surface patches generated for this example are displayed in the right part of Figure 5.

4. Generalized Lagrange Interpolation

Our first problem deals with constructing C^0 interpolatory hypersurfaces.

Problem 4.1 *Construct a single real algebraic hypersurface $\mathcal{H}$ in $\mathbb{R}^n$ which C^0 interpolates a collection of l_1 points $\mathbf{p}_i$, and l_k sub-varieties V_{j_k} of dimension k, $k = 1 \ldots n-2$ and degree $e[k]_{j_k}$.*

Since a point is a variety of dimension 0 and hypersurfaces in $\mathbb{R}^n$ are of dimension $n-1$, we note from Lemma 2.2 that a hypersurface in general will not contain a given point. However, the hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d, can be made to contain, i.e. C^0-interpolate, the point $\mathbf{p_i}$ if the coefficients of f satisfy the linear equation $f(\mathbf{p_i}) = 0$.

From Lemma 2.2 we note that a hypersurface in $\mathbb{R}^n$ will always intersect all sub-varieties of dimension h, for $h = 1 \ldots n-2$. To increase the dimension of the intersection or more precisely, to ensure that the hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d completely contains (i.e. C^0-interpolates) a sub-variety V of dimension h and degree $e[h]$ we use the following algorithm:

Algorithm 1 *1. Select any set L_V of $n_V = \binom{d*e[h]+h}{h}$ points on V, $L_V = \{\mathbf{p}_i = (x_i[i], \dots x_n[i]) | i = 1, \dots, n_V\}$. The set L_V may be computed by a straightforward generalization of computing random points on algebraic curves and surfaces. See [3] for discussion of such techniques.*

2. Next, set up n_V homogeneous linear equations $f(\mathbf{p}_j) = 0$, for $\mathbf{p}_j \epsilon L_V$. Any nontrivial solution of this linear system will represent an $\mathcal{H}$ which interpolates the entire subvariety V.

Correctness Proof: The proof of correctness of the above algorithm follows from Bezout's theorem 2.1. By Bezout's theorem $\mathcal{H}$ intersects V in a sub-variety of degree at most $d * e[h]$ and dimension $(n-1) + h - n = h - 1$. By making $\mathcal{H}$ contain $n_V = \binom{d*e[h]+h}{h}$ points of V, implies that $\mathcal{H} \cap V$ could also define a subvariety of dimension $h-1$ and degree greater than $d * e[h]$. This ensures that $\mathcal{H}$ must intersect V infinitely often and since V is irreducible, $\mathcal{H}$ must contain V. ♠

The irreducibility of the sub-variety is not a restriction, since an algebraic set can be handled by treating each irreducible component separately. The situation is more complicated in the real setting, if we wish to achieve separate containment of one of possibly several connected real components of a single sub-variety. There is first of course the nontrivial problem of specifying a single isolated real component of the sub-variety. One solution to the problem of interpolating only with a single real component, is given in [5] and uses weighted least squares approximation from additional data. See also [2] where a solution to isolating real components of varieties is derived in terms of a decomposition of space into sign-invariant cylindrical cells.

For the collection of l_1 points $\mathbf{p}$, and l_k sub-varieties V_{j_k} of dimension k, $k = 1 \dots n-2$ and degree $e[k]_{j_k}$ the above C^0 interpolation with a degree d hypersurface $\mathcal{H}$, yields a system $\mathbf{M_I}$ of $\sum_{k=1}^{n-1} l_k + \sum_{k=2}^{n-1} \sum_{j_k=1}^{l_k} \binom{d*e[k]_{j_k}+k}{k}$ linear equations. Remember $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d has $K = \binom{n+d}{n} - 1$ independent coefficient unknowns. C^0-interpolation of the entire collection of sub-varieties is achieved by selecting an algebraic hypersurface of the smallest degree n such that $K \geq r$, where r $(\leq k)$ is the rank of the system $\mathbf{M_I}$ of linear equations.

5. Generalized Hermite Interpolation

An algebraic hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ is said to Hermite interpolate or C^1-interpolate a sub-variety V with associated derivative or "normal" information $\mathbf{n}(\mathbf{p}) = (n_{x_1}(\mathbf{p}), \dots n_{x_n}(\mathbf{p}))$, defined for points $\mathbf{p} = (x_1, \dots x_n)$ on V if :

1. (containment condition) $f(\mathbf{p}) = 0$ for all points $\mathbf{p} = (x_1, \dots x_n)$ of V.

2. (tangency condition) $\nabla f(\mathbf{p})$ is not identically zero and $\nabla f(\mathbf{p}) = \alpha \mathbf{n}(\mathbf{p})$, for some $\alpha \neq 0$ and for all points $\mathbf{p} = (x_1, \dots x_n)$ of V.

Our second problem then deals with constructing C^1 interpolatory hypersurfaces.

Problem 5.1 *Construct a single real algebraic hypersurface $\mathcal{H}$ in $\mathbb{R}^n$ which C^1 interpolates a collection of l_1 points $\mathbf{p}_i$ with associated "normal" unit vectors $\mathbf{n}_i(\mathbf{p}_i)$, and l_k sub-varieties V_{j_k} of dimension k with $k = 1 \ldots n-2$ and degree $e[k]_{j_k}$ together with associated "normal" unit vectors $\mathbf{n}[k]_{j_k}$ for all points on each sub-variety of the given collection.*

In the previous section we have already shown that the containment condition reduces to solving a system of linear equations. We now prove that meeting the tangency condition for C^1-interpolation reduces to solving an additional set of linear equations.

A hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d, satisfies the tangency condition at the point $\mathbf{p}_i$ if the coefficients of f satisfy, without loss of generality, the $n-1$ homogeneous linear equations

$$n_{x_1} \cdot f_{x_j}(\mathbf{p}_i) - n_{x_j} \cdot f_{x_1}(\mathbf{p}_i) = 0 \quad j = 2 \ldots n \tag{2}$$

For the above equations we assumed, without loss of generality, that $n_{x_1} \neq 0$ as the given normal $\mathbf{n}$ is not identically zero at any point. To verify that the above equations correctly satisfy the tangency condition, it suffices to choose $\alpha = \frac{f_{x_1}}{n_{x_1}}$ for then each of the $f_{x_j} = \alpha n_{x_j}$. Also note that for the choice of $n_{x_1} \neq 0$, it must occur that $f_{x_1}(\mathbf{p}_i) \neq 0$, and hence $\alpha \neq 0$, for otherwise the entire $\nabla f(\mathbf{p})$ is identically zero.

To ensure that a hypersurface $\mathcal{H} : f(x_1, ..., x_n) = 0$ of degree d meets the tangency condition for C^1-interpolation of a sub-variety V of dimension h and degree $e[h]$ we use the following algorithm:

Algorithm 2

1. *Select a set of L_{NV} of $n_{NV} = \binom{(d-1)*e[h]+h}{h}$ point-normal pairs $[\mathbf{p}_j, \mathbf{n}[h]_j]$ on V where $\mathbf{p}_i \epsilon L_V$, with point set L_V on V computed to meet the containment condition.*

2. *Substitute each point-normal pair in L_{NV} into the $n-h-1$ equations*

   $$n_{x_1} \cdot f_{x_i}(\mathbf{p}) - n_{x_i} \cdot f_{x_1}(\mathbf{p}) = 0 \quad i = 2 \ldots (n-h) \tag{3}$$

   to yield additionally $(n-h-1) * n_{NV}$ linear equations in the coefficients of the $f(x, y, z)$.

Correctness Proof: The proof of correctness of the above algorithm follows from the following. We first note that even though each of the equations 3 above is evaluated at only $n_{NV} = \binom{(d-1)*e[h]+h}{h}$ points of V it holds for all points on V. Each equation (3) defines an algebraic hypersurface T of degree $(d-1)$ which intersects V of degree $e[h]$ in a sub-variety of degree at most $(d-1)e[h]$ and dimension $h-1$. Invoking Bezout's theorem, and from the irreducibility of V, it follows that V must lie entirely on the hypersurface T. Hence each equation (3) is satisfied along the entire sub-variety V.

We now show that the $n-h-1$ equations 3 satisfies the tangency condition as specified earlier. Again we assume, without loss of generality, that $n_{x_1} \neq 0$ as the given normal $\mathbf{n}$ is not identically zero at along V_h. Note that the containment i.e. C^0 interpolation of the dimension h variety V_h by the hypersurface $\mathcal{H}$ already guarantees that the h tangent directions on V_h at each point $\mathbf{p}$ of V_h are identical to h tangent directions of $\mathcal{H}$ at $\mathbf{p}$ on $\mathcal{H}$. Hence h components of the given normal vector $\mathbf{n}(\mathbf{p})$ (orthogonal to the tangent directions of V_h) are already matched with h components of the gradient vector $\nabla f(\mathbf{p})$ (orthogonal to the tangent directions of $\mathcal{H}$). Assume, without loss of generality, that these vector components are $f_{x_i} = \alpha n_{x_i}$, $i = (n-h+1)\dots n$, for any non-zero α. The remaining $n-h$ components of $\nabla f(\mathbf{p})$ of $\mathcal{H}$ are then matched up with the $n-h-1$ equations 3 as follows. Let $\alpha = \frac{f_{x_1}}{n_{x_1}}$. Then from the $n-h-1$ equations 3 we note that each of the $n-h-1$ $f_{x_i} = \alpha n_{x_i}$, $i = 2\dots(n-h)$ as required. Hence the entire vector $\nabla f(\mathbf{p}) = \alpha \mathbf{n}(\mathbf{p})$. Also note that for the choice of $n_{x_1} \neq 0$, it must occur that $f_{x_1}(\mathbf{p}_i) \neq 0$, and hence $\alpha \neq 0$, for otherwise the entire $\nabla f(\mathbf{p})$ is identically zero. ♠

For the collection of l_1 points $\mathbf{p}$, and l_k sub-varieties V_{j_k} of dimension k, $k = 1\dots n-2$ and degree $e[k]_{j_k}$ to achieve the tangency condition with a degree d hypersurface $\mathcal{H}$, requires satisfying an additionally system of $(n-1)*l_1 + \sum_{k=2}^{n-1}\sum_{j_k=1}^{l_k}(n-k-1)*\binom{(d-1)*e[k]_{j_k}+k}{k}$ linear equations. For C^1 interpolation we obtain a single homogeneous system $\mathbf{M_I}$ of linear equations consisting of the linear equations for C^0 interpolation of section 4 together with the above linear equations. Any non-trivial solution of this linear system $\mathbf{M_I}$, for which additionally ∇f is not identically zero for all points of the collection, (that is, the hypersurface $\mathcal{H}$ is not singular at all points or along any of the subvarieties V_k), will represent a hypersurface which Hermite interpolates the collection.

6. Least Squares Approximation

The result of a Hermite interpolation of a collection of sub-varieties with associated normals, is a family of algebraic hypersurfaces $f(x_1,\dots,x_n) = 0$ with extra degrees of freedom. This family is expressed as the nontrivial coefficients vectors in the nullspace of $\mathbf{M_I}$. To select a suitable hypersurface from the family, values must be specified for these extra degrees of freedom. We show that least squares approximation to additional points or sub-varieties around the original data can be used for selecting a suitable non-singular hypersurface from the family. Let $S_0 = \{v_i \in \mathbb{R}^n | i = 1,\cdots,l\}$ be a set of points which approximately describes a desirable hypersurface. (These points can be selected for example from a degree two hypersphere, or a hyperparaboloid etc., centered around the orginal data). A linear system $\mathbf{M_A x} = \mathbf{0}$, where each row of $\mathbf{M_A}$ is constructed from the linear conditions $f(v_i) = 0$ with $\mathbf{x}$ containing the undetermined coefficients of the family. Conventional least squares approximation is to minimize $\| \mathbf{M_A x} \|^2$ over the nullspace of $\mathbf{M_I}$. Though minimizing $\| \mathbf{M_A x} \|^2$ does yield a good distance approximation it does not prevent the resulting hypersurface from self-intersecting, pinching or splitting.

To rid our solution hypersurfaces of such singularities and provide more geometric control, we instead approximate a monotonic multivariate function $w = f(x_1,\dots,x_n)$ rather than just the implicit hypersurface $f(x_1,\dots,x_n) = 0$, the zero contour of the function. From simple degree two hypersurfaces we first generate $S_0 = \{(v_i, n_i) | i = 1,\cdots,l\}$ where v_i are approximating points, and n_i are approximating gradient vectors at v_i in $\mathbb{R}^n$. Then, from this set, we construct two more sets $S_1 = \{u_i | u_i = v_i + \alpha n_i, i = 1,\cdots,l\}$, and $S_{-1} = \{w_i | w_i = v_i - \alpha n_i, i = 1,\cdots,l\}$ for some small $\alpha > 0$. Next we set up the least squares system $\mathbf{M_A} = \mathbf{b}$ from the following three kinds of equations :

$f(v_i) = 0$, $f(u_i) = 1$, and $f(w_i) = -1$. These equations give an approximating contour level structure of the function $w = f(x_1, \ldots, x_n)$ near the orginal data. We found out that forcing well behaved contour levels rids the selected hypersurfaces of self-intersection in the spatial region enclosed by the points.

7. Conclusion

There are numerous open problems in the theory and application of multivariate interpolation. The primary problem amongst these stems from the non-uniqueness of interpolants in two and higher dimensions. There is an acute need for techniques of selecting a suitable candidate solution for the given input data, from the $K - r$ parameter family of C^1 interpolating hypersurfaces of degree d in n dimensional space. Here $K = \binom{n+d}{n} - 1$ and r is the rank of the system $\mathbf{M}$ of linear equations. We are still experimenting with the use of weighted least squares approximation on additional constructed data coupled with the interpolation of the given input data set as presented in sections 6, 3. One difficulty of the selection problem is exhibited in the right part of Figure 2 of example 3.1 of the previous section, where a certain choice of the approximating surface yields a degenerate joining solution in real space. This joining solution leaves a hole in the join and furthermore obstructs the holes of the original cylinders. Other difficulties arise from ensuring that the selected solution is also smooth (non-singular) in the domain of the input data. We are actively pursuing schemes which will give more user control in selecting desirable solutions of multivariate Hermite interpolation systems.

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