

# NURBS Approximation of Surface/Surface Intersection Curves

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## Abstract

We use a combination of both symbolic and numerical techniques to construct a degree bounded  $C^k$ -continuous, rational B-spline  $\epsilon$ -approximations of real algebraic surface-surface intersection curves. The algebraic surfaces could be either in implicit or rational parametric form. At singular points we use the classical Newton power series factorizations to determine the distinct branches of the space intersection curve. Besides singular points we obtain an adaptive selection of regular points about which the curve approximation yields a small number of curve segments yet achieves  $C^k$  continuity between segments. Details of the implementation of these algorithms and approximation error bounds are also provided.

## 1 Introduction

It is well known that the set of parametric algebraic curves and surfaces are a subset of algebraic curves and surfaces of the same degree. In particular the intersection curve of two parametric surfaces may not be parametric. See [2] for a discussion of these facts and the desired use of parametric representations for certain geometric modeling and display operations. To compute parametric representations for non-parametric algebraic intersection curves requires approximation. In this paper we present algorithms to construct a piecewise rational B-spline (a degree bounded, piecewise parametric,  $C^k$  continuous) approximation to a space curve, which comes from the intersection of two implicitly defined surfaces (IIS); or the intersection of two parametric defined surfaces (IPS). Though we restrict discussions to implicit and parametric algebraic surface-surface intersections,

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the algorithms can be directly extended to the intersection curve of arbitrary analytic surfaces. The results of this paper are a natural extension of piecewise rational approximation of non-parametric algebraic plane curves[5]. Piecewise rational B-spline approximations find increasing use in interactive geometric design[20] and in the contouring of scattered data[22].

There are mainly two solution approaches to the handling of the piecewise rational B-spline approximation problem. One is based on the subdivision of the enclosing three dimensional space into small cells where the intersection curve is evaluated in each cell by first transforming it into Bernstein-Bezier (BB) form. For a topologically correct approximation this method can only cope with a restricted class of point singularities with distinct tangents. The other method of tackling the approximation problem is based on a tracing of the intersection curve[7, 21] without any special consideration to singularities. In this paper, we too adaptively march along the intersection curve, paying special heed to singular points, followed by a corresponding stitching together of the approximating rational B-spline curve segments. We construct a piecewise rational B-spline approximations to two kinds of space intersection curves (IIS, IPS). Except at singular points, the composite curve keeps a simpler variant of Frenet frame continuity based on the curve arc length as the parameter. Among the various local parameterizations of the space curve, taking arc length as parameter has several advantages as indicated in section 3.

In recent years, several authors have discussed the notion of geometric continuity at a common point for two incident space curves[13, 15, 24, 25]. Furthermore there are a plenty of references which use different continuity criteria and construct parametric B-splines to approximate an ordered list of points (see for e.g. [9, 11, 25]). The frame continuity used in this paper is a simpler form of geometric continuity with the connection matrix being diagonal, and differs from the well known Frenet-frame continuity that has a lower triangular connection matrix. We also exhibit how Padé approximation can be adapted to yield very natural Hermite approximations.

This paper is organized as follows: Section 2 formally defines the piecewise approximation problem and an outline of our piecewise rational B-spline approximation algorithm. Section 3 presents some mathematical preliminaries and also motivates our choice of frame continuity based on arc length parameterizations. Section 4 and 5 discuss the expansion of the two different surface-surface intersection curves into power series based on arc length. In Section 6, we illustrate how to use Newton iterations to cope with the solution of under-determined system of non-linear equations at regular curve points and over-determined system of non-linear equations at singularities. Section 7 presents details of several approaches to construct a rational parametric curve segment which is  $C^k$  frame continuous at end points. Here we extend traditional Padé approximation techniques [19] to develop a two point Padé rational parametric curve interpolant. We also utilize the transformation technique of [6] to convert the piecewise rational curve segments into piecewise rational B-splines and standard NURBS representations. Sections 8 and 9 treat the problem of isolating singular points on the intersection curve and the use of Newton factorizations to construct the approximation of the distinct branches of the curve at these singular points. Finally, in Section 10 we discuss the implementation of our algorithm and present several examples.

The transformation of the piecewise rational approximation into piecewise rational B-splines and standard NURBS representations are treated in a separate paper[6].

## 2 The Problem and the outline of the Algorithm

The Rational Approximation Problem. Given a real intersection space curve  $SC$  which is either

- (a) the intersection of implicit surfaces (IIS) defined by  $f_1(x, y, z) = 0$ ,  $f_2(x, y, z) = 0$ , and within a bounding box  $B = \{(x, y, z) : x_0 \leq x \leq x_1, \quad y_0 \leq y \leq y_1, \quad z_0 \leq z \leq z_1\}$  or
- (b) the intersection of parametric surfaces(IPS) defined by

$$\begin{aligned} X_1(u_1, v_1) &= [G_{11}(u_1, v_1) \ G_{21}(u_1, v_1), \ G_{31}(u_1, v_1)]^T \\ X_2(u_2, v_2) &= [G_{12}(u_2, v_2) \ G_{22}(u_2, v_2), \ G_{32}(u_2, v_2)]^T \end{aligned}$$

and within a bounding box

$$B = \{(u_1, v_1, u_2, v_2) \ : \ \begin{array}{ll} u_{10} \leq u_1 \leq u_{11}, & v_{10} \leq v_1 \leq v_{11} \\ u_{20} \leq u_2 \leq u_{21}, & v_{20} \leq v_2 \leq v_{21} \end{array}\}$$

and an error bound  $\epsilon > 0$ , a continuity index  $k$ , construct a  $C^k$  (or  $G^k$ ) continuous piecewise parametric rational  $\epsilon$ -approximation of all portions of  $SC$  within the given bounding box  $B$ .

The Outline of the Algorithm. The approximation process is a tracing procedure along the curve. It consists of the following steps:

1. Form a starting point list (SPL) by computing the boundary points containing the intersection points of the curve  $SC$  and the bounding box  $B$ . Further the SPL is made to contain at least one point for each inner loop component of  $SC$  i.e. a curve loop completely inside the given box  $B$ . Tracing directions are also provided at each of these points in the SPL. (See section 10 for implementation details).
2. Test if SPL is empty. If yes, the tracing is finished. Otherwise, starting from a point  $p$  in SPL, trace the curve along the given direction until either of the following tests in step 3 or step 4 are true. The tracing step consists of the following sub-steps:
  - (a). Compute an arc length based power series expansion (see sections 4 and 5) up to  $k + 1$  terms at the given point  $p$ .
  - (b). Determine a step-length and a point  $\hat{q}$  on the above expansion curve in the tracing direction within a step-length of  $p$ , and then starting from  $\hat{q}$  refine to a new point  $q$  on the curve  $SC$  by Newton iterations (see section 6).
  - (c). Compute an arc length power series expansion up to  $k + 1$  terms at the new point  $q$ .
  - (d). Construct an approximating rational parametric curve segment by  $C^k$  Hermite interpolation (see section 7 ) of the two end points  $p$  and  $q$ .
  - (e). Add the rational curve approximant into the piecewise approximation list and return to step 2, to continue the tracing from the newly constructed point  $q$ .

3. Test if a singular point is met. If yes, stop the present tracing and put the end point of the tracing into SPL (we may delete a few approximation segments from the present approximation list, because the step length near a singular point is small). Then locate the singular point(see section 8), and obtain a finite set of power series expansion at the singular point corresponding to the distinct curve branches (see section 9). Trace each branch one or two steps, and then put the end points of the tracing into SPL. Then return to Step 2.
4. Test if another point in SPL is met(see section 10). If yes, we have stitched together one continuous segment of the curve. Delete the two end points of the traced segment from SPL and return to Step 2.

### 3 Mathematical Preliminaries

In this paper, we will express a space curve as a power series, locally at a point and with its arc length as a parameter. We refer to [10] for some intrinsic parameters of space curves. Let  $r(s) = [x(s), y(s), z(s)]^T$  be a space curve, where  $s$  is arc length of the curve measured from some fixed point. The tangent vector  $t(s) = r'(s)$  has unit length;  $k(s) = \|r''(s)\|$  is the curvature, where  $\|\cdot\|$  is the Euclidean norm in  $\mathbb{R}^3$ . Further,  $n(s) = r''(s)/k(s)$  is the principle normal;  $b(s) = t(s) \times n(s)$  is the binormal, where  $\times$  denotes the cross product of two vectors. Finally, the number  $T(s)$  defined by  $b'(s) = -T(s)n(s)$  is the torsion. The three orthogonal vectors  $t(s)$ ,  $n(s)$  and  $b(s)$  form the so called Frenet frame. These vectors are related by the Frenet formulas:  $t' = kn$ ,  $b' = -Tn$ ,  $n' = -kt + Tb$ . The derivatives of  $r(s)$  are therefore given by

$$r'(s) = t, \quad r''(s) = kn, \quad r'''(s) = k'n + kTb - k^2t \quad (3.1)$$

Since  $t = r'(s)$ ,  $k = \|r''(s)\|$  and  $T = r'(s) \times r''(s) \cdot r'''(s) / \|r''(s)\|^2$  then the curve is obviously tangent, or curvature or torsion continuous if  $r'(s)$ , or  $r'(s)$  and  $r''(s)$ , or  $r'(s)$ ,  $r''(s)$  and  $r'''(s)$  is continuous respectively. In this paper, we construct a piecewise approximation of the given curve such that the composite curve is tangent ( $t(s)$ ), normal ( $n(s)$ ) and binormal ( $b(s)$ ) continuous.

Among the various local parameterizations of the space curve, taking arc length as parameter has several advantages.

- A. If  $r(s)$  is the parameterization of the given curve and  $s$  is arc length starting from some point, then  $r'(s)$ ,  $r''(s)$ ,  $r'''(s)$  is equivalent to  $t(s)$ ,  $n(s)$ ,  $b(s)$  in the sense that the continuity of  $r'(s)$ ,  $r''(s)$ ,  $r'''(s)$  are equivalent to the continuity of  $t(s)$ ,  $n(s)$ ,  $b(s)$  where the triple  $t(s)$ ,  $n(s)$ ,  $b(s)$  is the Frenet frame. Therefore, we need only to force the composite curve's first three derivatives to be continuous at the break points without considering the connection matrix as in the case of geometric continuity.
- B. Since the arc length of the curve is independent of any coordinate system, then the expansion of power series may have larger convergence radius. This will, in turn, lead to fewer segments of approximation.

- C. From the geometric point of view, Frenet frame continuity is the most natural and useful requirement. It keeps the tangent, principle normal and binormal varying continuously, while other types of geometric continuity cannot achieve this conclusion.

## 4 Local Expansion of the Intersection Curve of Implicit Surfaces

Let  $f_1(p), f_2(p)$  be two algebraic polynomials with  $p = [x, y, z]^T \in \mathbb{R}^3$ . The intersection of implicitly defined surfaces(IIS)  $f_1(p_1) = 0, f_2(p_1) = 0$  is defined by  $f_1(p) = f_2(p) = 0$ . In this paper, we assume the defining surfaces are smooth, i.e., the normals of the surfaces are not equal to zero at any point on the surface in the region considered. Now let  $F(p) = [f_1(p), f_2(p)]^T, p_0 \in \mathbb{R}^3$  be a point on the intersection curve  $r(s)$ , where  $s$  is the arc length measured from  $p_0 = r(0)$  with prescribed direction. Then, as in [7],  $r'(0), r''(0)$  and  $r'''(0)$  are computed as follows:

$$F(r)(s) = F(r)(0) + s \frac{dF(r)(0)}{ds} + \frac{s^2}{2!} \frac{d^2F(r)(0)}{ds^2} + \dots \quad (4.1)$$

where

$$\frac{d^k F(r)(0)}{ds^k} = V_k(0) + \nabla F(p_0) r^{(k)}(0) \quad (4.2)$$

$$V_1(s) = 0, \quad V_k(s) = V_{k-1}'(s) + [\nabla F(r)]' r^{(k-1)}(s), \quad k = 1, 2, \dots \quad (4.3)$$

and  $\nabla F(p) = \left[ \frac{\partial F(p)}{\partial x}, \frac{\partial F(p)}{\partial y}, \frac{\partial F(p)}{\partial z} \right] \in \mathbb{R}^{2 \times 3}$ . It follows from  $F(r(s)) \equiv 0$  that

$$\nabla F(p_0) r^{(k)}(0) = -V_k(0) \quad (4.4)$$

The system of equation (4.4) has three unknowns and two equations. It has in general infinitely many solutions. Now we assume  $\nabla f_1(p_0)$  and  $\nabla f_2(p_0)$  are linearly independent and illustrate how to get  $r^{(k)}(0)$  such that the equations in the last section are satisfied. Let  $t$  be a vector such that

$$t \in \nabla F(p_0)^\perp, \quad \|t\| = 1 \quad (4.5)$$

and its sign is so chosen that  $t$  gives the correct direction along the same line we are going. Then for any vector  $x \in \mathbb{R}^3$  there exist unique  $\alpha \in \mathbb{R}$  and  $y \in \text{range}(\nabla F(p_0)^T)$ , such that  $x = \alpha t + y$ . Let

$$r^{(k)}(0) = \alpha_m t + \nabla F(p_0)^T \beta_m \quad (4.6)$$

Then by (4.4), we have  $\beta_m$  is uniquely defined by

$$\nabla F(p_0) \nabla F(p_0)^T \beta_m = -V_k(0) \quad (4.7)$$

and  $\alpha_m$  is arbitrary. Now we determine  $\alpha_m$  ( $m = 1, \dots, 4$ ), such that  $r^{(m)}(0)$  ( $m = 1, \dots, 4$ ) satisfy (3.1).

- A.  $\mathbf{m} = 1$ .** Since  $V_1(0) = 0$ , then  $\beta_1 = 0$ . Hence  $r'(0) = \alpha_1 t$ . According to the definition of  $t$ , we choose  $\alpha_1 = 1$ .
- B.  $\mathbf{m} = 2$ .** Since we want the  $r''(0)$  orthogonal to  $r'(0) = t$ , i.e.,  $r''(0) \in \text{range}(\nabla F(p_0)^T)$ , the only choice is  $\alpha_2 = 0$ . We then have  $k = \|r''(0)\|$ .
- C.  $\mathbf{m} = 3$ .** It follows from (3.1) that  $k'n + kTb \in \text{range}(\nabla F(p_0)^T)$ . Then  $\alpha_3 = -k^2$ , and further  $k' = r''(0)^T r'''(0)/k$ .
- D.  $\mathbf{m} = 4$ .** From (3.1),  $r^{(4)}(s) = (k'' - kT^2 - k^3)n + [k'T + (kT)']b - 3kk't$ . Then  $\alpha_4 = -3kk'$ . Finally we obtain the approximate expansion  $r(s) \approx \sum_{i=0}^4 (r^{(i)}(0)/i!)s^i$ .

## 5 Local Expansion of the Intersection Curve of Parametric Surfaces

Let

$$\begin{aligned} X_1(u_1, v_1) &= [G_{11}(u_1, v_1) \ G_{21}(u_1, v_1), \ G_{31}(u_1, v_1)]^T \\ X_2(u_2, v_2) &= [G_{12}(u_2, v_2) \ G_{22}(u_2, v_2), \ G_{32}(u_2, v_2)]^T \end{aligned}$$

be two parametric surface, where  $G_{ij}$  are given smooth functions. The intersection curve of the parametric surface (IPS) is defined by  $r(s) = X_1(u_1(s), v_1(s))$  (or  $X_2(u_2(s), v_2(s))$ ) with  $X_1(u_1(s), v_1(s)) = X_2(u_2(s), v_2(s))$  where the parameter  $s$  is the arc length measured from some point on the curve. Let  $Q_1 = (u_1, v_1)^T$ ,  $Q_2 = (u_2, v_2)^T$ , and  $Q_1^*, Q_2^*$  be the points in  $\mathbb{R}^2$  such that  $X_1(Q_1^*) = X_2(Q_2^*)$ . At point  $X_1(Q_1^*)$ , we want to expand  $r(s)$  into power series  $r(s) = r(0) + r'(0)s + \frac{r''(0)}{2}s^2 + \dots$ . On the curve  $r(s)$ ,  $Q_1$  and  $Q_2$  are functions of  $s$ , we can express them as

$$Q_j(s) = \sum_{i=0}^{\infty} \frac{Q_j^{(i)}(0)}{i!} s^i, \quad j = 1, 2 \quad (5.1)$$

As the case of IIS, we expand  $X_j(Q_j(s))$  (for  $j = 1, 2$ )

$$X_j(Q_j(s)) = X_j(Q_j(0)) + \frac{dX_j(Q_j)(0)}{ds} s + \frac{d^2 X_j(Q_j)(0)}{ds^2} \frac{s^2}{2} + \dots$$

$$\frac{d^k X_j(Q_j)(0)}{ds^k} = V_{kj}(0) + \nabla X_j(Q_j^*) Q_j^{(k)}(0), \quad j = 1, 2 \quad (5.2)$$

where

$$V_{1j}(s) = 0, \quad j = 1, 2, \quad V_{kj}(s) = \frac{d}{ds} V_{k-1,j}(s) + \frac{d}{ds} [\nabla X_j(Q_j)] Q_j^{(k-1)}(s)$$

By  $X_1(Q_1(s)) \equiv X_2(Q_2(s))$ , we have  $\nabla X_1 Q_1^{(m)} - \nabla X_2 Q_2^{(m)}(0) = V_{m1}(0) - V_{m2}(0)$ . Let  $n_i = X_{iu_i} \times X_{iv_i}$ ,  $X_{iu_i} = \left[ \frac{\partial G_{1i}}{\partial u_i}, \frac{\partial G_{2i}}{\partial u_i}, \frac{\partial G_{3i}}{\partial u_i} \right]$ . Then  $n_1, n_2$  are the normals of the two surfaces. Suppose

$n_1$  and  $n_2$  are linearly independent. Let  $t \in \mathbb{R}^3$  such that  $t \in [n_1 n_2]^\perp$ ,  $\|t\| = 1$ , and its sign is properly chosen such that it points to the correct direction. Then we have the expression

$$\begin{aligned} \nabla X_1 Q_1^{(m)}(0) + V_{m1}(0) &= \nabla X_2 Q_2^{(m)}(0) + V_{m2}(0) \\ &= \alpha_m t + [n_1, n_2] \beta_m \end{aligned} \quad (5.3)$$

Since  $n_1^T \nabla X_1 = 0$ ,  $n_2^T \nabla X_2 = 0$ , we have from (5.3)

$$[n_1, n_2]^T [n_1, n_2] \beta_m = \begin{bmatrix} n_1^T V_{m1} \\ n_2^T V_{m2} \end{bmatrix}. \quad (5.4)$$

Therefore  $\beta_m$  is uniquely determined by the nonsingularity of the matrix  $[n_1, n_2]^T [n_1, n_2]$ , and  $\alpha_m$  is arbitrary. From (5.3), we determine  $\alpha_m$ , such that  $r^{(m)}(0) = \alpha_m t + [n_1, n_2] \beta_m$ . This can be done exactly the same as the case of IIS by regarding  $[n_1, n_2]$  as  $\nabla F(p_0)^T$ .

After  $r^{(m)}(0)$  are received, we can compute  $Q_j^{(m)}(0)$ ,  $j = 1, 2$ . From (5.3),

$$\nabla X_j^T \nabla X_j Q_j^{(m)}(0) = \nabla X_j^T (r^{(m)}(0) - V_{mj}(0)), \quad j = 1, 2. \quad (5.5)$$

Solving these equations, we get  $Q_j^{(m)}(0)$ . The purpose of computing  $Q_j^{(m)}(0)$  is to compute the approximate value of  $Q_j(s)$  by (5.1). This approximate value serves as the initial value for the Newton method to get an accurate value  $Q_j$  on the curve.

## 6 Newton Iterations

While tracing a surface-surface intersection curve  $SC$ , at simple (regular) points of  $SC$  we need to solve an undetermined nonlinear system that has more unknowns than equations. At singular points on the curve, we need to solve an overdetermined nonlinear system that has more equations and less unknowns. Consider in general an arbitrary system of nonlinear equations  $F(x) = [f_1(x_1, \dots, x_m) \cdots f_n(x_1, \dots, x_m)]^T = 0$ . We need to determine a solution of the system  $F(x) = 0$  by Newton iterations from a given initial value  $p_0 \in \mathbb{R}^m$ . In our tracing procedure these initial values are points on the local expansion curves, within an adaptively computed step length. These initial values are then refined back to the original intersection curve  $SC$  to yield the actual interpolating points for the rational curve segment approximation. The Newton iteration used is

$$\nabla F(p_k) \Delta_k = -F(p_k), \quad p_{k+1} = p_k + \Delta_k \quad (6.1)$$

where  $\nabla F = \left[ \frac{\partial F}{\partial x_1} \quad \frac{\partial F}{\partial x_2} \quad \cdots \quad \frac{\partial F}{\partial x_m} \right] = \left[ \nabla f_1 \quad \cdots \quad \nabla f_n \right]^T$  is a  $n \times m$  matrix.

**Case A:  $m = n + 1$ .** Here equation (6.1) is a under-determined linear system. Suppose the set of  $\nabla f_i$  is linearly independent, then the general solution of (6.1) is

$$\Delta_k = \alpha_k t + \nabla F(p_k)^T \beta_k \quad (6.2)$$

where  $t \in \nabla F(p_k)^\perp$ ,  $\alpha_k \in \mathbb{R}$  is arbitrary and  $\beta_k \in \mathbb{R}^n$  satisfies the following equation

$$\nabla F(p_k) \nabla F(p_k)^T X = -F(p_k) \quad (6.3)$$

This has a unique solution since  $\nabla F(p_k)$  is of full rank. Finally,  $\alpha_k$  is chosen as follows:

1. **IIS case.** In this case,  $m = 3$ ,  $n = 2$  and  $t$  in (6.2) is the tangent direction of the curve. The change of  $p_k$  in the direction of  $t$  should be as small as possible. Therefore, we set  $\alpha_k = 0$  ([7]).
2. **IPS case.** Now  $m = 4$ ,  $n = 3$ ,  $p = (x_1, x_2, x_3, x_4)^T := (u_1, v_1, u_2, v_2)^T$  and  $f_i(x_1, x_2, x_3, x_4) = G_{i1}(x_1, x_2) - G_{i2}(x_3, x_4)$ , for  $i = 1, 2, 3$ . The initial value is given by (5.1), i.e.,  $p_0 = (Q_1(s_0)^T, Q_2(s_0)^T)^T$ , where  $s_0$  is the step length of the approximation of  $r(s)$ . In order to determine  $\alpha_k$  in (6.2), we project  $p_{k+1} \in \mathbb{R}^4$  (domain space) into  $\mathbb{R}^3$  (value space) by  $X_1(p_k^{(1)}) + \nabla X_1(p_k^{(1)}) \Delta_k^{(1)}$ , where  $p_k^{(1)}$  (or  $p_k^{(2)}$ ) and  $\Delta_k^{(1)}$  (or  $\Delta_k^{(2)}$ ) are the first (or last) two components of  $p_k$  and  $\Delta_k$ , respectively. Let  $n_1 = X_{1u_1}(p_k^{(1)}) \times X_{1v_1}(p_k^{(1)})$ ,  $n_2 = X_{2u_2}(p_k^{(2)}) \times X_{2v_2}(p_k^{(2)})$  and  $n_3 = n_1 \times n_2$ . Then there exist  $\tilde{\alpha}_k \in \mathbb{R}$ ,  $\tilde{\beta}_k \in \mathbb{R}^2$  such that  $X_1(p_k^{(1)}) + \nabla X_1(p_k^{(1)}) \Delta_k^{(1)} = \tilde{\alpha}_k n_3 + [n_1, n_2] \tilde{\beta}_k$  and  $\tilde{\beta}_k$  is determined uniquely by
 
$$\begin{bmatrix} n_1^T X_1(p_k^{(1)}) \\ n_2^T X_2(p_k^{(2)}) \end{bmatrix} = [n_1, n_2]^T [n_1, n_2] \tilde{\beta}_k \text{ and}$$

$$\begin{aligned} \tilde{\alpha}_k &= n_3^T X_1(p_k^{(1)}) + n_3^T \nabla X_1(p_k^{(1)}) \Delta_k^{(1)} \\ &= n_3^T X_1(p_k^{(1)}) + n_3^T \nabla X_1(p_k^{(1)}) [\alpha_k t^{(1)} + \nabla F^{(1)}(p_k)^T \beta_k] \\ &= a(p_k) \alpha_k + b(p_k) \end{aligned}$$

where  $a(p_k)$  and  $b(p_k)$  are constants depending on  $p_k$ . For the same reason as IIS, we take  $\tilde{\alpha}_k = 0$ . Hence  $\alpha_k = -\frac{b(p_k)}{a(p_k)}$ .

**Case B:  $n > m$ .** This case happens when we arrive at a singular point on the intersection curve  $SC$  (see Section 9). Now system (6.1) is over-determined. So we find the least squares approximate solution, i.e.,

$$\nabla F(p_k)^T \nabla F(p_k) \Delta_k = -\nabla F(p_k)^T F(p_k) \quad (6.4)$$

## 7 Rational Curve Hermite Interpolation between Simple Points

Let  $r_1(u)$ ,  $r_2(v)$  be two space curves, where  $u$  and  $v$  are arc lengths of the curves measured from some point on the respective curve. At point  $u = u_0$ ,  $v = v_0$ , if  $r_1^{(i)}(u_0) = r_2^{(i)}(v_0)$ ,  $i = 0, 1, \dots, k$ , we say that  $r_1$  and  $r_2$  are  $k$ -frame connected, or the composite curve is  $k$ -frame continuous. In particular, if  $k = 3$ , we say the curve is frame continuous. Given a point  $p_0$  on the curve  $r(s)$ , which is either IIS or IPS, the arc length  $s$  is measured from  $p_0$  (i.e.,  $r(0) = p_0$ ) in the given direction.

**Step Length.** From the approximation  $r(s) \approx \sum_{i=0}^{k+1} r^{(i)}(0) \frac{s^i}{i!}$ , we compute a trial step length  $\beta > 0$  such that

$$\frac{\|r^{(k+1)}(0)\| \beta^{k+1}}{(k+1)!} / \left\| \sum_{i=0}^k \frac{r^{(i)}(0) \beta^i}{i!} \right\| < \epsilon \quad (7.1)$$

For such a  $\beta$ , using  $\sum_{i=0}^{k+1} r^{(i)}(0) \beta^i / i!$  (for IIS), or  $[\sum_{i=0}^{k+1} Q_1^{(i)}(0)^T \beta^i / i!, \sum_{i=0}^{k+1} Q_2^{(i)} \beta^i / i!]^T$  (for IPS) as an initial value, we compute a new point  $p_1$  on the curve by Newton iterations (section 6). We then construct rational approximations as follows:

### A. Rational Hermite interpolation

Let  $m, n$  be two nonnegative integers with  $m + n = 2k + 1$ . We construct a rational vector function  $R(s) = [R_1(s), R_2(s), R_3(s)]^T$ , where  $R_i(s) = P_{mi}(s)/Q_{ni}(s)$ ,  $i = 1, 2, 3$  are  $(m, n)$  type rational functions, such that

$$R^{(i)}(s) = r^{(i)}(s), \quad i = 0, 1, \dots, k \quad (7.2)$$

for  $s = 0$  and  $s = \beta$ . If either  $Q_{ni}(s)$  has zeros in  $[0, \beta]$  or the error  $\max_{s \in [0, \beta]} \|r(s) - R(s)\| > \epsilon$ , we halve the  $\beta$ . The approximation error is bounded in the following way: Since  $e_i(s) = r_i(s)Q_{ni}(s) - P_{mi}(s) = O(s^{k+1}(s - \beta)^{k+1})$ , by the remainder formula of Hermite interpolation [9], we have

$$e_i(s) = [s(s - \beta)]^{k+1} (r_i Q_{ni}) \underbrace{[0, \dots, 0]_{k+1}}_{k+1} \underbrace{[\beta, \dots, \beta]_{k+1}}_{k+1}, s],$$

where  $f[t_0, \dots, t_r]$  stands for divided difference of  $f$  on  $t_0, \dots, t_r$ . Hence

$$|r_i(s) - R_i(s)| \leq \left(\frac{\beta}{2}\right)^{2k+2} \frac{|D_{ki}(s)|}{\min_{s \in [0, \beta]} |Q_{ni}(s)|} \quad (7.3)$$

where  $D_{ki}(s) = (r_i Q_{ni})[0, \dots, 0, \beta, \dots, \beta, s]$  is a function in  $s$ . That can be bounded approximately by either

$$|D_{ki}(0)| + |D_{ki}(\beta)| \quad \text{or} \quad \max_{s \in [0, \beta]} |\tilde{D}_{ki}(s)|$$

where  $\tilde{D}_{ki}(s)$  is the interpolation polynomial of degree 2 at  $D_{ki}(0)$ ,  $D_{ki}(\frac{\beta}{2})$  and  $D_{ki}(\beta)$ . Let  $g = r_i/Q_{ni}$ . Then the divided difference can be computed by the following well known recurrence

$$g[t_0, \dots, t_k] = \begin{cases} g^{(k)}(t_0)/k! & \text{if } t_0 = \dots = t_k \\ \frac{g[t_0, \dots, t_{r-1}t_{r+1}, \dots, t_k] - g[t_0, \dots, t_{s-1}, t_{s+1}, \dots, t_k]}{t_s - t_r} & \text{if } t_r \neq t_s \end{cases}$$

### B. Rational Vector Hermite Interpolation

We construct a rational function  $R(s) = [P_{m_1}(s), P_{m_2}(s), P_{m_3}(s)]^T / Q_n(s)$  such that (7.2) holds and

$$m + n/3 = 2k + 1 \quad (7.4)$$

where  $n$  is divisible by 3. Now each component of the vector rational function has the same denominator. But the degree  $m + n$  of each component is higher than the previous case. However, if we transform the vector rational function in case 1 into a rational function that has common denominator, then the degree is higher than in case 2. This transform is necessary when we represent the curve in rational Bernstein-Bézier form. The error bound of the approximation can be estimated in the same way as before.

### C. Two Point Padé Approximation

The two point Padé approximation method discussed here consists of the following two steps. First, compute the Padé approximation  $P_{m_1i}(s)/Q_{n_1i}(s)$  at  $s = 0$ , such that  $r_i(s) - P_{m_1i}(s)/Q_{n_1i}(s) = O(s^{k+1})$ ,  $i = 1, 2, 3$ , and

$$m_1 + n_1 = k. \quad (7.5)$$

Second, compute the Padé approximation  $P_{m_2i}(s)/Q_{n_2i}(s)$  at  $s = \beta$  to the function  $\tilde{r}_i(s) = (r_i(s) - P_{m_1i}(s)/Q_{n_1i}(s))/s^{k+1}$  such that  $\tilde{r}_i(s) - P_{m_2i}(s)/Q_{n_2i}(s) = O((s - \beta)^{k+1})$ ,  $i = 1, 2, 3$ , and

$$m_2 + n_2 = k \quad (7.6)$$

The required two point approximation is

$$R_i(s) = \frac{P_{m_1i}(s)Q_{n_2i}(s) - s^{k+1}P_{m_2i}(s)}{Q_{n_1i}(s)Q_{n_2i}(s)}$$

which is  $(\max\{m_1 + n_2, k + m_2 + 1\}, n_1 + n_2)$  type rational function and satisfies condition (7.2). For example, if  $k = 3$ , take  $m_1 = m_2 = 2$ ,  $n_1 = n_2 = 1$ , then  $R_i(s)$  is a (6,2) type rational function. Since the denominator of  $R_i(s)$  is a product of two polynomials, it is easy to check the appearance of the poles of  $R_i(s)$  in  $[0, \beta]$  when  $n_i$  is small, say  $n_i \leq 2$ .

Denote  $Q_{ni}(s) = Q_{n_1i}(s)Q_{n_2i}(s)$  ( $n = n_1 + n_2$ ), the error can be estimated as in the rational Hermite interpolation case.

### D. Two Point Vector Padé Approximation

Similar to the rational vector Hermite interpolation, we can also consider a two point vector Padé approximation. Now conditions (7.5) and (7.6) should be replaced by  $m_1 + n_1/3 = k$ ,  $m_2 + n_2/3 = k$  respectively, and further we require that  $n_1$  and  $n_2$  are divisible by 3. The error can be computed as before.

## 8 Isolating the Singular Points

During the tracing of an intersection space curve, one may encounter singular points<sup>1</sup>. Near these points, the coefficient matrix of the systems (4.7) for IIS, (5.4) and (5.5) for IPS, are nearly singular.

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<sup>1</sup>The intersection space curve is assumed to be singly defined and hence may possess only isolated singular points

When a near singular condition of the coefficient matrix is detected, the tracing procedure is temporarily suspended and the singular point is accurately isolated as follows.

### A. Singular Point of IIS

Let  $p_0 = (x_0, y_0, z_0)^T \in \mathbb{R}^3$  be a singular point of the intersection curve of  $f_i(p) = 0$ ,  $i = 1, 2$ . That is  $f_i(p_0) = 0$ ,  $i = 1, 2$  and the surfaces  $f_1(p) = 0$  and  $f_2(p) = 0$  are tangent at  $p_0$ . Hence there exist constants  $\alpha_1 \neq 0$ ,  $\alpha_2 \neq 0$  such that

$$\alpha_1 F_1^{(1)}(p - p_0) = \alpha_2 F_2^{(1)}(p - p_0) \quad (8.1)$$

where  $f_i(p) = \sum_{s=0} F_i^{(s)}(p - p_0)$  and  $F_i^{(s)}(u, v, w)$  is a homogeneous polynomial of degree  $s$ . If the order of the singularity is greater than two, then equation (8.1) is replaced by  $\alpha_1 F_1^{(s)}(p - p_0) = \alpha_2 F_2^{(s)}(p - p_0)$ ,  $s = 1, 2, \dots, L$  or equivalently

$$\alpha_1 \frac{\partial^s f_1(p_0)}{\partial x^i \partial y^j \partial z^k} = \alpha_2 \frac{\partial^s f_2(p_0)}{\partial x^i \partial y^j \partial z^k}, \quad \forall (i, j, k) \quad (8.2)$$

with  $i + j + k = s$ ,  $s = 1, 2, \dots, L$ , where  $L + 1$  is the order of the singular point. In order to eliminate  $\alpha_1$  and  $\alpha_2$ , use one equation of (8.2), say  $\alpha_2 \frac{\partial f_2(p_0)}{\partial x} = \alpha_1 \frac{\partial f_1(p_0)}{\partial x}$  if  $\frac{\partial f_2(p_0)}{\partial x} \neq 0$ , to obtain

$$f_{i,j,k}(p_0) = \frac{\partial f_2(p_0)}{\partial x} \frac{\partial^s f_1(p_0)}{\partial x^i \partial y^j \partial z^k} - \frac{\partial f_1(p_0)}{\partial x} \frac{\partial^s f_2(p_0)}{\partial x^i \partial y^j \partial z^k} = 0$$

$\forall (i, j, k) \in \{(i, j, k) : i + j + k = s, s = 1, 2, \dots, L\} \setminus \{1, 0, 0\}$ .

Now use Newton iterations (Section 6) to solve the system of equations

$$\begin{cases} f_i(p) & = 0 \\ f_{i,j,k}(p) & = 0, \quad i + j + k \leq s \end{cases} \quad (8.3)$$

Use  $s = 1$  if the resulting matrix(see (6.3) ) is nonsingular, otherwise increase  $s$  by 1 until the matrix is nonsingular.

### B. Singular Points of IPS

Let  $Q_1^*, Q_2^* \in \mathbb{R}^2$  be the points such that  $X_1(Q_1^*) = X_2(Q_2^*)$ , i.e.,  $p^* = X_1(Q_1^*)$  on the intersection curve. We use the definition of the singularity for IIS curve to define the singularity for an IPS curve. For this we need to determine the partial derivatives of the implicit form of parametric surfaces, as described below. We exhibit this for surface  $X_1$ . Surface  $X_2$  can be treated in the same way.

Under the assumption of smoothness of the surface  $X_1$ , the vectors  $\frac{\partial X_1(Q_1^*)}{\partial u_1}$  and  $\frac{\partial X_1(Q_1^*)}{\partial v_1}$  are linearly independent. We may thus assume, without loss of generality, that the Jacobian matrix

$$J(G_{11}, G_{21}) = \begin{bmatrix} \frac{\partial G_{11}(Q_1^*)}{\partial u_1} & \frac{\partial G_{11}(Q_1^*)}{\partial v_1} \\ \frac{\partial G_{21}(Q_1^*)}{\partial u_1} & \frac{\partial G_{21}(Q_1^*)}{\partial v_1} \end{bmatrix}$$

is nonsingular. The inverse functions of

$$x = G_{11}(u_1, v_1), \quad y = G_{21}(u_1, v_1) \quad (8.4)$$

also exist and are given by

$$u_1 = \tilde{G}_{11}(x, y) \quad v_1 = \tilde{G}_{21}(x, y) \quad (8.5)$$

around  $Q_1^*$ . Substitute (8.5) into  $z = G_{31}(u_1, v_1)$ , to obtain an implicit representation of the parametric surface.

$$f_1(x, y, z) = G_{31}(\tilde{G}_{11}(x, y), \tilde{G}_{21}(x, y)) - z = 0 \quad (8.6)$$

Now compute the partial derivatives of  $f_1$ . The derivative about  $z$  is trivial, so consider  $\frac{\partial f_1}{\partial x}$  first. It follows from (8.6) and (8.4) that

$$\frac{\partial f_1}{\partial x} = \frac{\partial G_{31}}{\partial u_1} \frac{\partial u_1}{\partial x} + \frac{\partial G_{31}}{\partial v_1} \frac{\partial v_1}{\partial x} \quad (8.7)$$

$$J(G_{11}, G_{21}) \begin{bmatrix} \frac{\partial u_1}{\partial x} \\ \frac{\partial v_1}{\partial x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad (8.8)$$

Solving (8.8), we get  $\frac{\partial u_1}{\partial x}$ ,  $\frac{\partial v_1}{\partial x}$ , from (8.7) we get  $\frac{\partial f_1}{\partial x}$ . Similarly,  $\frac{\partial f_1}{\partial y}$  can be computed. Knowing the partials one can compute the singular points as in the IIS case. For higher order singularities the higher order partial derivatives can be computed similarly to the computation of second order derivatives shown below. From (8.7), we have

$$\begin{aligned} \frac{\partial^2 f_1}{\partial x \partial y} &= \left( \frac{\partial^2 G_{31}}{\partial u_1^2} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{31}}{\partial u_1 \partial v_1} \frac{\partial v_1}{\partial y} \right) \frac{\partial u_1}{\partial x} + \frac{\partial G_{31}}{\partial u_1} \frac{\partial^2 u_1}{\partial x \partial y} \\ &+ \left( \frac{\partial^2 G_{31}}{\partial u_1 \partial v_1} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{31}}{\partial v_1^2} \frac{\partial v_1}{\partial y} \right) \frac{\partial v_1}{\partial x} + \frac{\partial G_{31}}{\partial v_1} \frac{\partial^2 v_1}{\partial x \partial y} \end{aligned} \quad (8.9)$$

and differentiating (8.4), we have

$$J(G_{11}, G_{21}) \begin{bmatrix} \frac{\partial^2 u_1}{\partial x \partial y} \\ \frac{\partial^2 v_1}{\partial x \partial y} \end{bmatrix} = - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad (8.10)$$

where

$$v_i = \left( \frac{\partial^2 G_{i1}}{\partial u_1^2} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{i1}}{\partial u_1 \partial v_1} \frac{\partial v_1}{\partial y} \right) \frac{\partial u_1}{\partial x} + \left( \frac{\partial^2 G_{i1}}{\partial u_1 \partial v_1} \frac{\partial u_1}{\partial y} + \frac{\partial^2 G_{i1}}{\partial v_1^2} \frac{\partial v_1}{\partial y} \right) \frac{\partial v_1}{\partial x}$$

From (8.10) we get  $\frac{\partial^2 u_1}{\partial x \partial y}$ ,  $\frac{\partial^2 v_1}{\partial x \partial y}$ , from (8.9) we get  $\frac{\partial^2 f_1}{\partial x \partial y}$ .

## 9 The Local Approximation at Singular Points

At the singular points, simple Taylor series expansions fail and we must use special methods to tackle the approximation problem.

1. **IIS.** Let  $p_0 = (x_0, y_0, z_0)^T \in \mathbb{R}^3$  be a singular point on the curve. Since the matrix  $\nabla f_i(p_0) \neq 0$ , we may assume, without loss generality, that  $\frac{\partial f_1}{\partial z} \neq 0$ . Then we can express  $z$  by a power series  $z = \phi(x, y)$  in  $x$  and  $y$  from  $f_1(x, y, z) = 0$  around the point  $p_0$ . Substituting  $z$  into  $f_2(x, y, z) = 0$ , we get  $h(x, y) = f(x, y, \phi(x, y)) = 0$ . As in the plane curve case [5], expanding  $h(x, y) = 0$  at point  $(x_0, y_0)^T$  by Weierstrass and Newton factorization, we obtain

$$\begin{cases} x = x_0 + t^{k_i} \\ y = \psi_i(t) \end{cases} \quad i = 0, 1, \dots, m$$

where  $\psi_i(t)$  is a power series in  $t$  and  $m$  is the number of the branches of the curve  $h(x, y) = 0$ . We then have  $z = \phi(x_0 + t^{k_i}, \psi_i(t)) = \theta_i(t)$ , for  $i = 0, 1, \dots, m$ . Therefore we get the local parametric form of the space curve as

$$\begin{cases} x = x_0 + t^{k_i} \\ y = \psi_i(t) \\ z = \theta_i(t) \end{cases} \quad i = 0, 1, \dots, m$$

For each branch, use the two point interpolating condition to get a rational approximation.

2. **IPS.** Let  $Q_1^* = (u_1^*, v_1^*)^T$ ,  $Q_2^* = (u_2^*, v_2^*)^T$  be the points in  $\mathbb{R}^2$  such that  $X_1(Q_1^*) = X_2(Q_2^*)$  and  $X_1(Q_1^*)$  is a singular point of the curve IPS. Since the matrices  $\nabla X_1(Q_1^*)$  and  $\nabla X_2(Q_2^*)$  are full rank in column, we may assume  $J(G_{11}, G_{21})$  is nonsingular at  $Q_1^*$ . By one of the first two equations, say the first,  $G_{11}(u_1, v_1) = G_{12}(u_2, v_2)$ , we can express  $u_1$  as

$$u_1 = \phi^{(1)}(v_1, u_2, v_2) \quad (9.1)$$

Substituting it into another equation of the first two, we get

$$v_1 = \phi^{(2)}(u_2, v_2) \quad (9.2)$$

Substituting  $u_1$  and then  $v_1$  into the last equation  $G_{31}(u_1, v_1) = G_{32}(u_2, v_2)$ , we have  $\phi^{(3)}(u_2, v_2) = 0$ . Now, using plane curve factorization techniques for dealing with the singularities, we get  $u_2 = u_2^* + t^{k_i}$ ,  $v_2 = \phi_i(t)$ ,  $i = 0, 1, \dots, m$ . Substituting them back to (9.2) and (9.1), we have  $v_1 = \phi^{(2)}(u_2^* + t^{k_i}, \phi_i(t)) = \psi_i(t)$ ,  $u_1 = \phi^{(1)}(\psi_i(t), u_2^* + t^{k_i}, \phi_i(t)) = \theta_i(t)$ . Then the local parameterization is obtained by

$$\begin{aligned} r_i(t) &= X_1(\theta_i(t), \psi_i(t)) \\ &\text{or } X_2(u_2^* + t^{k_i}, \phi_i(t)), \quad i = 0, 1, \dots, m \end{aligned}$$

The next step for getting the approximation is the same as IIS.

## 10 Implementation Details and Examples

### 1. Starting Points

In order to trace the intersection curve  $SC$ , we need to provide a starting point on each real component of the curve. Besides the boundary points which are straightforward roots of univariate or coupled bivariate polynomial equations[1] one computes a starting point on each real component completely inside the given box. For IIS this can be done by projecting the intersection curve (via resultant elimination) into a plane and then finding a coordinate axis extreme point on the projection curve of that component. See [1] for details of such resultant elimination schemes. For IPS, papers [8] [18] provide some numerical methods for computing these starting points.

### 2. Curve Interpolation Points

When we march along the curve, we encounter precomputed points on the way. An encountered point may be a boundary point, a starting point on a closed loop or may be an end point of the prior segment tracing. Suppose  $p_0 \in \mathbb{R}^3$  is a point on the curve,  $r(s)$  ( $s \in [0, \beta]$ ) is a segment of the curve, which approximates the original curve. Then a possible question is whether  $r(s)$  passes through  $p_0$  within the allowable error? We answer this question by computing the distance between  $p_0$  and  $r(s)$ :

$$\text{dis}(p_0, r) = \min_{s \in [0, \beta]} \|r(s) - p_0\| \quad (10.1)$$

Since  $r(s)$  is a rational function in  $s$ , the minimum point of (10.1) can be computed by  $\frac{d}{ds} \|r(s) - p_0\|^2 = 0$ . If  $s = s^* \in [0, \beta]$  is the minimum point, then if  $\|r(s^*) - p_0\| < \epsilon$ ,  $r(s)$  passes through  $p_0$ . Then we modify  $r(s)$  such that  $r(s)$  is frame continuous at  $s^*$  and replaces  $\beta$  by  $s^*$ . Otherwise,  $r(s)$  does not pass through  $p_0$ .

### 3. Solving Linear System of Equations

In all the cases in this paper (see (4.7), (5.4), (5.5), (6.3) and (6.4)), we always solve the linear system  $Ax = b$  with a positive definite coefficient matrix  $A$ . The size of matrix  $A$  is as small as one, and as large as four. A stable method to solve this equation is to use singular value decomposition  $A = U^T \Sigma U$ , where  $U$  is an orthogonal matrix and  $\Sigma$  is a diagonal matrix. The solution is  $x = U^T \Sigma^{-1} U b$ .

### 4. Tangent Direction

In Sections 4 and 5, we have mentioned that the sign of the tangent vector  $t$  at an expansion point should be properly chosen. Now we will make this point clear.

- a. If the expansion point is a boundary point, then  $t$  points to the interior of the box.
- b. If the point is a starting point on a loop, then the sign can be any.

Figure 1: Piecewise Rational B-spline Approximation of the Intersection Curve of Two Implicit Surfaces

- c. If the point is an end point of a previous approximation  $\tilde{r}(s) = \sum_{i=0}^{k+1} r^{(i)}(0)s^i / i!$  ( $s \in [0, \beta]$ ), then we choose the sign of  $t$  such that  $\tilde{r}'(\beta)^T t \geq 0$ .

**Examples** We present several examples of piecewise rational approximations of surface-surface intersection curves and parametric space curves implemented in GANITH[4], an X-11 based interactive algebraic geometry toolkit, using Common Lisp for the symbolic computation and C for all numeric and graphical computation. Though we did not discuss the approximation of high degree parametric space curves, the derivation follows similar lines to the IIS and IPS cases.

**Example 10.1** *The example is shown in Figure 1. The intersection curve is given by the two implicit surfaces  $x^2 + y^2 - 1 = 0$  and  $y^2 + z^2 - 1 = 0$  within the bounding box  $[-2, 2, -2, 2, -2, 2]$  as shown in the bottom left window. The approximation error epsilon is 0.1 for each of the piecewise rational B-spline approximations. The left window is a  $C^1$  continuous piecewise parametric cubic B-spline curve while the right window is a  $C^3$  continuous piecewise parametric degree 7 B-spline curve. The bottom right window shows the input window of GANITH.*

**Example 10.2** *The example is shown in Figure 2. The intersection curve is given by the an implicit surface  $x^2 + z^2 + 2 * z = 0$  and a parametric surface  $(x = \frac{s+s*t^2}{1+t^2}, y = \frac{2-2*t^2}{1+t^2}, z = \frac{4*t-2-2*t^2}{1+t^2})$  within the bounding box  $[-2, 2, -2, 2, -2, 2]$  as shown in the top left window. The approximation*

Figure 2: Piecewise Rational B-spline Approximation of the Intersection Curve of an Implicit and a Parametric Surface

*error epsilon is 0.1 for the piecewise rational B-spline approximation in the top right and the bottom left window and 0.01 in the bottom right window. The top right window is a  $C^1$  continuous piecewise parametric cubic B-spline curve while the bottom windows are  $C^3$  continuous piecewise parametric degree 7 B-spline curve.*

**Example 10.3** *The example is shown in Figure 3. The intersection curve is given by the two parametric surfaces  $(x = s, y = t, z = 2 * s^4 + t^4)$  and  $(x = s, y = t, z = 3 * s^2 * t - t^2 + 2 * t^3)$  within the bounding box  $[-2, 2, -2, 2, -2, 2]$ . The approximation error epsilon is 0.01 for each of the piecewise rational B-spline approximations. The top two windows are  $C^2$  continuous piecewise parametric quintic B-spline curve while the bottom windows are  $C^3$  continuous piecewise parametric degree 7 B-spline curve.*

**Example 10.4** *This series of examples are shown in Figure 4. The original parametric space curve of the top left window is given by  $(x = t, y = t^2, z = t^3)$ . The original parametric space curve of the top right window is given by  $(x = t^2, y = t^3, z = t^4)$ . The original parametric space curve of the bottom left window is given by  $(x = t * (t - 1) / (1 + t^2), y = t * (1 - t) * (1 + t), z = t * (1 - t) * t^2 / (1 + 2 * t^2))$ . The original parametric space curve of the bottom right window is given by  $(x = t * (t - 2) * (t + 2), y = t * (t - 2) * (t + 2) * (1 + t), z = t * (t - 2) * (t + 2) * (1 - t))$ . For each of the windows the approximation error epsilon is 0.01 and the approximation is by  $C^3$  continuous piecewise parametric degree 7 B-spline curve.*

Figure 3: Piecewise Rational B-spline Approximation of the Intersection Curve of Two Parametric Surfaces

Figure 4: Piecewise Rational B-spline Approximation of Parametric Space Curves

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