The Combinatorics of Real Algebraic Splines over a Simplicial Complex

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Abstract

We bound the dimension of fixed degree real algebraic interpolatory spline spaces. For a given planar triangulation $\mathcal{T}$ real algebraic splines interpolate specified z_i values at the vertices $v_i = (x_i, y_i)$ of $\mathcal{T}$. For a three dimensional simplicial complex $\mathcal{ST}$, real algebraic splines interpolate the boundary vertices $v_j = (x_j, y_j, z_j)$ of $\mathcal{ST}$. The main results of this paper are lower bounds on the dimension of degree m real algebraic C^0 and C^1 interpolatory splines over $\mathcal{T}$ and implicit real algebraic C^1 interpolatory splines over $\mathcal{ST}$.

1 Introduction

Piecewise polynomial functions or surfaces of some fixed degree m and continuously differentiable upto some order r (i.e. C^r-continuous or C^r-smooth) are known as splines or finite elements. Splines are used in applications ranging from image processing, computer aided design, to the solution of partial differential equations via finite element analysis. C^r-continuous splines are traditionally defined over a given planar triangulation $\mathcal{T}$, with a bivariate polynomial function $z = f(x, y)$ for each triangular face. Note that since functions $z = f(x, y)$ are single valued, functional splines cannot be defined for a general simplicial complex $\mathcal{ST}$. See below for a summary of prior results on multivariate splines[11].

In this paper we extend these past results in two directions. First we consider real algebraic splines defined over $\mathcal{T}$, with an implicit real algebraic surface, the real zeros of a trivariate polynomial $f(x, y, z) = 0$, for each triangular face. Next we also consider implict real algebraic splines defined over a simplicial complex in three dimensional space $\mathcal{ST}$, with an implicit real algebraic surface for each triangular face of the boundary of $\mathcal{ST}$. The main complication that arises from using implicit real algebraic surfaces $f(x, y, z) = 0$ is that the representation being multivalued may cause the zero contour surface to have multiple real sheets, self-intersections and several other undesirable singularities. In section 3, we characterize the real Bernstein-Bezier (BB) form of a trivariate polynomial within a tetrahedron such that the zero-contour of the polynomial within the tetrahedron is smooth (non-singular) and a single sheeted real algebraic surface. We call this a triangular algebraic patch and lower

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bound its dimension. We then use the dimension of this algebraic patch to lower bound the dimension of the C^0 and C^1 interpolatory real algebraic spline spaces over $\mathcal{T}$ and $\mathcal{ST}$. These algebraic splines are then guaranteed to be smooth and without multiple sheets. Having explicit bounds on the dimension of these interpolatory algebraic spline spaces provides an answer to the question of "What is the lowest degree algebraic spline that would be required to fit given arbitrary data over $\mathcal{T}$ and $\mathcal{ST}$?. The motivation and advantages of using algebraic surfaces and splines in geometric modeling and computer graphics are detailed in [5, 6].

Let the set of all C^r-*smooth* bivariate polynomial function splines, and implicit algebraic splines, of degree m over $\mathcal{T}$ be denoted by $SP^r_m(\mathcal{T})$, and $SI^r_m(\mathcal{T})$ respectively. Additionally let $SI^r_m(\mathcal{ST})$ be the set of C^r-*smooth* implicit algebraic splines of degree m over $\mathcal{ST}$. All these sets form a vector space over the field of coefficients of the polynomials.

There are two basic problems dealing with spline spaces:

1. Estimate the dimension of the spline sets $SP^r_m(\mathcal{T})$, $SI^r_m(\mathcal{T})$, and $SI^r_m(\mathcal{ST})$ over triangulations $\mathcal{T}$ and a simplicial complex $\mathcal{ST}$ respectively, for various $m \geq 1$ and $r \geq 0$.

2. Construct an explicit interpolatory (and approximatory) basis for each of the $SP^r_m(\mathcal{T})$, $SR^r_m(\mathcal{T})$, $SI^r_m(\mathcal{T})$, and $SI^r_m(\mathcal{ST})$ spline sets for various $m \geq 1$ and $r \geq 0$.

Additional variants of the above problems are those involving special triangulations (e.g. regular grids). All these problems are higher degree analogues of combinatorial relations developed for planar triangulations and simplicial complexes[13].

Facts & Prior Results

Fact 1.1 *Let v, e and f be the number of faces, vertices and edges of a finite planar triangulation $\mathcal{T}$ where the outside face is not counted. Let α be the number of vertices of the outer face. Then from Euler's formula $v - e + f = 1$. Since each face other than the outside is triangular $3f + \alpha = 2e$. Thus $e = 3v - \alpha - 3$, and $f = 2v - \alpha - 2$.*

Fact 1.2 *Let v, e and f be the number of boundary faces, vertices and edges of a simplicial complex $\mathcal{ST}$ which is closed and of genus g, Then from Euler's formula $v - e + f = 2 - 2g$. Since each face is triangular and each edge has exactly two incident faces $3f = 2e$. Thus $e = 3(v + 2g - 2)$, and $f = 2(v + 2g - 2)$.*

Fact 1.3 *The dimension of $SP^0_1(\mathcal{T})$ = dimension of $SI^0_1(\mathcal{T}) = v$.*

Lemma 1.1 (Billera [9]) *(1) For a generic triangulation $\mathcal{T}$, the dimension of $SP^1_2(\mathcal{T}) = 3v$ - $5e + 6f + 2\alpha$ $= \alpha + 3$.*

(2) For a generic triangulation $\mathcal{T}$, the dimension of $SP^1_3(\mathcal{T}) = 3v$ - $7e + 10f + 4\alpha = f + 2\alpha + 3 = 2v + \alpha + 1$.

Lemma 1.2 (Morgan & Scott [14]) *For $m \geq 5$ and any triangulation $\mathcal{T}$, the dimension of $SP^1_m(\mathcal{T}) = 3v$ - $(2m+1)e + \binom{m+2}{2} f + 2(m-1)\alpha + \beta$ where β = the number of rectangles triangulated with cross diagonals.*

Morgan and Scott [14] also construct an explicit basis for $SP_m^1(\mathcal{T})$.

Lemma 1.3 (Alfeld & Schumaker [3]) *For $m \geq 4r+1$ and any triangulation $\mathcal{T}$, the dimension of $SP_m^r(\mathcal{T})$ = $3v$ - $(2m+1)e$ + $\begin{pmatrix} m+2 \\ 2 \end{pmatrix} f$ + $2(m-1)\alpha + \beta$ where β = the number of rectangles triangulated with cross diagonals.*

Alfeld, Piper & Schumaker [2] also construct explicit bases for $SP_m^r(\mathcal{T})$ for $m \geq 4r+1$ and for SP_4^1 [1]. Algorithms for the explicit construction and graphics display of C^1 algebraic splines over $\mathcal{ST}$ are given in [7, 8].

Main Results

The main results of this paper are as follows. In section 3 we prove that the dimension of a smooth and single sheeted algebraic surface patch within a bounding tetrahedron is $\geq \begin{pmatrix} m+1 \\ 2 \end{pmatrix}$. In section 4.1 we prove that (a) the dimension of $SI_m^0(\mathcal{T})$ is $\geq \begin{pmatrix} m+1 \\ 2 \end{pmatrix} f - (2m+1)e + (2m+1)\alpha - v + 2$ and (b) the dimension of $SI_m^1(\mathcal{T})$ is $\geq \begin{pmatrix} m+1 \\ 2 \end{pmatrix} f - (6m+1)e + 3(2m+1)\alpha + v + 2$. In section 4.2 we prove that (a) the dimension of $SI_m^0(\mathcal{ST})$ is $\geq$ $\begin{pmatrix} m+1 \\ 2 \end{pmatrix} f - (2m+1)e - 2v - 4g + 2$ and (b) the dimension of $SI_m^1(\mathcal{ST})$ is $\geq \begin{pmatrix} m+1 \\ 2 \end{pmatrix} f - 2(3m-1)e - v - 2g + 2$.

2 Interpolatory Curves in Space

We state here some straightforward lemmas which we shall use in subsequent sections.

A point $\mathbf{p} = (p_x, p_y, p_z)$ in space with a prespecified 'normal' vector $\mathbf{n} = (n_x, n_y, n_z)$ defines a unique oriented plane at $\mathbf{p}$ given by $T_{\mathbf{p}}(x, y, z) = n_x(x - p_x) + n_y(y - p_y) + n_z(z - p_z) = 0$. Let $(P_0 = (p_0, n_0), P_1 = (p_1, n_1))$ be an edge-normal-pair. A conic segment $S(P_0, P_1)$ is said to C^1-interpolate P_0 and P_1 if there exists a non-degenerate quadratic surface (quadric) $F : ax^2 + by^2 + cz^2 + dxy + eyz + fzx + gx + hy + iz + j = 0$ such that

- $S(P_0, P_1)$ is a singly connected conic segment on F,
- p_0 and p_1 are the end points of $S(P_0, P_1)$,
- the gradient of $f(x, y, z) = 0$ at p_0 and p_1 have the same directions as n_0 and n_1, respectively. In other words $\nabla f(p_0) = \alpha n_0$ and $\nabla f(p_1) = \beta n_1$ for constants $\alpha, \beta > 0$.

Lemma 2.1 *Irreducible conics can C^1 interpolate the two endpoints $\mathbf{p}_0$ and $\mathbf{p}_1$ of any edge with prespecified 'normal' vectors $\mathbf{n}_0$ and $\mathbf{n}_1$ respectively, iff $T_{\mathbf{p}_0}(\mathbf{p}_1) \cdot T_{\mathbf{p}_1}(\mathbf{p}_0) > 0$.*

Proof : See [7]. $\diamond$

The geometric interpretation of the inequality $T_{(p_0, \nabla f(p_0))}(p_1) \cdot T_{(p_1, \nabla f(p_1))}(p_0) > 0$ is that p_0 is on the positive (negative) halfspace of T_{P_1} if and only if p_1 is on the positive (negative) halfspace of T_{P_0}.

Lemma 2.2 *Irreducible space cubic curves can C^1 interpolate the two endpoints of any edge with any prespecified 'normal' vectors.*

Proof: The set of all (irreducible) parametric space cubic curves $x(t), y(t), z(t)$ form a vector space of dimension 12. The C^1 interpolation of the endpoints of an edge with prespecified 'normal' vectors impose 10 linear conditions on the coefficients of such cubics. $\diamondsuit$

Theorem 2.1 (Bezout) *An algebraic space curve of degree n intersects an algebraic surface of degree m in either at most $m * n$ points, or infinitely often.*

3 Triangular Algebraic Patches

Let $\{p_1, \ldots p_j\} \in I\!R^3$. Then the *convex hull* of these points is defined by $[p_1 p_2 ... p_j] = \{p \in I\!R^3 : p = \sum_{i=1}^{j} \alpha_i p_i, \alpha_i \geq 0, \sum_{i=1}^{j} \alpha_i = 1\}$. Let $p_1,\ p_2,\ p_3,\ p_4 \in I\!R^3$ be affine independent. Then the tetrahedron with vertices $p_1,\ p_2,\ p_3$, and p_4, is $V = [p_1 p_2 p_3 p_4]$. For any $p = \displaystyle\sum_{i=1}^{4} \alpha_i p_i \in V$, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p. Let $p = (x, y, z)^T$, $p_i = (x_i, y_i, z_i)^T$. Then the barycentric coordinates relate to the Cartesian coordinates via the following relation

$$\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} \tag{1}$$

Any polynomial $f(p)$ of degree m can be expressed in Bernstein-Bezier(BB) form over V as $f(p) = \sum_{|\lambda|=m} b_\lambda\, B_\lambda^m(\alpha),\quad \lambda \in \mathcal{Z}_+^4$ where $B_\lambda^m(\alpha) = \frac{m!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!}\ \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\alpha_4^{\lambda_4}$ are the trivariate Bernstein polynomials for $|\lambda| = \sum_{i=1}^{4} \lambda_i$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$. Also $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ is the barycentric coordinate of p, $b_\lambda = b_{\lambda_1\lambda_2\lambda_3\lambda_4}$(as a subscript, we simply write λ as $\lambda_1\lambda_2\lambda_3\lambda_4$) are called control points, and $\mathcal{Z}_+^4$ stands for the set of all four dimensional vectors

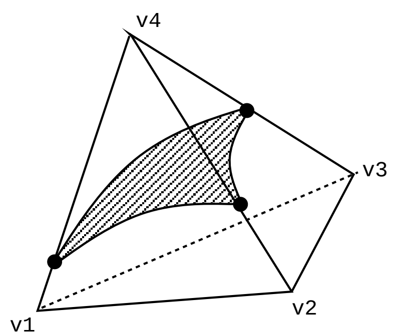

FIGURE 1: A Smooth and Single-Sheeted Triangular Algebraic Surface Patch

with nonnegative integer components. Let

$$F(\alpha) = \sum_{|\lambda|=m} b_\lambda B_\lambda^m(\alpha), \ |\alpha| = 1, \tag{2}$$

be a given polynomial of degree m on the simplex $S = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \in I\!R^4 \ : \ \sum_{i=1}^4 \alpha_i = 1, \quad \alpha_i \geq 0\}$ The surface patch within the simplex is defined by $S_F \subset S \ : \ F(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = 0$.

Definition 3.1 *Triangular algebraic patch.*

If any line segment passing through the j-th vertex v_j of S and its opposite face $S_j = \{(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \ : \ \alpha_j = 0, \ \ \alpha_i > 0, \ \sum_{i \neq j} \alpha_i = 1\}$ intersects S_F only once, then we call S_F a *triangular j-patch* (see Figure 1). Note, for a given simplex, we can have triangular j-patches for $j = 1, 2, 3, 4$.

Lemma 3.1 *The triangular j-patch is smooth (non-singular) and single-sheeted.*

Proof. Let $g(\alpha_1, \alpha_2, \alpha_3) = F(\alpha_1, \alpha_2, \alpha_3, 1 - \alpha_1 - \alpha_2 - \alpha_3)$. The single-sheeted property is trivially satisfied by the single intersection definition of the triangular j-patch. The smoothness of the surface patch S_F requires that $\nabla g(\alpha_1, \alpha_2, \alpha_3) \neq 0$ for every $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$ on S_F. Suppose the triangular j-patch is not smooth, there will be a point $\alpha^* = (\alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*)^T \in S_F$ such that $\nabla g = 0$. Since

$$\frac{\partial g}{\partial \alpha_i} = \frac{\partial F}{\partial \alpha_i} - \frac{\partial F}{\partial \alpha_4}, \qquad i = 1, 2, 3.$$

we have

$$\frac{\partial F}{\partial \alpha_1} = \frac{\partial F}{\partial \alpha_2} = \ldots = \frac{\partial F}{\partial \alpha_4}.$$

By the Euler formula for homogeneous polynomials $\sum_{i=1}^4 \alpha_i \frac{\partial F}{\partial \alpha_i} = 4F$ and $\sum\limits_{i=1}^{4} \alpha_i = 1$, we have $\frac{\partial F}{\partial \alpha_i} = 0, \ i = 1, \ldots, 4$. It follows from the definition of a triangular j-patch that there exists a point $p_1 \in S_j$ and $t = t^* \in [0, 1]$ such that $\alpha^* = t^* v_j + (1 - t^*) p_1 = \alpha(t^*)$. That is $F(\alpha(t^*)) = 0$. And further

$$\left. \frac{\partial F(\alpha(t))}{\partial t} \right|_{t=t^*} = \sum_{i=1}^{4} \frac{\partial F}{\partial \alpha_i} \, \frac{\partial \alpha_i}{\partial t} = 0.$$

This means that t^* is a double zero of $F(\alpha(t))$. A contradiction to the single intersection definition of the triangular j-patch. $\diamond$

Theorem 3.1 (Bajaj et. al. [8]) *Let $F(\alpha)$ be defined as (2), and $j (1 \leq j \leq 4)$ be a given integer. If there exists an integer $k (0 < k < m)$ such that*

$$b_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \geq 0, \qquad \lambda_j = 0, 1, \ldots, k - 1, \tag{3}$$

$$b_{\lambda_1 \lambda_2 \lambda_3 \lambda_4} \leq 0, \qquad \lambda_j = k + 1, \ldots, m. \tag{4}$$

and $\sum\limits_{\substack{|\lambda|=m \\ \lambda_j = 0}} b_\lambda > 0, \ \sum\limits_{|\lambda|=m} b_\lambda < 0$ *for at least one $\lambda_j (k < \lambda_j \leq m)$. Then S_F is a triangular j-patch.*

Proof. Without loss of generality assume $j = 4$. Let $p = (y_1, y_2, y_3, 0)^T \in S_4$ (i.e., $y_i > 0, \sum_{i=1}^{3} y_i = 1$),

$$\alpha(t) = tv_4 + (1-t)p = ((1-t)y_1, (1-t)y_2, (1-t)y_3, t)^T$$

for $t \in (0, 1)$. Then

$$\begin{aligned} F(\alpha(t)) &= \textstyle\sum_{|\lambda|=m} \frac{b_\lambda m!}{\lambda!} y_1^{\lambda_1} y_2^{\lambda_2} y_3^{\lambda_3} (1-t)^{\lambda_1+\lambda_2+\lambda_3} t^{\lambda_4} \\ &= \textstyle\sum_{|\lambda|=m} \frac{b_\lambda(\lambda_1+\lambda_2+\lambda_3)!}{\lambda_1!\lambda_2!\lambda_3!} y_1^{\lambda_1} y_2^{\lambda_2} y_3^{\lambda_3} \frac{m!}{(\lambda_1+\lambda_2+\lambda_3)!\lambda_4!} (1-t)^{\lambda_1+\lambda_2+\lambda_3} t^{\lambda_4} \\ &= \textstyle\sum_{\ell=0}^{m} \left(\sum_{\substack{|\lambda|=m \\ \lambda_4=\ell}} b_\lambda B_{\lambda_1\lambda_2\lambda_3}^{m-\ell}(y_1, y_2, y_3) \right) B_\ell^m(t) \\ &= \textstyle\sum_{\ell=0}^{m} C_\ell(y_1, y_2, y_3) B_\ell^m(t). \end{aligned} \tag{5}$$

By (3) and (4), $C_0 > 0$, $C_\ell \geq 0$, for $\ell = 1, \ldots, k-1$, $C_\ell \leq 0$, for $\ell = k+1, \ldots, m$. If $C_m = \ldots = C_{m-q+1} = 0$ and $C_{m-q} < 0$ for some q with $0 \leq q \leq m-k-1$. Then $F(\alpha(t))$ can be written as

$$F(\alpha(t)) = (1-t)^q \sum_{\ell=0}^{m-q} C_\ell(y_1, y_2, y_3) B_\ell^{m-q}(t). \tag{6}$$

where $C_0 > 0$, $C_{m-q} < 0$, and the sequence $C_0, C_1 \ldots C_{m-q}$ has one sign change. By the variation diminishing property of the functional BB form, the equation $F(\alpha(t))$ has exactly one root in (0,1). $\diamond$

Theorem 3.2 *The dimension of a single triangular j-patch is* $\geq \binom{m+1}{2}$.

Proof. For $F(\alpha)$ defined as (2), and $j (1 \leq j \leq 4)$ a given integer, assign $b_{\lambda_1\lambda_2\lambda_3\lambda_4} > 0$ for $\lambda_j = 0$ and $b_{\lambda_1\lambda_2\lambda_3\lambda_4} < 0$ for $\lambda_j = 2, \ldots, m$. Then from Theorem 3.1 it follows that S_F is a triangular j-patch with free coefficients $b_{\lambda_1\lambda_2\lambda_3\lambda_4}$ for $\lambda_j = 1$. $\diamond$

4 Implicit Algebraic Splines

4.1 Planar Triangulations $\mathcal{T}$

Consider first the question of constructing a C^0-smooth algebraic spline over a triangulation $\mathcal{T}$ in the $z = 0$ plane. If over each triangular face of $\mathcal{T}$, we have a single triangular algebraic patch S_F of degree m, then the set $SI_m(\mathcal{T})$ of all S_F over $\mathcal{T}$ is a vector space of dimension $k = \binom{m+1}{2} f - f$. If we require the patch S_F defined per face of $\mathcal{T}$ to be continuous across face boundaries, then the dimension of $SI_m(\mathcal{T})$ is further restricted and a lower bound is given by Theorem 4.1. This subspace is $SI_m^0(\mathcal{T})$.

Theorem 4.1 *The dimension of* $SI_m^0(\mathcal{T})$*, the set of* C^0*-smooth algebraic splines of degree* $m > 1$ *is* $\geq \binom{m+1}{2} f - (2m+1)e + (2m+1)\alpha - v + 2$.

Proof: At each vertex the patches must have the same z value. Thus C^0 continuity at vertices requires satisfaction of at most $k_v = \sum_{int}(d_i - 1) + \sum_{ext}(d_i - 2)$ linear constraints where d_i is the degree of the ith vertex. Algebraic manipulation yields $k_v = 2e - v - \alpha$. Conics are the lowest degree curves which can form a triangular patch on a surface of degree $m > 1$. By Lemma 2.1 conics suffice for C^0 interpolation of the endpoints of an edge. Since a conic can intersect a degree m algebraic surface in at most $2m$ points (via Bezout), C^0 continuity across each interior edge requires $(2m - 1)$ additional linear constraints for a total of $k_e = (2m - 1)(e - \alpha)$. The total number of independent linear constraints are thus no larger than $(2m + 1)e - 2m\alpha - v$. $\diamondsuit$

If we further require that at each vertex v_i the patches S_F of incident faces have a specified value z_i, then the degrees of freedom of $SI_m^0(\mathcal{T})$ are further restricted by v.

Corollary 4.1 *The dimension of $SI_m^0(\mathcal{T})$, the set of C^0-smooth algebraic splines of degree $m > 1$ having a specified z_i value at each vertex v_i is* $\geq \begin{pmatrix} m+1 \\ 2 \end{pmatrix} f - (2m+1)e + (2m+1)\alpha - 2v + 2$.

Consider next the question of constructing a C^1-smooth algebraic splines over the triangulation $\mathcal{T}$ in the $z = 0$ plane. Since we require each of the triangular algebraic patches S_F defined per face of $\mathcal{T}$, to be C^1 continuous across face boundaries, the dimension of $SI_m^0(\mathcal{T})$ of lemma 4.1 is further restricted and a lower bound is now given by Theorem 4.2. This subspace is $SI_m^1(\mathcal{T})$.

Theorem 4.2 *The dimension of $SI_m^1(\mathcal{T})$, the set of C^1-smooth algebraic splines of degree $m > 1$ is* $\geq \begin{pmatrix} m+1 \\ 2 \end{pmatrix} f - (6m+1)e + 3(2m+1)\alpha + v + 2$.

Sketch of Proof: At each vertex the patches of $SI_m^0(\mathcal{T})$ must now also have the same derivative value. Thus C^1 continuity (including C^0) at vertices requires in total the satisfaction of at most $k_v = 3\sum_{int}(d_i - 1) + 3\sum_{ext}(d_i - 2)$ linear constraints where d_i is the degree of the ith vertex. Algebraic manipulation yields $k_v = 6e - 3v - 3\alpha$. Now the gradient of an algebraic surface of degree m is a (3×1) vector of algebraic surfaces of degree at most $m - 1$. Further by Lemma 2.2 degree three curves suffice for C^1 interpolation of the endpoints and normals of an edge. Since a cubic curve can intersect a degree m algebraic surface in at most $3m$ points (via Bezout), C^0 continuity across each interior edge requires at most $(3m - 1)$ linear constraints. Surface patches which have C^0 continuity of these cubics already satisfy one of the components of the gradient along the edge curves. Hence C^1 continuity across each interior edge requires $(3m - 4)$ additional linear constraints for a total of $k_e = (6m - 5)(e - \alpha)$. The total number of additional linear constraints are thus no larger than $(6m + 1)e - 2(3m - 1)\alpha - 3v$. $\diamondsuit$

If we further require that at each vertex v_i the patches S_F of incident faces have a specified value z_i and a specified derivative vector $\mathbf{n}_i$, then the degrees of freedom of $SI_m^1(\mathcal{T})$ are further restricted by $3v$.

Corollary 4.2 *The dimension of $SI_m^1(\mathcal{T})$, the set of C^1-smooth triangular algebraic splines of degree $m > 1$ having a specified value z_i and a specified derivative vector $\mathbf{n}_i$ at each vertex v_i, is of dimension* $\geq \begin{pmatrix} m+1 \\ 2 \end{pmatrix} f - (6m+1)e + 3(2m+1)\alpha - 2v + 2$.

4.2 Simplicial Complexes $\mathcal{ST}$

Consider first the question of constructing a C^0-smooth algebraic splines over the boundary faces of a spatial triangulation $\mathcal{ST}$ in (x, y, z) space. If over each triangular face of $\mathcal{ST}$, we have a single triangular algebraic patch S_F of degree m, then the set $SI_m(\mathcal{ST})$ of all S_F over $\mathcal{ST}$ is a vector space of dimension $k = \binom{m+1}{2} f - f$. If we require the patches S_F defined per face of $\mathcal{ST}$ to be continuous across face boundaries, then the dimension of $SI_m(\mathcal{ST})$ is further restricted amd a lower bound is given by Theorem 4.3. This subspace is $SI_m^0(\mathcal{ST})$.

Theorem 4.3 *The dimension of $SI_m^0(\mathcal{ST})$, the set of C^0-smooth algebraic splines of degree $m > 1$ is $\geq \binom{m+1}{2} f - (2m+1)e - 2v - 4g + 2$.*

Sketch of Proof: At each boundary vertex the surface patches must be zero. Thus C^0 continuity at boundary vertices requires satisfaction of at most $k_v = \sum_{boundary} d_i = 2e$ linear constraints where d_i is the degree of the ith vertex. Conics are the lowest degree curves which can form a triangular patch on a surface of degree $m > 1$. By Lemma 2.1 conics suffice for C^0 interpolation of the endpoints of an edge. Since a conic can intersect a degree m algebraic surface in at most $2m$ points (via Bezout), C^0 continuity across each boundary edge requires at most $(2m - 1)$ additional linear constraints for a total of $k_e = (2m - 1)e$. The total number of independent linear constraints are thus no larger than $(2m + 1)e$. $\diamondsuit$

Consider next the question of constructing a C^1-smooth algebraic spline over the boundary faces of a spatial triangulation $\mathcal{ST}$. Since we require each of the triangular algebraic patches S_F defined per face of $\mathcal{ST}$, to be C^1 continuous across face boundaries, the the dimension of $SI_m^0(\mathcal{ST})$ of lemma 4.3 is further restricted and a lower bound is now given by Theorem 4.4. This subspace is $SI_m^1(\mathcal{ST})$.

Theorem 4.4 *The dimension of $SI_m^1(\mathcal{ST})$, the set of C^1-smooth algebraic splines of degree $m > 1$ is $\geq \binom{m+1}{2} f - 2(3m-1)e - v - 2g + 2$.*

Sketch of Proof: At each vertex the patches of $SI_m^0(\mathcal{ST})$ must now also have the same derivative value, the choice of which is arbitrary. Thus C^1 continuity (including C^0) at boundary vertices requires in total the satisfaction of at most $k_v = \sum_{boundary} d_i = 2e$ linear constraints where d_i is the degree of the ith vertex. These are the same number of constraints required to enforce C^0 continuity at the vertex. The reason why these constraints suffice even for C^1, is because C^1 continuity of each patch along the two edge curves incident at the vertex (which we consider below) automatically guarantee the patch to be C^1 at the vertex.

Now the gradient of an algebraic surface of degree m is a (3×1) vector of algebraic surfaces of degree at most $m - 1$. Further by Lemma 2.2 degree three curves suffice for C^1 interpolation of the endpoints and normals of an edge. Since a cubic curve can intersect a degree m algebraic surface in at most $3m$ points (via Bezout), C^0 continuity across each boundary edge requires $(3m + 1 - 2)$ linear constraints. Surface patches which have C^0 continuity of these cubics already satisfy one of the components of the gradient along the edge curves. Hence C^1 continuity across each interior edge requires $(3(m - 1) + 1 - 2)$ additional linear constraints for a total of $k_e = (6m - 5)e$. The total number of linear constraints are thus no larger than $3(2m - 1)e$. $\diamondsuit$

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